MECHANICS
for
ENGINEERS

MECHANICS for ENGINEERS

Statics and Dynamics

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PREFACE

This is a textbook on theoretical mechanics for students of engineering; it covers statics and dynamics as usually taught at the undergraduate level. It is the outgrowth of repeated revisions of *Technical Mechanics*, a textbook of long standing, which was published originally in 1903, rewritten in 1914 and in 1925, and which has now been rewritten again. In this third rewriting the book has been so changed and amplified that the adoption of a new and more accurately descriptive title seemed advisable.

The scope of the book was determined by consideration of present teaching practice in American schools of engineering and of the trend toward more thorough and more advanced instruction in mechanics, especially in dynamics. The arrangement and method of presentation represent the consensus of the authors and their colleagues, based on a collective experience of many years in the teaching of mechanics and of subjects for which theoretical mechanics is a prerequisite — strength of materials, structural analysis, machine design, hydraulics, mechanical vibrations, and aerodynamics. Readers familiar with *Technical Mechanics* will readily apprehend the changes that characterize this new book. In statics, more attention is given to problems that involve non-coplanar forces. In dynamics, kinematics and kinetics are treated concurrently instead of separately. It is believed that the early introduction of the force-acceleration relationship greatly helps the student to understand acceleration, lends reality to concepts that might otherwise seem mere mathematical abstractions, and sustains interest by making possible the consideration of complete problems of a practical nature. The methods of vector analysis have been used more generally in the discussion of curvilinear motion of a particle and of spherical and general motion of a rigid body. Plane motion and relative motion are dealt with more extensively, and d'Alembert's principle receives greater emphasis in view of the increasing importance of such subjects as engine balancing and dynamic stress. Topics that have been added are virtual work and mechanical vibrations, an introduction to the latter being given in the Appendix.

Every effort has been made to present essential principles with clarity and rigor, and to illustrate their applications and emphasize their physical significance by a great number of examples and problems taken from the various fields of engineering and from the everyday experience of the student. Especial pains were taken in the preparation of an extensive collection of carefully graded problems, which will be found in the back of the book. Problems intended to illustrate the application of some single principle are grouped in the

approximate order of difficulty under the appropriate article heading. Problems the solution of which requires the application of several different principles, and problems that are comparatively long or difficult, are designated as "General," under the appropriate chapter heading. Complete or partial answers to many of these problems are given, for it is the authors' experience that a student is more likely to attempt the solution of a difficult problem, and that he derives more satisfaction from success, when he knows what the answer should be. The instructor who prefers problems to which answers are not given will find these in ample number. Indeed, many more of both kinds of problems are provided than it would be reasonable to assign to a class, the collection having been made large enough to permit freedom of selection and changes in assignments from term to term.

The authors wish to thank their colleagues, Professor M. O. Withey, Professor J. B. Kommers, Professor K. F. Wendt, and Mr. H. R. Puckett, for suggestions and constructive criticism. In the securing of data for examples and problems the cooperation of many correspondents was received, and in particular grateful acknowledgment is made of the helpfulness of the late Mr. David Lindquist, chief engineer of the Otis Elevator Company; of the General Motors Corporation; of the Army Ordnance Department; and of Mr. Harry Bauer of the Bucyrus-Erie Company.

INTRODUCTION

The purpose of this introduction is to give the student some idea as to the nature of mechanics — to tell him what it is about, and to indicate its uses in engineering.

Mechanics is the science that treats of the action of forces on bodies; it comprises statics, which deals with the action of forces on bodies at rest or in equilibrium, and dynamics, which deals with the action of forces in producing or modifying the motion of bodies. Dynamics in turn is sometimes subdivided into kinematics, which treats of motion as such, without regard to its cause, and kinetics, which treats of the influence of forces on the motion of bodies. Although mechanics comprehends the action of forces on bodies of any kind, it is usually understood to relate particularly to solid bodies; the mechanics of gases and liquids is usually called fluid mechanics, or, more specifically, aerodynamics, hydrodynamics, hydrostatics, or hydraulics, as may be appropriate.

It is expedient to distinguish theoretical mechanics and applied mechanics. The former comprises the body of exact laws and principles that have been mathematically deduced from certain fundamental facts known through observation and experiment; the latter consists of the application of these laws to practical problems. In engineering, these problems are those encountered in the design of structures and machines, and almost always involve the answering of these questions: What are the forces (loads) that come upon the structure or machine and on its parts; and how large, of what form, and of what material should the parts be made in order that they may sustain these forces with safety? Some of the forces in question will be known at the outset; some can be assumed in accordance with engineering judgment and experience; and some must be calculated. Consider, for example, the problem of designing a bridge. The weights of the vehicles it is to carry are known; the wind pressure it must withstand can be assumed on the basis of past records and experiment, but the supporting reactions of the piers and the mutual pressures and tensions of the members must be calculated. Since the bridge is stationary, the unknown forces are found by the principles of statics. On the other hand, in designing a steam engine or other machine, the parts of which are to move in a specified manner, the unknown forces are found by the principles of dynamics.

When all the forces that act on a given part are known, there has yet to be ascertained the effect these forces have in stretching, bending, twisting, or breaking it. The study of the relations between the forces that act on a body and the changes they produce in its size or form, or the tendency they have to break it, is the province of that branch of applied mechanics generally called

"mechanics of materials" or "strength of materials." We are not, in this book, concerned with these effects. The scope of that part of mechanics with which we here deal is limited to the definition and analysis of the relations existing between balanced forces, between the various aspects of pure motion, and between the forces that act on a body and the motion of that body.

The thoughtful student will not fail to perceive as he pursues the study of the subject that many of the simpler relations stated were already known to him through experience, though probably in a general and qualitative rather than precise and formal way. Indeed, great progress was made in the application of the principles of mechanics long before the precise nature of these principles was understood. Experience, observation, intuitive understanding all played a part and enabled the earlier builders to make elaborate structures and to employ successfully mechanical devices of some complexity. But it was only with the recognition of the simple and precise principles involved that mechanics as a science can be said to have begun. The early history of the development of that science is a record of the perception, by different thinkers or observers, and at long intervals, of principles that appear now to be simple. Often these principles were of such a nature as to appear in some commonly used mechanical device, for example, the lever and the inclined plane, the principles of which respectively were first perceived by Archimedes (287–212 B.C.) and Stevin (1548–1620). Again, they might be of a nature to be perceived from familiar occurrences or easily performed experiments; thus Galileo (1564–1642) proved by his famous experiments at the Tower of Pisa that bodies fall with equal rapidity regardless of their weight. The first broad systematization and formal statement of mechanical laws was the work of Newton (1642–1727). "No essentially new principle of mechanics has been stated since Newton's day. All that has been accomplished has been a deductive, formal and mathematical development on the basis of his laws."* This development is relatively enormous, as are also the number and variety of the adaptations to engineering. Only a small part, such as is likely to be of most interest and use to engineers, is included in this book.

* From Mach's *Science of Mechanics*. But since that critical work was written, a new system of mechanics has been invented and developed by Einstein and others, which in some respects supersedes the classical or Newtonian mechanics, though not in the fields of engineering.

NOTATIONS

A acceleration regarded as a vector quantity.
a acceleration in rectilinear motion; magnitude of acceleration.
B a constant (in vibration theory).
b distance from axis of suspension to mass-center of a pendulum.
C a couple; constant of integration.
c coefficient of rolling resistance; ratio of tension at lowest point of catenary cable to its weight per unit length; coefficient of damping.
d distance from mass-center to an axis for transfer of axes formula in moment of inertia theory; distance from axis of suspension to center of percussion.
E energy.
e coefficient of restitution; spring elongation.
F general symbol for force.
G center of gravity or mass-center.
g acceleration of gravity.
H horizontal component of cable tension; angular momentum regarded as a vector quantity.
h magnitude of angular momentum.
I moment of inertia.
J angular impulse.
K product of inertia.
k radius of gyration; spring constant or factor.
L length; linear impulse regarded as a vector quantity.
l length; magnitude of linear impulse.
M moment of a force or force system; bending moment.
m mass.
N normal pressure; angular velocity of precession.
n frequency of a simple harmonic motion; angular velocity of spin.
O origin; base point or other reference point.
P general symbol for point, particle or force; power.
Q center of percussion; center of gyration.
q circular frequency of a free undamped vibration.
R resultant force; radius.
r radius.
S stress in a truss member; stress in a spring, or spring force.
s position coordinate.

T tension in a cable; period of a simple harmonic motion; period of a pendulum; twisting moment in a bar.
t time.
U linear momentum regarded as a vector quantity.
u magnitude of linear momentum; relative jet velocity; circular frequency of damping force.
V velocity regarded as a vector quantity.
v velocity in rectilinear motion; magnitude of velocity.
W weight of a body.
w weight of an elementary particle; weight per unit volume or unit length.
X, Y, Z rectangular coordinate axes.
x, y, z rectangular coordinates.

GREEK LETTER SYMBOLS

α (l. c. alpha) angular acceleration; direction angle of a vector with respect to the x axis; general symbol for an angle; symbol for a constant in vibration theory.
β (l. c. beta) direction angle of a vector with respect to the y axis; general symbol for an angle; magnification factor (in vibration theory).
γ (l. c. gamma) direction angle of a vector with respect to the z axis.
δ (l. c. delta) logarithmic decrement of a vibration.
η (l. c. eta) efficiency.
θ (l. c. theta) general symbol for an angle.
κ (l. c. kappa) torsional spring factor for shaft of unit length.
μ (l. c. mu) coefficient of friction.
ρ (l. c. rho) radius vector; density; radius of gyration in vibration theory.
Σ (cap. sigma) symbol for summation.
τ (l. c. tau) one-fourth the period of a pendulum.
ϕ (l. c. phi) general symbol for an angle.
Ψ (cap. psi) angle between the instantaneous axis and the axis of precession
Ω (cap. omega) angular velocity regarded as a vector quantity.
ω (l. c. omega) magnitude of angular velocity; circular frequency.

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STATICS

CHAPTER I

FORCES AND FORCE SYSTEMS

1. Nature of a Force. The word **force** is a general term for any push or pull. A force is always exerted on a body by another body, or on a part of a body by another part. Though a force is really an action of one body on another, it is customary and convenient to speak of the force itself as acting on the body to which it is applied.

A force may act through contact like the pressures of a crankshaft on its bearings, or it may act from a distance like gravitational or magnetic attraction. It may act on or be distributed over a considerable area of contact like the thrust of earth against a retaining wall, or it may act on so small an area as to be practically concentrated at a point like the pressure of a locomotive wheel on a rail. But whether exerted through contact or from a distance, whether distributed or concentrated, a force is always exerted *on* something *by* something. You should avoid speaking of a supposed force that cannot be thus accounted for.

The *gravitational force* exerted on a body by the earth acts toward the earth's center; it is a distributed force, acting on all the particles that make up the body, but for many purposes it is convenient and correct to regard it as a concentrated force acting at a point called the *center of gravity* of the body. The exact significance of the term "center of gravity" and the way in which the position of the point is determined for any given body are explained in Chapter VI.

The gravitational attraction or earth-pull on a body is commonly called the **weight** of the body. Generally we use "weight" in this sense; but occasionally we use "earth-pull" instead to avoid any possible misunderstanding; and sometimes, following common usage, we speak of a relatively heavy body as "a weight."

2. Description and Representation of a Force. Your earliest notions about forces were based on your own experience with forces exerted by or on yourself. For example, when moving a heavy body you perceived that the force you applied had (i) *magnitude*, according to how hard you pushed or pulled; (ii) *direction*, according to whether you pushed or pulled up, down, to the right, or left; (iii) *place of application*, according to where you grasped the body. These three attributes — magnitude, direction, and place of applica-

tion — serve to describe a force and are called the elements or characteristics of a force.

The *line of action* of a concentrated force, or a force so considered, is a line of indefinite length, parallel to the direction of the force, and containing its point of application. In other words, it is the line along which the force acts. A force may act along its line of action in either of two ways: to the right or left, up or down, etc. Accordingly, the *sense* of a force is right or left, up or down. For many purposes a force is sufficiently described by its magnitude, line of action, and sense, because, as will be presently explained, the position of the point of application in the line of action is of no consequence in the problems of statics.

It is convenient and customary to speak of the position or location of a concentrated force, or one so considered, meaning thereby the position or location of the line of action of the force.

REPRESENTATION OF A FORCE. The obvious and natural way of representing a force is this: first the body on which the force acts is represented by a sketch; then the line of action of the force is indicated by a line drawn in the correct place, the sense being shown by an arrowhead on the drawn line; and finally a letter, F, P, Q, etc., is affixed to serve as a name or label for convenient reference. If the numerical magnitude of the force is known it should be indicated, and it will usually serve to identify the force in lieu of a letter. For example, suppose that a ring is suspended by two chains and supports a load of 100 lb by means of a third chain, as shown in Fig. 1*a*. The forces exerted on the ring by the chains are represented in Fig. 1*b*. The sketch of the body, in this example the ring, is important; it adds reality to the representation and serves as a reminder that all forces act on *bodies*.

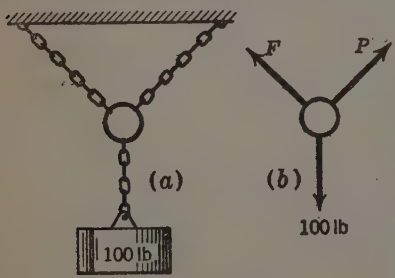


FIG. 1.

For simplicity, in the discussion of principles and methods we shall often represent the body on which the forces act by a subdivided square or rectangle. But you should think of the square or rectangle as a physical body — a board or block or structure or machine part — and not as a mere geometrical figure.

Two different meanings of the word "direction" should be noted. For example, the direction of a certain street is north and south, but the direction of a wind blowing along the street is to the north or south. Similarly, the direction of the line of action of a certain force is, say, vertical; but the direction of the force is vertically downward or upward. Thus, direction *includes* sense when used with reference to anything that *has* sense, and otherwise means simply orientation or alignment.

3. Units of Force. To express the magnitude of a force, one must, of course, compare it with some other force regarded as a unit. Many units of force are in use; the most convenient are the *gravitational units*, which are the earth-pulls on certain control standards, pieces of metal skillfully prepared and carefully preserved, for measuring quantity of material (as iron, coal, etc.) "by weight."

The earth-pull on any of these standards is called by the name of the standard: thus, the earth-pull on the kilogram standard (described in Art. 87) is called a kilogram force; the earth-pull on the pound standard is called a pound force, etc.

Since the earth-pull on any given thing is slightly different in different places,* each place really has its own local gravitational unit force (kilogram, pound, etc.). Fortunately the difference between the local pound forces for any two places is negligible in most engineering calculations. The extreme variation in any gravitational unit is that between its magnitudes at the highest elevation on the equator and at the poles; this difference is but 0.6 per cent; for points within the United States the extreme variation is about one-half as much.

It seems desirable to establish gravitational units of force that are absolute, or single valued. This has not been done officially. But many writers have defined these units as earth-pulls on the kilogram, pound, etc., at any place where the acceleration due to gravity has the standard value, namely, 9.80665 meters (or 32.1738 ft) per sec per sec. This value, which has some legislative sanction, corresponds closely to the actual value at sea level in latitude 45°.

4. The Effects of Forces. In the introduction to this book, mechanics is defined as the science that treats of the effects of forces, in particular the effects of forces on solid bodies. It is explained that, in general, forces have two effects on a body, namely (i) to cause it to move (if at rest but free to move) or to move differently (if in motion), and (ii) to deform it. It is also explained that the study of the relations between forces and the deformations produced by them is the province of mechanics of materials; that the study of the relations between forces and motion is the province of dynamics; and that the study of the circumstances under which a number of forces acting simultaneously on a body mutually counteract one another and result in equilibrium is the province of statics.

In statics, therefore, one may distinguish as an effect of a force the tendency to preserve or destroy equilibrium; and, since supporting or equilibrating forces are generally produced by the application of a force to a body not free

* The discovery of this fact was made in 1671, when a pendulum was transported from Paris, where its period had been accurately determined, to French Guiana, and there found to vibrate more slowly. The difference in periods was correctly ascribed to a lesser earth-pull on the pendulum at its new location. (See Art. 116 for pendulum theory.)

to move, it may be said also that one of the effects of a force is to produce, or bring into action, other forces. Statics deals mainly with the equilibrating and force-producing effects of forces.

5. Principles of Transmissibility and Action and Reaction. *The tendency of a concentrated force to maintain or destroy the equilibrium of a rigid body does not depend on the position of the point of application in the line of action.* This fact is known as the **principle of transmissibility.**

As an example, consider the block (Fig. 2) which is prevented from sliding down the smooth plane by a force P. The force required is the same, whether applied as a pull at A or as a push at B. Obviously, also, the pull could be

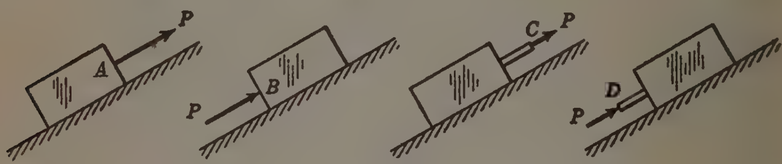


FIG. 2.

applied at C, the upper end of a weightless cord, or the push at D, the rear end of a weightless rod; and the length of such cord or rod would make no difference. This could be proved by measuring P under each of the conditions described, using a spring balance, but to most students the truth of the principle will appear self-evident.

The cord or rod mentioned may be thought of as a weightless, rigid extension of the block. This concept of a weightless and rigid extension of a body is useful in some discussions, as it enables one to think of a body as acted on by a force whose line of action does not pass through the body; one simply imagines the force to act on such an extension of the body.

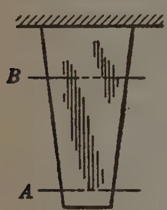


FIG. 3.

It should be noted that the principle of transmissibility as stated refers to the equilibrating effect of a force. The effect of a force in deforming or breaking a body may depend upon where the force is applied along its line of action; thus it is obvious that a tapered strip of rubber hanging as shown in Fig. 3 would be stretched farther, and be more likely to be broken, by a load applied at A than by the same load applied at B.

ACTION AND REACTION. *When one body exerts a force upon another body then the latter also exerts one on the former; and these two forces are equal in magnitude, are opposite in sense, and, if concentrated or so regarded, have a common line of action.* These facts may be expressed by saying that action and reaction are equal, opposite, and collinear, action meaning either of the two forces and reaction the other. This last sentence is really a brief statement of Newton's third law of motion (Art. 88), given here because it is essential

in statics. For an illustration of this law refer to Fig. 1*b*, where the forces exerted on the ring by the chains are represented. The ring exerts forces on the chains equal and opposite to those represented. Or suppose that you push with your hand against a wall; the wall exerts an equal, opposite, and collinear force against your hand. This statement is true whether the wall stands or topples.

6. Force Systems; Description. Mechanics deals largely, as you will see, with groups or systems of forces. The forces of a system are called *coplanar* if their lines of action are in the same plane, and *noncoplanar* if they are not in the same plane; they are called *concurrent* if their lines of action intersect at a point, and *nonconcurrent* if they do not so intersect; they are called *parallel* if their lines of action are parallel, and *nonparallel* if their lines of action are not parallel. A given force system is described, in accordance with the foregoing definitions, as concurrent, noncoplanar parallel, etc., according as the forces of the system are concurrent, noncoplanar parallel, etc.

To facilitate systematic discussion, we classify force systems as follows:

- Coplanar
  - Concurrent
    - Collinear
    - Nonparallel
  - Nonconcurrent
    - Parallel
    - Nonparallel
- Noncoplanar
  - Concurrent
  - Nonconcurrent
    - Parallel
    - Nonparallel

COUPLES. A force system consisting of a pair of forces that are equal, parallel, and opposite in sense is called a **couple.** The two forces of a couple can be dealt with individually, like the forces of any other system, but it is usually convenient to consider the couple as a unit. So considered, a couple has noteworthy properties, and these are discussed in Art. 21.

7. Vector Representation of Forces. Representation of a force as described in Art. 2 is in part pictorial or graphical. But, in mechanics, the word graphical is reserved for representations of forces and other quantities by vectors, a **vector** being defined as a straight line of definite length and direction, including sense, generally indicated by an arrowhead on the line but sometimes for convenience by the alphabetical order of letters at the ends of the line.

Thus to describe a force completely, a scale drawing is made of the body on which the force acts, and on this drawing is shown the line of action of the force; the force is further described by a vector whose direction is the same as that of the force and whose length is equal to the magnitude of the force according to some specified scale.

This vector may be drawn on the line of action of the force, in which event it is a *localized vector* and completely represents the force; or it may be drawn

at any convenient place, in which event it is a *free* or *unlocalized* vector and represents only the magnitude and direction of the force, the position of the force being given by the drawn line of action. The first method is illustrated in Fig. 4*a*, where the localized vector AB, made 0.8 in. long, completely represents, to a scale of 1 in. = 125 lb, the 100-lb force acting on the ring of Fig. 1*a*. The second method is illustrated in Fig. 4*b*, where the free vector AB represents the magnitude and direction of W and the indicated line of action represents the position of W.

The first method requires using two different scales in one figure, which is likely to be confusing to beginners. For this and other reasons that will

FIG. 4.

appear later, the second method is usually preferable, and it is generally employed in this book. When this second method is used, the scale drawing that shows the body and the line of action of the force is called **space diagram**; the scale drawing that shows the vector is called **vector diagram.**

8. Note on Vector Analysis. Only a few vectorial operations are employed in this book, namely, addition, subtraction, and differentiation. The first and second are explained below, and the third in Art. 99. The symbols for vectorial addition and subtraction are $\mapsto$ and $\rightarrow$, respectively.

VECTOR ADDITION. The rule for adding two vectors is: draw the vectors, tail to tip; then draw a line joining their free ends, and place an arrowhead on this line pointing from the free tail toward the free tip. This third vector is defined to be the sum of the given two. See Fig. 5*a*, where two additions of V_1 and V_2 are shown. Obviously, the sums agree.

Addition of any number of vectors is accomplished by adding to the sum of the first two the third one; to the sum of the first three the fourth one, etc., until the sum of all is arrived at. See Fig. 5*b* for the summation of four vectors, V_1, V_2, V_3, and V_4. You see that the vectors representing the intermediate sums are unnecessary.

You see also that the desired sum may be found by drawing the vectors, tail to tip, in succession in *any* order. Any such succession of vectors is called a **vector polygon.** Observe that the arrowheads of a vector polygon are confluent, that is, point the same way around; observe also that a vector polygon is not, in general, a closed figure. The tail of the first vector drawn and

the tip of the last vector are called *the beginning* and *the end*, respectively, of the vector polygon. In these terms, the rule for adding several vectors is this: draw any vector polygon for the vectors; the vector extending and pointing from the beginning to the end of the polygon is the desired sum.

The sum of the vectors that represent any vector quantities (alike in kind, of course) is called the *vector sum* of the quantities. Such a sum usually has important or significant meaning; the ordinary or *scalar sum* of the vector quantities generally has no meaning. For example, the vector sum of any number of forces gives the magnitude and direction of the resultant of the forces (Arts. 26 and 28); the scalar sum of the forces is without meaning unless they are parallel.

If you are not already thoroughly familiar with vector addition, you should master the process at once. Test yourself: on a suitable sheet of paper draw four vectors at random; then find their sum at least three times from three different vector polygons; and finally compare your answers. How many different polygons can be drawn for four vectors?

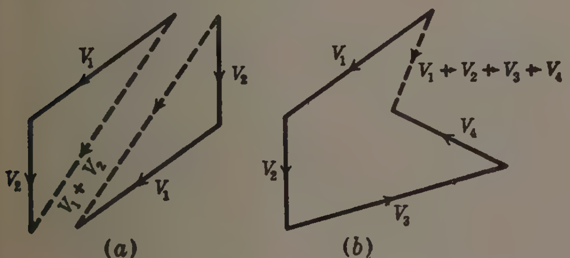


FIG. 5.

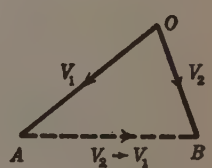


FIG. 6.

VECTOR SUBTRACTION. Basically, subtraction of vectors is like subtraction of scalars. Subtraction of scalar 2 from scalar 7 may be looked upon as a search for a scalar which if added to 2 gives 7; similarly, subtraction of vector V_1 from vector V_2 is a search for a vector which if added to V_1 gives V_2. The rule for actually finding $V_2 \rightarrow V_1$ is: from any chosen point O, Fig. 6, draw the vectors V_1 and V_2, and mark their tips A and B, respectively; then vector AB is $V_2 \rightarrow V_1$, for AB added to V_1 equals V_2.

9. Note on the Quantities Dealt with in Mechanics. The quantities of mechanics (force, mass, velocity, energy, time, etc.) are distinguished as vector or scalar. A *vector quantity* has both magnitude and direction; a *scalar quantity* has only magnitude, with or without sign. Force and velocity are vector quantities; mass and time are scalar quantities.

Vector quantities that are regarded as having position are said to be *localized;* those regarded as without position are said to be *unlocalized.* Force and linear momentum (Arts. 7 and 156) are localized; velocity and angular momentum (Arts. 95 and 161) are unlocalized.

CHAPTER II

COMPOSITION AND RESOLUTION OF FORCES

A. Definitions and General Principles

10. Equilibrium and Equivalence of Force Systems. A body at rest is said to be in **equilibrium**; and the force system comprising all the forces that act on the body is also said to be in equilibrum or to be balanced.

The conditions under which various kinds of force systems are in equilibrium are discussed fully in Chapter III, but two relevant and important facts that may be taken as axiomatic are stated here: (i) *a single force cannot be in equilibrium*, and (ii) *if two forces are in equilibrium, they are equal, opposite, and collinear.*

Any number of the forces of a system in equilibrium may be regarded as holding in equilibrium, or balancing, the remainder; thus the given system may be regarded as composed of two systems, A and B, which balance each other. Any other system C that would balance A is equivalent to B; any other system D that would balance B is equivalent to A. *Two force systems are, then, equivalent when they have the same balancing or equilibrating effect.*

11. The Resultant of a Force System. By **resultant** of a given force system is meant the simplest system equivalent to the given system, simplest system meaning the one containing the fewest forces. It will be shown that the resultant of a given force system is either a force, a couple, or two noncoplaner forces, depending upon the nature of the system. By **antiresultant** of a force system is meant the reversed resultant. Obviously, the antiresultant would balance the resultant and hence balance the system. Therefore the antiresultant of a system is called also the **equilibrant** of the system. If the resultant of a force system is zero, or nil, the forces of the system mutually balance.

The following illustration may serve to make clear the meaning of the terms just defined. Figure 7 represents a ring which is supported by two cords and to which is attached, by means of a third cord, a heavy body of weight W. The ring is acted on by the pulls F_1 and F_2 of the two inclined cords and by the pull W of the vertical cord. These three forces are in equilibrium, since the ring remains at rest. Any one of the three (say W) is the equilibrant of the other two (F_1 and F_2). W is also the antiresultant of F_1 and F_2; the resultant R of F_1 and F_2 is equal and opposite to and collinear with their antiresultant W.

The process of determining the resultant of a given force system is called

composition; the forces are said to be compounded or reduced to their resultant. The methods by which composition is effected are discussed in the next article and in other articles.

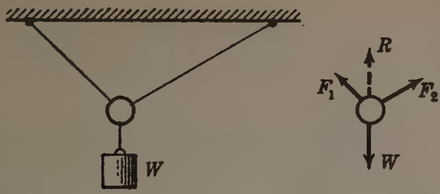


FIG. 7.

12. Two Methods of Analysis. We digress here to explain that two distinct methods may be employed in the solution of problems of statics, namely, the algebraic and the graphical methods. In the algebraic method any drawings used may be sketches; any calculations are of the ordinary kind, made with slide rule, tables, etc. In the graphical method, drawings must be accurate and to scale; any calculations are geometrical or vectorial. How these calculations are performed is explained later.

Though the algebraic method is better adapted to some problems and the graphical to others, the consideration of both methods usually contributes to the thorough understanding of a problem, and in this book the application of each is explained and illustrated except where the graphical method is impracticable. Choice of method may depend on the facilities (slide rule, tables, drawing instruments, etc.) at hand, or simply on personal preference, for or against drawing. Occasions sometimes arise when the two methods may be used to advantage.

13. Composition of Two Concurrent Forces. In approaching the problem of composition of concurrent forces we consider first two forces, say F_1 and F_2 (Fig. 8*a*), which act on a board at points 1 and 2, in the plane of the board. For such conditions there are four rather obvious important facts concerning the resultant R, namely: (i) R lies in the plane of F_1 and F_2; (ii) R acts through the point of concurrence of F_1 and F_2; (iii) R lies somewhere between F_1 and F_2 as shown — in the angle formed by the two lines drawn from the point of concurrence in the directions in which the two forces act; (iv) the sense of R is as indicated, that is toward the right. If you regard these facts as obvious, supported by common sense and experience, you might go further, and estimate with fair accuracy the magnitude and direction of R. However, precise determination is generally desired. This can be accomplished by means of either law following.

PARALLELOGRAM LAW. *If two concurrent forces be represented in magnitude and direction by vectors OA and OB, their resultant is represented by the vector forming the diagonal OC of the parallelogram constructed on OA and OB.*

Referring to Fig. 8*b*, suppose that $F_1 = 30$ and $F_2 = 50$ lb. In the vector diagram, OA and OB are drawn to represent F_1 and F_2, respectively, to a scale of 1 in. = 50 lb. Then the resultant R is represented in magnitude and direction by OC. OC is 1.00 in. long; therefore, $R = 1.00 \times 50 = 50.0$ lb. The line of action of R is indicated in the space diagram; it is drawn through

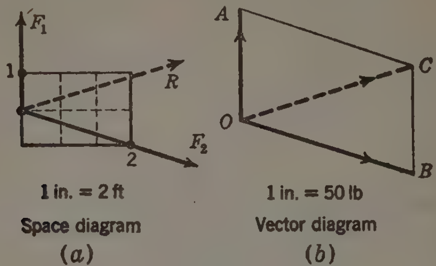


FIG. 8.

the point of concurrence of F_1 and F_2, parallel to OC. The resultant is thus completely determined and graphically described. (The vectors OA and OB could be laid off on the lines of action of F_1 and F_2 in the space diagram, but, as already stated in Art. 7, it is better to keep the space and vector diagrams entirely separate.)

The parallelogram law is generally taken as the basis for the whole science of statics. Substantially, it was first perceived by Stevin (1548–1620) and utilized by him to solve some simple problems of statics. Many mathematical proofs of it have been devised.* We regard it as sufficiently established through many years of practical use without discovered error.

TRIANGLE LAW. *The resultant of two concurrent forces is represented, in magnitude and direction, by the sum of the vectors that represent the two forces.*

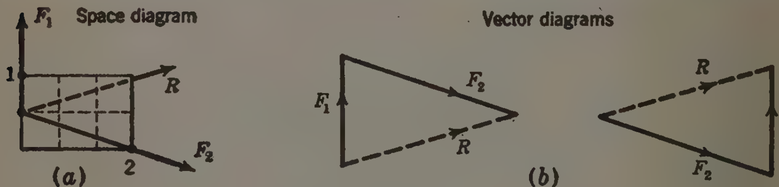


FIG. 9.

For example, consider the forces F_1 and F_2 acting on the body shown in Fig. 9*a* (the same as Fig. 8*a*). Two vector summations are shown in Fig. 9*b*; each sum gives the magnitude and direction of the resultant R. The line

* Authors of eighteen proofs are listed in Whewell's *Philosophy of the Inductive Sciences*, Vol. 1, p. 205.

of action of R passes, as already pointed out, through the intersection of the lines of action of F_1 and F_2, *in the space diagram.*

The triangle law is a modern equivalent of the much older parallelogram law, from which it follows readily. The triangle of forces is to be preferred both because it involves the drawing of fewer lines and because it can be extended easily, as shown in Art. 26, to effect the composition of any number of concurrent forces.

An easy algebraic method for determining the resultant of two given concurrent forces is suggested by the triangles of Fig. 9*b*. The magnitude and direction of the two given forces being known, two sides and the included angle of the triangle are known; from these data, the length and direction of the third side can be determined by trigonometry. Obviously, this method does not require that the triangle be drawn to scale. It is advisable to make a good sketch from which you can estimate the magnitude and direction of the resultant as a check on the calculated values.

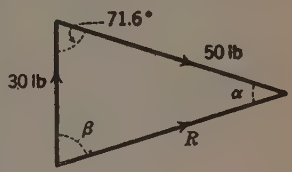


FIG. 10.

For example, see Fig. 10, which is a sketch of the first triangle of Fig. 9*b*. The given numerical values of F_1 and F_2 are indicated. The value of the angle opposite the side R was computed from Fig. 9*a*. The angles opposite the sides 30 and 50, respectively, equal α and β of that figure and are so indicated. Finally,

$$R^2 = 30^2 + 50^2 - 2 \times 30 \times 50 \cos 71.6 \quad \text{whence} \quad R = 49.5 \text{ lb}$$

$$\frac{\sin \alpha}{\sin 71.6} = \frac{30}{49.5} \quad \text{and} \quad \frac{\sin \beta}{\sin 71.6} = \frac{50}{49.5}$$

$$\alpha = 35.0° \quad \text{and} \quad \beta = 73.4°$$

14. Components of a Force. The forces of a system that is equivalent to a given force are called **components** of that force. The process of determining, for a given force, an equivalent system of components is called **resolution**; the given force is said to be resolved into components. If the given force and its components are concurrent, the force is said to be resolved *at* the point of concurrence.

Obviously there are many different force systems which are equivalent to any given force, and any such equivalent system may have any number of forces; hence there are, for a given problem of resolution, many different solutions, unless restrictions are imposed as to the number, direction, magnitude, etc., of the components. Such restrictions are usually indicated by the conditions of the problem in hand.

Methods for resolving a force into components are explained in the next few articles.

15. Resolution of a Force into Two Concurrent Components. Note the word "concurrent." Two such components must intersect on the line of action of the given force; thus, as in Fig. 11*a*, where P and Q are components of F, the three forces have a common point of concurrence. This follows from the fact that F is the resultant of P and Q, and, as you saw in Art. 13, two concurrent forces and their resultant do have a common point of concurrence.

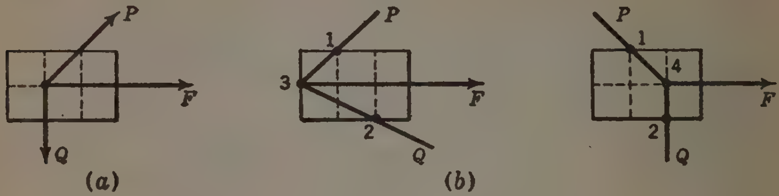


FIG. 11.

To illustrate further, suppose that it is required to indicate two components, P and Q, of the force F of Fig. 11*b*, it being specified only that P and Q respectively act through points 1 and 2. The common point of concurrence may be anywhere on F, as 3 or 4. (Judge the senses of P and Q, and place arrowheads accordingly. Remember that P and Q are equivalent to F; they do not balance F.) The magnitudes of P and Q depend on the point of concurrence chosen, of course. Generally, one resolves a force into components that satisfy given specifications which completely determine them.

The graphical method for finding the magnitude (and sense) of required components consists in applying the parallelogram or triangle law inversely. In that way one finds two forces (satisfying given specifications, if any) whose resultant is the given force. These forces are the required components.

EXAMPLE 1. It is required to resolve F, 100 lb (Fig. 12), into two components, parallel to the diagonals of the rectangle. *Solution* by the parallelogram construction: O was chosen as common point of concurrence; lines parallel to the diagonals were drawn through O; these are

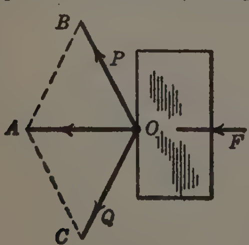


FIG. 12.

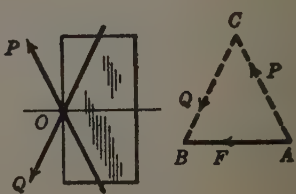


FIG. 13.

the lines of action of the components, say P and Q respectively. Obviously the senses of P and Q are as indicated by the arrowheads. Then the vector OA was laid off to represent 100 lb according to the force scale, and the parallelogram $OBACO$ was completed. Finally,

vectors OB and OC represent P and Q completely. A solution by the triangle construction, separated from the space diagram, is shown in Fig. 13. This is certainly the better solution where many forces are to be resolved.

EXAMPLE 2. It is required to resolve F, 100 lb (Fig. 14), into two components, one of which would balance G, 50 lb. Obviously this specified component P must be collinear with, opposite to, and equal to G. The second component Q must be concurrent with F and P. The magnitude and direction of Q were found in the vector diagram: AB and AC were drawn in the directions of F and P and equal to them. Since $AC \mathbin{+\!\!\!\!\rightarrow} CB = AB$, CB represents the magnitude and direction of Q. The line of action of Q passes through the point of concurrence of F and P and is parallel to CB. CB scales 140 lb.

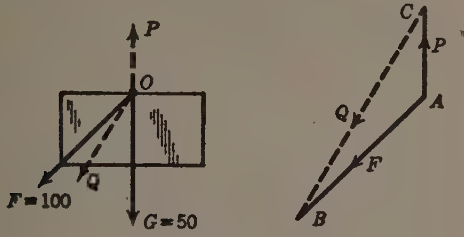


FIG. 14.

The algebraic method consists in sketching the vector triangle and solving it trigonometrically for the desired quantities. For example, see Fig. 15, which is a sketch of the triangle of Fig. 14. The angle at A is 135°, AC is 50 lb, and AB is 100 lb. Solution of the triangle gives Q = 140 lb, α = 30.4°, and β = 14.6°. This method involves more labor than the graphical one

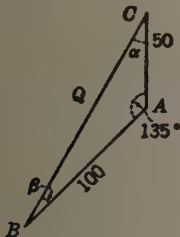


FIG. 15.

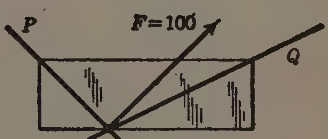


FIG. 16.

unless the triangle of resolution is a right triangle, as in Fig. 16, for example. (Sketch the triangle, and compare the magnitude of each component with the given 100-lb force. Suppose that P is not perpendicular to F but to Q; then sketch the triangle and compare as before.)

16. Rectangular Components of a Force and of a System. Resolution of a force into two components perpendicular to each other is a special case of great importance. The components are called *rectangular components;* and, if parallel to coordinate axes, they are called also *axial* or *x and y components.*

Let F (Fig. 17) be a given force, α and β the acute angles between F and the coordinate axes indicated, and F_x and F_y the axial components of F. (These components intersect on F, of course.) From the sketch of the triangle of resolution you see that

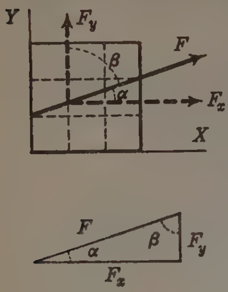


FIG. 17.

$$F_x = F \cos \alpha \qquad \text{and} \qquad F_y = F \cos \beta$$

that is, the rectangular component of a force parallel to any line is equal to the product of the force and the cosine of the acute angle between the force and the line. F_x and F_y can be expressed in terms of a single angle; thus

$$F_x = F \cos \alpha \qquad \text{and} \qquad F_y = F \sin \alpha$$

or

$$F_x = F \sin \beta \qquad \text{and} \qquad F_y = F \cos \beta^*$$

Signs are given to axial components in specific or numerical examples; x and y components acting to right and up respectively are positive; others are negative.

Resolution into rectangular components is the most common kind of resolution. Hence, by "the component of a force along or parallel to a given line," the rectangular component is meant.

COMPONENT OF A SYSTEM OF FORCES. It is often desirable to make use of the algebraic sum of the (rectangular) components, along or parallel to a coordinate axis, of all the forces of a given system. Such a sum is called the component of the system along or parallel to that axis.

17. Resolution of a Force into x, y, and z Components. It is often desirable to resolve a force into components along or parallel to three rectangular axes, x, y, and z. We first *indicate* a graphical method; it is not practical, but it leads to our explanation of the algebraic method which is practical.

Let the vector PQ (Fig. 19) represent a force F to be resolved into components parallel to the coordinate axes indicated. A, B, and C are projections

* *Direction-angles, -sines, and -cosines.* We call the acute angle between a given line and a coordinate axis a direction-angle of the line. (This differs somewhat from standard usage.) The sine and cosine of such an angle we call the direction-sine and direction-cosine, respectively, of the line.

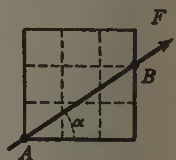


FIG. 18.

The line of action of a force is given sometimes not by a direction-angle but by means of two points, as A and B (Fig. 18), on the line. Then the direction-sine and -cosine can be readily and conveniently expressed as ratios. Thus, suppose that the horizontal and vertical projections of AB are 3 and 2 in. long, respectively. Then $AB = (3^2 + 2^2)^{\frac{1}{2}} = 3.61$ in., and

$$\sin \alpha = \frac{2}{3.61} \qquad \text{and} \qquad \cos \alpha = \frac{3}{3.61}$$

From these, F_x and F_y can be computed quickly by slide rule for any given value of F.

of Q on lines through P parallel to the axes. PQ is a diagonal of the indicated parallelepiped. The parallelogram $PAQD$ shows resolution of F into two components represented by PA and PD. The parallelogram $PBDC$ shows resolution of PD into two components represented by PB and PC. Hence PA, PB, and PC represent the three desired (x, y, z) components of F.

This representation is by a sort of perspective only and is not to scale, but it shows clearly that PA, PB, and PC are projections of PQ, and that therefore

$$F_x = F \cos \alpha \qquad F_y = F \cos \beta \qquad F_z = F \cos \gamma$$

where α, β, and γ are the direction-angles of F with respect to the x, y, and z axes, respectively.

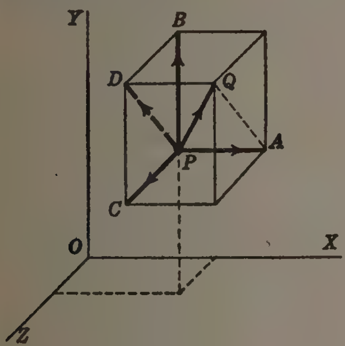


FIG. 19.

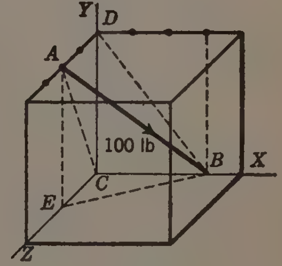


FIG. 20.

If the direction of F is given, not by direction-angles, but by two points on the line of action, then the direction-cosines of F can be expressed after the manner explained in Art. 16. To illustrate, we take a 100-lb force which acts through points A and B (Fig. 20) of the 4-in. cube, downward, to the right and rear. The projections of A on the x, y, and z axes, respectively, are C, D, and E; the projections of B are B, C, and C. The lengths of the projections of AB on the axes are 3, 4, and 2 in.; hence $AB = (3^2 + 4^2 + 2^2)^{\frac{1}{2}}$ or 5.38 in., and

$$\cos \alpha = \frac{3}{5.38} \qquad \cos \beta = \frac{4}{5.38} \qquad \cos \gamma = \frac{2}{5.38}$$

Therefore, the x, y, and z components of the 100-lb force, with sign, are, respectively, $+55.8$, -74.4, and -37.2 lb.

18. Moment or Torque of a Force. There are two somewhat different simple ways in which the equilibrium of a body may be disturbed: (i) it may be moved as a whole, up or down, to the right or left; or (ii) it may be rotated about some fixed line or axis.

The tendency of a force to cause displacement of the first sort depends

wholly on the magnitude of the force, but the tendency to turn a body about a given axis depends not only on the magnitude of the force but also on the position of its line of action with respect to the axis. For example, you know from experience that to unscrew a tight nut (Fig. 21) you should grasp the applied wrench as near the end as practicable, and should pull or push in the direction of P rather than in the direction of P_1 or P_2. Obviously, an end force directed along the wrench would not have any turning effect at all, and neither would a transverse force parallel to the axis of the bolt.

Experience suggests, and careful experiments would prove, that the tendency of a force to turn a body about a line or axis that is perpendicular to the force* is directly proportional to (i) the magnitude of the force, and (ii) the perpendicular distance from the axis to the line of action of the force, and, therefore, to the product of these quantities. This product is taken as the measure of the turning effect of the force and is called the **moment of the force** about or with respect to the axis. The perpendicular distance from the axis to the force is called the *moment arm* or *arm* of the force with respect to that axis; thus, in Fig. 21, a, a_1, and a_2 are the arms of P, P_1, and P_2 with respect to the axis of the bolt O, and Pa, P_1a_1, and P_2a_2 are the corresponding moments.

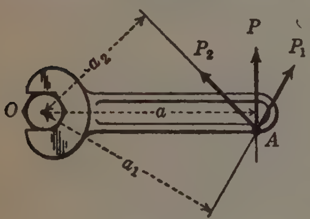


FIG. 21.

In dealing with coplanar forces, only moments about a line or axis perpendicular to the plane of the forces are ever desired. Such moments are commonly said to be taken about the point where the line intersects the plane of the forces. A point about which moments are so taken is called a *center* or *origin of moments*, and the perpendicular distances from this center to the forces are called the *arms of the forces*. It is convenient to denote the point or center and the line by the same letter.

Units of moments. Since a moment is the product of a force and a distance, the unit of moment is the unit of force times the unit of distance; thus, the magnitude of a moment may be expressed in inch-pounds, foot-pounds, centimeter-kilograms, etc.

Signs of moments. When the combined turning effect of a number of forces is to be considered, it is necessary to distinguish between the moments of the several forces in respect to the direction or sense of the rotation which each one tends to produce. Moments are accordingly described as *clockwise* or *counterclockwise;* this distinction requires the adoption of an arbitrary point of view. We use the common convention, namely: rotation about an axis perpendicular to the printed or written page is viewed from the reading side of the page; rotation about a coordinate axis is viewed from the positive end of

* The tendency of a force to turn a body about an axis not perpendicular to the line of action of the force is discussed in Art. 24.

that axis looking toward the origin. For convenience, the sense of a moment is indicated by giving it algebraic sign; the usual rule, which is followed in this book, is to consider clockwise moments negative and counterclockwise moments positive.

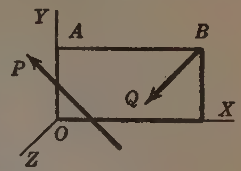


FIG. 22.

For example, suppose that the z axis (Fig. 22) is perpendicular to the plane of the paper, the force P in the plane, and Q parallel to the z axis. The moment of P about point A is negative; about point O positive. The moment of Q about the x axis is positive; about the y axis negative; about the z axis zero.

MOMENT OF A SYSTEM OF FORCES. By this is meant the measure of their combined turning effect with respect to a given line. Obviously, the effect is expressed by the algebraic sum of the moments, about that line, of the individual forces of the system. We write it ΣM_x, ΣM_0, etc., the subscript indicating the line, or point if the system is coplanar, about which the moments are taken.

19. Law of Moments. *If two systems of forces are equivalent, the moments of the systems about any given line or point are equal.* This follows directly from the definition of equivalent (Art. 10), for, if two force systems are equivalent, they have the same effect in preserving or destroying equilibrium; therefore they have the same turning effect about any given line; therefore they have equal moments with respect to any given line.

The foregoing law is applied most frequently in two special cases, for which it takes this form: *The moment of any system of forces and the moment of their resultant about any given line, or point if the system is coplanar, are equal*; *or, conversely, the moment of any force and the moment of its components about any given line, or point if the system is coplanar, are equal.*

The law of moments leads to an alternative method for calculating the moment of a force about a point. In this method the force is resolved into components, and then their moments are added algebraically; according to the law of moments, this sum is the desired moment. When using this method, it is advantageous to resolve the given force in such a way that the line of action of one component passes through the given center or origin of moments. The moment of this component is zero, and the moment of the other one is equal to the desired moment. To illustrate, we calculate the moment M of F (Fig. 23) about O. With A as point of concurrence of F_x and F_y, F_y has no moment about O, and so $M = -(F_x \times 4)$. With B as point of concurrence, we find that $M = -(F_y \times 2)$. (You should calculate F_x and F_y and then compare the two values of M. Next choose C as point of concurrence; calculate M, and compare. Calculate M directly, without resolving F, and compare.)

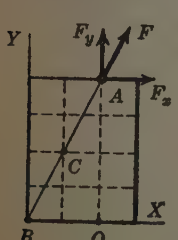


FIG. 23.

20. Moment of a Couple. Since a couple is a system of forces, the definition of moment of a system (Art. 18) applies to the moment of a couple. That is, the moment of a couple about any line (or point in its plane) is equal to the algebraic sum of the moments of the forces of the couple about that line (or point). Almost always it is desired to consider only the moment of a couple about a line perpendicular to its plane or about a point in its plane.*

The moment of a couple about any point in its plane is equal to the product of the magnitude of either force and the perpendicular distance between the forces. This distance is called the *arm* of the couple. You should prove this proposition by calculating the moment of a couple, having, say, vertical forces F and an arm a, for three different points, one between the forces, one to the right of both forces, and the third to the left of both forces.

For brevity, the moment of a couple is often said to be "the force times the arm." It should be noted that this is correct only for a point in the plane of the couple or a line perpendicular to it. "Moment of a couple," without mention of an axis or center of moments, means moment of the couple about *any* point in or *any* line perpendicular to its plane.

By *magnitude of a couple* some writers mean the moment of the couple. Thus they say a couple has magnitude, plane (of action), and sense, quite as a force has magnitude, line of action, and sense. This usage is convenient and is followed at times in this book.

The signs of the moments of a couple about different points in the plane (or about lines perpendicular to the plane) are the same. (You observed this fact in the proof you just made.) It follows that, if the body to which the couple is applied were made free to rotate in succession about various lines perpendicular to the plane of the couple, then all such rotations, due to the couple, would be in the same direction. *Sense of a couple* refers to the direction, clockwise or counterclockwise, in which the couple turns or tends to turn the body on which the couple acts. The sense of a given couple should be obvious to you on inspection. From this sense, you know the sign of the moment of the couple, positive or negative according as the sense is counterclockwise or clockwise.

21. Equivalent Couples. *Couples in the same or parallel planes, of equal moment and like sense, are equivalent.*

The truth of this proposition may seem obvious; it could be demonstrated by means of the apparatus represented in Fig. 24. The shaft-pulley-disk system hangs freely, the shaft vertical. Suppose that it is held in this position (say by hand) while a couple C is applied to the pulley by means of cords carrying equal weights as shown, and that a second couple C_1 is similarly applied to pegs A and B set in the disk. If the plane of C_1 is parallel to the plane of C, and the moment of C_1 is made equal and opposite to that of C, then on being released the system will not move; it is in equilibrium under the

* For moment of a couple about a line not perpendicular to its plane, see Art. 24.

action of its own weight W, the supporting force F of the chain, and the two couples. Since W and F are the only vertical forces, they are equal and opposite; and, since they are also collinear, they balance each other. Therefore the two couples C and C_1 balance each other. Now in this experiment the disk may be at any distance below or above the pulley, and the pegs A and B may be anywhere in the disk; hence any couple C_1, C_2, C_3, etc., whose plane is parallel to that of C and whose moment is equal and opposite to that of C will balance C; and so C_1, C_2, C_3, etc., are equivalent. Hence couples whose planes are parallel, moments equal, and senses alike are equivalent.

PRINCIPLE OF TRANSMISSIBILITY. The equivalent couples C_1, C_2, C_3 may be thought of not as different couples, but as the same couple in different positions; and so it appears that *a couple may be shifted about in its own plane or into any parallel plane, and its forces may be changed in magnitude and direction and moved closer together or farther apart, all without changing the effect of the couple, so long as the moment and sense of the couple are kept the same.* This is a statement of the principle of transmissibility for couples.

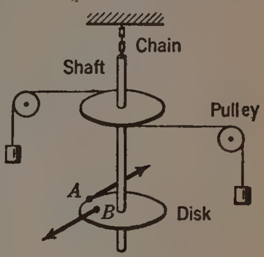


FIG. 24.

22. Representations of a Couple. Heretofore we have always represented a couple by completely describing both its forces. But, according to the principle of transmissibility just stated, all that matters in respect to a couple is the magnitude and sense of its moment and the aspect of its plane (the tilt or inclination of its plane with reference to a fixed plane of reference).

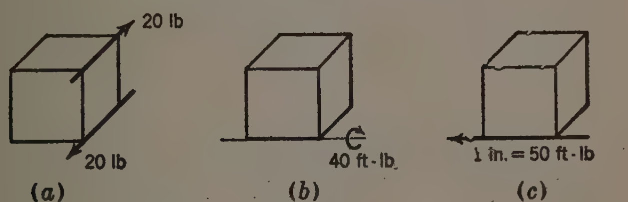


FIG. 25.

There are two schemes for representing these significant characteristics of a couple. For example, the couple acting on the 2-ft cube of Fig. 25*a* is represented by: (i) a curved arrow (Fig. 25*b*) in a plane parallel to the plane of the couple with a number expressing the moment of the couple and pointed to indicate the sense of the couple; (ii) a free vector (Fig. 25*c*) which is perpendicular to the plane of the couple, has a length equal, by a stated scale, to the moment of the couple, and points to the place from which the sense of the couple appears counterclockwise, or in the direction a right-hand screw would advance in a nut fixed on a line parallel to the vector if turned by the

couple ("screw rule"). Vector representation of couples is especially advantageous when dealing with couples in nonparallel planes (Art. 25).

23. Composition of Couples in Parallel Planes. *The resultant of any number of couples in parallel* planes is a couple in any parallel plane; and the moment of the resultant is equal to the algebraic sum of the moments of the given couples.*

Consider any such system of couples, in parallel planes. Imagine all couples respectively replaced by other (equivalent) couples having equal arms; then imagine these new couples shifted into any plane parallel to the planes of the given couples, and placed so that their forces lie in two parallel lines. The couples now consist of two sets of collinear forces. Since the forces respectively of one set are equal and opposite to the forces of the other, the resultants of the two sets are equal and opposite, and so the resultants constitute a couple, which is the resultant of the given couples.

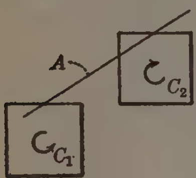


FIG. 26.

To prove the second part of the proposition, let C_1, C_2, etc. (Fig. 26) denote the given couples, C the resultant couple, and A a line perpendicular to the planes of the given couples. According to the principle of moments (Art. 19) the moment of C about A is equal to the algebraic sum of the moments of the given couples about that line. But the moments of these couples about line A are the moments of the couples respectively (Art. 20).

24. Moment of a Force and of a Couple (Continued); General Case. It is shown in Art. 18 how to calculate the moment of a force about a line perpendicular to the force, and in Art. 20 how to calculate the moment of a couple about a line perpendicular to the plane of the couple. Here it is shown how to calculate such a moment about a line inclined to the force or plane of the couple.

MOMENT OF A FORCE. There are three ways in which the calculation can be made. In each, the given force is resolved into concurrent components whose moments can be easily determined, and then these moments are added algebraically. This sum is the desired moment (see law of moments, Art. 19). The point of concurrence of the components is on the line of action of the given force, of course.

1. The force is resolved into three rectangular components, one parallel to the given line and the other two perpendicular to it. The parallel component has no moment about the given line; the moments of the perpendicular components can be calculated as explained in Art. 18.

For example, consider the force F exerted by the prop OA (Fig. 27) on the

* It is shown in the next article that the resultant of any number of couples in nonparallel planes is a couple; but this resultant cannot be determined so simply as for couples in parallel planes.

lid of the chest, and suppose that it is required to compute the moment of F about the axis of the hinges (line L). We show how to arrive at the desired components of F and its moment. The plane of O and L is taken as a coordinate plane, O as origin, x axis parallel to L, and y and z axes as indicated. Now F can be resolved into axial components F_x, F_y, F_z at any point in its line of action. First suppose that point A is chosen. Then F_x has no moment about L; F_y has an arm a and a clockwise moment; F_z has an arm b and a clockwise moment. Hence the combined moment of the components is $-(F_y \times a) - (F_z \times b)$, and this is the desired moment. Next, suppose that O is chosen as the point at which F is resolved into axial components. Again, F_x has no moment about L, and now F_z has no moment because it intersects L; F_y has an arm c and a clockwise moment. Hence the moment of the components is $-(F_y \times c)$, and this is the desired moment. This result is, of course, equal to the one first obtained. The point of concurrence of the components can always be chosen so that only one of them has a moment about the given line.

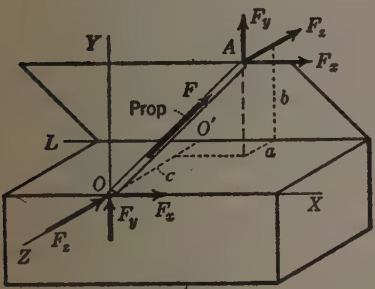


FIG. 27.

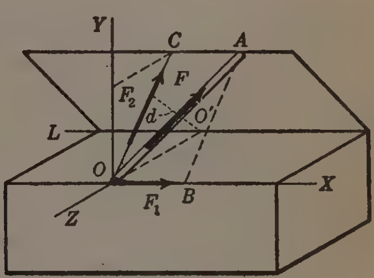


FIG. 28.

2. The force is resolved into two components, one parallel to the given line and one perpendicular to it. (They are obviously rectangular components.) The moment of the first component is zero, and so the moment of the second is the desired moment. We show how to arrive at the desired components of F and its moment. B and C respectively are the projections of A on the x axis and the yz plane (Fig. 28). (Any other point of F might be used in place of A.) $ABOC$ in space is a rectangle. Desired components, F_1 and F_2, are indicated at O; F_1 is parallel to L, and F_2 is perpendicular to it. F_2 has an arm d and a clockwise moment; hence its moment is $-(F_2 \times d)$, and this is the desired moment.

3. The force is resolved into two rectangular components; one intersects the given axis, and the other is perpendicular to it. The moment of the first is zero, and so the moment of the second is the desired moment. We show how to arrive at the desired components of F and its moment. D and E respec-

tively are the projections of A on the y axis and the zx plane (Fig. 29). (Any other point on F might be used in place of A.) $ADOE$ in space is a rectangle. Desired components F_1 and F_2 are indicated at O. F_1 intersects L and has no moment about L; F_2 (equal to F_y) is perpendicular to L and has an arm c and a clockwise moment. Hence its moment is $-(F_y \times c)$, and this is the desired moment.

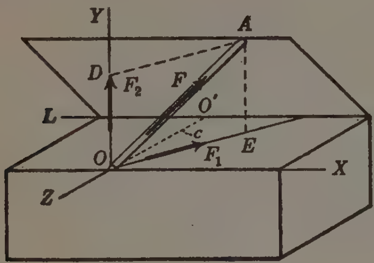


FIG. 29.

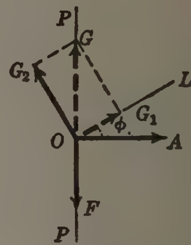


FIG. 30.

MOMENT OF A COUPLE. Let L denote the line about which the moment of a couple C is desired, O the point in which L pierces the plane of C, and OA a vector representing C. Imagine the body on which C acts turned, together with C and the lines OL and OA, till the lines lie in the plane of the paper (see Fig. 30). Since the plane of the couple is perpendicular to OA, it is represented in this figure by the line PP. Let F and G denote the equal forces of C, and a the arm of C. The magnitude or moment of C is Fa or Ga. Imagine C turned about in its plane till one of its forces F falls in PP; G lies behind the paper but is shown in PP for explanatory purposes. We are after the moment of F and G about OL. Clearly the moment of F is zero; the moment of G is the sum of the moments of the components G_1 and G_2, parallel and perpendicular respectively to OL. Clearly G_1 has no moment about OL; the moment of G_2 is G_2a, and this is the moment of C about OL. Since $G_2 = G \cos \phi$,

$$G_2a = (G \cos \phi) a = (Ga) \cos \phi$$

that is, *the moment of a couple C about any line is equal to the product of the magnitude or moment of C and the cosine of the angle between the line and the vector of C.*

25. Composition of Couples Continued, and Resolution. *The resultant of any two couples is a couple; it is represented by the sum of the vectors that represent the given couples.* For proof, see below. From this proposition it appears that two couples may be compounded just as two concurrent forces are (Art. 13). Therefore any number of couples may be compounded just as any number of concurrent forces are (Art. 26). Moreover, a couple can be

resolved into component couples, just as a force is resolved into concurrent component forces (Arts. 15 and 17).*

Proof. Consider the two couples C_1, C_2 in the nonparallel planes M, N (Fig. 31*a*). To find their resultant, let C_1 be represented by the forces F_1, F_2, of magnitude F, with arm a; and let this couple be shifted so that F_2 lies

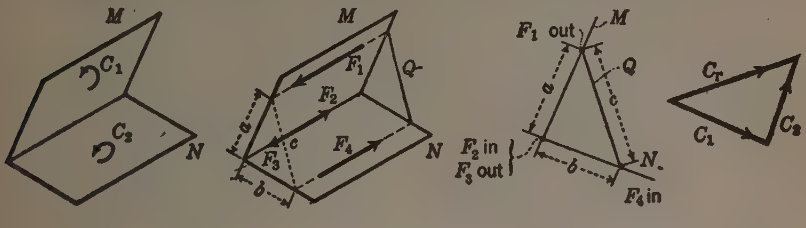


FIG. 31.

along the intersection of planes M and N as shown in (*b*). Let C_2 be represented by the forces F_3, F_4, also of magnitude F, with arm b; and let this couple be shifted so that F_3 lies along the intersection of M and N. The equal, opposite, and collinear forces F_2, F_3 cancel, leaving the forces F_1 and F_4, and these forces constitute a couple C_r which is the resultant of C_1 and C_2. (Note that, since F_1, F_2, F_3, and F_4 all have the common magnitude F, the couples C_1, C_2, and C_r are proportional to their respective arms a, b, and c.)

We now show that the resultant couple C_r is represented by the vector sum of vectors representing C_1 and C_2. Figure 31*c* is an end view of (*b*), and (*d*) shows vectors representing C_1 and C_2, and marked accordingly. These vectors are drawn perpendicular to the planes of the couples they represent, and are placed tail to tip so that the vector marked C_r is their sum. That this vector represents the resultant couple C_r is apparent from the fact that the triangle formed by the arms a, b, c and the triangle formed by the vectors C_1, C_2, C_r are similar. Vector C_r is proportional to arm c and hence to couple C_r; it is normal to the plane Q in which couple C_r lies, and it points in the direction from which the sense of C_r appears counterclockwise.

B. Composition of Various Kinds of Force Systems

26. Concurrent System; Graphical Composition. *The resultant of a concurrent system is a single force. The line of action of the resultant passes through*

* Composition of couples in parallel planes (Art. 23) and composition of collinear forces are strictly analogous. And composition of couples in nonparallel planes is, as just explained, strictly analogous to composition of concurrent noncoplanar forces. Since there are no other kinds of couple systems than these mentioned, there are no calculations relating to couples that correspond to composition of nonconcurrent forces. It follows that the composition and resolution of couples is simpler than the composition and resolution of forces.

the point of concurrence of the forces of the given system, because the moment of the system is zero about any line through that point; and the principle of moments (Art. 19) requires that the moment of the resultant, likewise, be zero about any such line.

The magnitude and direction of the resultant can be determined by the method of progressive composition which may be described as follows: By the triangle law (Art. 13) find the resultant R_1 of any two of the forces of the given system, then find the resultant R_2 of R_1 and any other given force; and so on until the resultant of all is found.

This graphical method is practical only for coplanar forces. We illustrate it now for four such forces applied to a 2-ft square board (Fig. 32). Taking any two of the forces, say F_1 and F_2, their resultant R_1 is given in magnitude

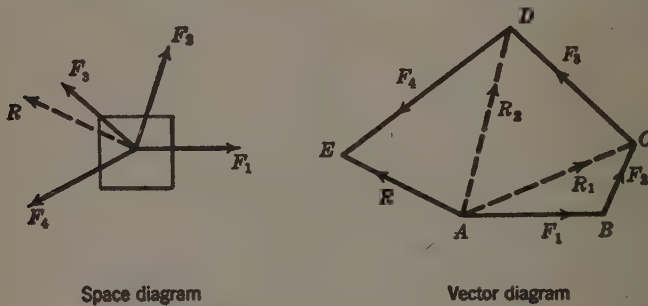


FIG. 32.

and direction by their vector sum AC; the resultant R_2 of R_1 and F_3 is given by their vector sum AD; finally, the resultant R of R_2 and the remaining force F_4 is given by their vector sum AE. The line of action of R, shown in the space diagram, is parallel to AE.

It is apparent that there is no need actually to determine the successive or partial resultants R_1 and R_2. The desired or final resultant is given, in magnitude and direction, by the vector sum of the given forces.

A vector polygon for a system of forces is also called a *force polygon* for the system, or for the forces. Many different force polygons can be drawn for a given system of forces (Art. 8).

27. Concurrent System; Algebraic Composition. The resultant of such a system is a single force concurrent with the forces of the system (see preceding article). Methods for compounding coplanar and noncoplanar systems are the same in principle but differ in detail; they are explained below separately.

COPLANAR SYSTEM. To determine the resultant algebraically proceed as follows: Choose x and y axes with origin at the point of concurrence of the forces to be compounded; then resolve each force into its x and y components at

the origin, and imagine it replaced by them; next find the resultant of the forces acting in the x axis, and the resultant of those acting in the y axis; finally, find the resultant of these two rectangular resultants; this is the resultant sought.

The foregoing process is so simple and obvious that formulas for the results are unnecessary. However, formulas may be useful for reference. So we let F_x and F_y respectively denote the x and y components of any given force F, R_x and R_y the resultants of the x and of the y components, R the resultant sought, and θ the acute angle between R and the x axis. Then, obviously, $R_x = \Sigma F_x$, $R_y = \Sigma F_y$,

$$R = (R_x^2 + R_y^2)^{\frac{1}{2}} \qquad \text{and} \qquad \theta = \tan^{-1} (R_y \div R_x)$$

Note that the signs of ΣF_x and ΣF_y indicate the directions of R_x and R_y respectively. From these directions and the numerical values of R_x and R_y you can make a fair estimate of the direction of R, which may serve as a check on the calculated angle θ.

EXAMPLE. Six forces act upon a 4-by-4-ft board as shown in Fig. 33. It is required to determine their resultant algebraically.

Solution. We take x and y axes respectively horizontal and vertical. The components of the two forces in the axes (8 lb and 7 lb) require no calculation; their values are immediately written into the table below, with signs according to the usual rule (see Art. 16). Then for any inclined force we calculate

$$F_x = F \cos \alpha \qquad \text{and} \qquad F_y = F (\cos \beta \text{ or } \sin \alpha)$$

where α and β are the acute angles between F and the x and y axes, respectively.

The direction-angles of the 4-lb force being 45°, the direction-cosines are found to be 0.707 and are recorded in the table below. The direction-cosines of the other three forces are calculated and recorded as ratios. Thus, for the 6-lb force: the projections of OA on the x and y axes are 1 and 2 ft, respectively; hence $OA = 2.24$, and the direction-cosines of the force are as recorded. In a similar manner the direction-cosines of the remaining two forces are found. Then the remaining x and y components are computed, and finally R_x and R_y are found to be +5.00 and −6.02 lb. These signs respectively indicate that R acts to the right and downward. Since R_y is greater than R_x, the direction of R is about as indicated. The exact angle between R and the x axis is $\tan^{-1} (6.02 \div 5.00) = 50.3°$. The magnitude of R is $(5.00^2 + 6.02^2)^{\frac{1}{2}} = 7.82$ lb.

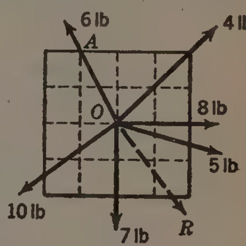


FIG. 33.

F	$\cos \alpha$	$\cos \beta$	F_x	F_y
8	....	....	+8	0
7	....	....	0	−7
4	0.707	0.707	+2.83	+2.83
6	1 ÷ 2.24	2 ÷ 2.24	−2.68	+5.36
10	2 ÷ 2.50	1.5 ÷ 2.50	−8.00	−6.00
5	2 ÷ 2.06	0.5 ÷ 2.06	+4.85	−1.21
			+5.00	−6.02

NONCOPLANAR SYSTEM. The procedure for determining the resultant algebraically is this: Choose x, y, and z (rectangular) axes with origin at the point of concurrence of the forces to be compounded; then resolve each force into its x, y, and z components at the origin (see Art. 17), and imagine it replaced by them; next find the resultant of the forces in the x axis, of those in the y axis, and of those in the z axis. These are, respectively,

$$R_1 = \Sigma F_x \qquad R_2 = \Sigma F_y \qquad R_3 = \Sigma F_z$$

Finally find the resultant R of R_1, R_2, and R_3.

Progressive composition of these axial resultants is indicated in Fig. 34, where they are represented by the vectors OA, OB, and OC, respectively. The resultant R' of R_1 and R_2 is represented by the vector OP, and the resultant of R' and R_3 (or of R_1, R_2, and R_3) is represented by the vector OQ, which you see is a diagonal of the parallelepiped $OABC$. Hence

$$R = (R_1^2 + R_2^2 + R_3^2)^{\frac{1}{2}}$$

and its direction-angles are respectively

$$\cos^{-1}(R_1 \div R) \qquad \cos^{-1}(R_2 \div R) \qquad \cos^{-1}(R_3 \div R)$$

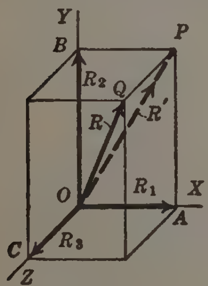


FIG. 34.

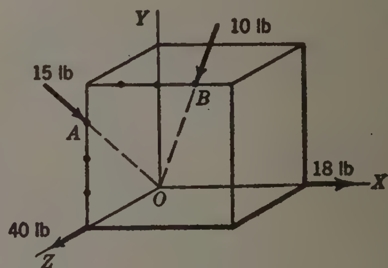


FIG. 35.

EXAMPLE. Four forces act on a 4-ft cube as shown in Fig. 35. It is required to determine their resultant algebraically.

Solution. Rectangular axes are assumed as indicated on the figure. Each force is resolved at the origin into x, y, and z components. For the 40- and 18-lb forces, the components are obviously as given in the table below. Since the 15-lb force is perpendicular to the x axis, its x component is zero. The projections of OA on the coordinate axes are, respectively, 0, 3, and 4 ft. Hence $OA = 5$ ft, and the direction-cosines of the 15-lb force are as recorded. The projections of OB are 3, 4, and 4 ft; hence $OB = 6.40$ ft, and the direction-cosines of the 10-lb force are as recorded. Next the axial components of these forces are computed and a sign given each, and ΣF_x, ΣF_y, and ΣF_z are found.

F	$\cos \alpha$	$\cos \beta$	$\cos \gamma$	F_x	F_y	F_z
18	....	....	....	+18.00	0.00	0.00
40	....	....	....	0.00	0.00	+40.00
15	$0 \div 5$	$3 \div 5$	$4 \div 5$	0.00	−9.00	−12.00
10	$3 \div 6.4$	$4 \div 6.4$	$4 \div 6.4$	−4.69	−6.25	−6.25
				+13.31	−15.25	+21.75

The resultant of the x components $= +13.31$; the resultant of the y components $= -15.25$; the resultant of the z components $= +21.75$ lb. The resultant of these resultants (and hence of the system) acts through the point of concurrence O. It is directed to the right, down, and forward. Its magnitude is

$$R = (13.31^2 + 15.25^2 + 21.75^2)^{\frac{1}{2}} = 29.7 \text{ lb}$$

The angles it makes with the x, y, and z axes are, respectively,

$$\cos^{-1}(13.31 \div 29.7) = 63.4^\circ \qquad \cos^{-1}(15.25 \div 29.7) = 59.1^\circ$$
$$\cos^{-1}(21.75 \div 29.7) = 42.9^\circ$$

Queries. What can you say about the resultant of a coplanar concurrent system in the following cases? (i) $\Sigma F_x = 0$; (ii) $\Sigma F_x = \Sigma F_y = 0$. What can you say about the resultant of a noncoplanar concurrent system in the following cases? (i) $\Sigma F_x = 0$; (ii) $\Sigma F_x = \Sigma F_y = 0$; (iii) $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$.

28. Coplanar Nonconcurrent Systems; Graphical Composition. Two methods are explained below; they are quite different. The first is simpler in principle; the second is more comprehensive, and preferable for compounding many forces.

FIRST METHOD. The resultant is found by progressive composition as for concurrent forces (Art. 26). But in the present problem there is no common point of concurrence of the given forces, and so nothing is known in advance about the line of action of the resultant. The position of this line must be determined by drawing the lines of action of the partial resultants R_1, R_2, · · ·, and finally the line of action of the desired resultant R.

Since the line of action of the resultant of two concurrent forces passes through their point of concurrence, the two forces chosen for first compounding, into R_1, should intersect within the limits of the space diagram. (If all given forces are parallel or nearly so, use the scheme described below.) Similarly, the force chosen for compounding with R_1 should intersect R_1 within the stated limits, and so on. You are advised to determine R_1 completely; then R_2 completely, · · ·; and finally R. When all is done you will have drawn a vector or force polygon for the given forces. The sum of the vectors represents the magnitude and direction of the resultant just as it does for concurrent forces.

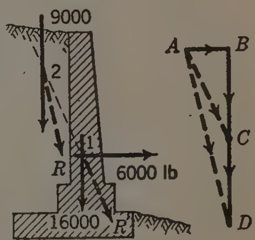


FIG. 36.

EXAMPLE 1. Figure 36 represents a section of a retaining wall, a 1-ft portion of which is considered. The forces that act on this portion are: its own weight (16,000 lb); the earth pressures on the back (6000 lb), on the top of the base (9000 lb), and on the bottom of the base. It is required to determine the resultant of the first three forces, whose lines of action are given as indicated.

Solution. The resultant of the 6000- and the 16,000-lb forces is determined by means of the force triangle ABC drawn to a certain convenient scale; AB represents the 6000-lb force, BC represents the 16,000-lb force, and AC gives the magnitude and direction of their resultant R'.

The line of action of R' passes through point 1 and is parallel to AC; it is drawn and produced to intersect the line of action of the 9000-lb force. The resultant of R' and the 9000-lb force is next determined by means of the force triangle ACD; AC represents R', CD represents the 9000-lb force, and AD gives the magnitude and direction of their resultant R, which is the resultant of the three given forces. The line of action of R passes through point 2 and is parallel to AD, as shown.

If the forces to be compounded are parallel or nearly so, augment the given system by adding two equal, opposite, and collinear forces, taking their common line of action somewhat across those of the given forces. Obviously, the resultant of the enlarged system is just like that of the given system. Therefore you find the resultant of the enlarged system by progressive composition, compounding first any two forces whose intersection is accessible for the first partial resultant R_1, etc.

Special case; a force polygon for the system closes. The resultant is, in general, a couple. The system is not necessarily in equilibrium. To show this, let R' denote the resultant of all the forces except one, F_n. It is obvious from the closed polygon that R' and F_n are equal, parallel, and opposite. A construction for the line of action of R' would, in general, show R' not coin-

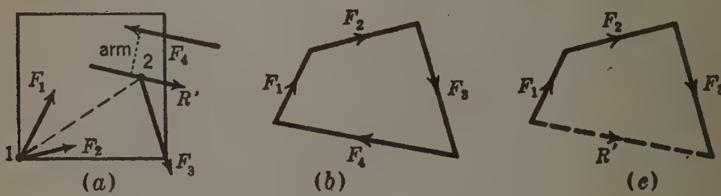


FIG. 37.

ciding with F_n. Hence R' and F_n constitute a couple, the resultant of the given system. For example, F_1, F_2, F_3, and F_4 (Fig. 37*a*) constitute a system whose force polygon (Fig. 37*b*) closes. The magnitude and direction of the resultant R' of the first three forces is represented in Fig. 37*c*. The line of action of the resultant, in the space diagram, is marked R'; and the arm of the couple $R'F_4$ is indicated. You see also from this diagram that the sense of the couple is counterclockwise. (The moment of the couple is, of course, equal to the algebraic sum of the moments of the given forces about any point in their plane.*)

* Three other resultant couples could be found by successive constructions for the resultant R' of all but one of the given forces. Another could be found by augmenting the system, as explained above in the paragraph on forces parallel or nearly so, and then finding the resultant R' of the enlarged system minus one of the added forces. Still other couples could be found by this device. All these resultant couples are equivalent, since each is the resultant of the given system; and hence the moments of the couples are equal. (We have here a proof that equivalent coplanar couples have equal moments; and it is a fair inference that couples whose moments are equal are equivalent.)

SECOND METHOD. The method consists of progressive resolution, followed, in general, by a simple composition. Each force of the given system is resolved into two concurrent components in such manner that all but two of the components balance. The resultant of these two, easily found, is the resultant of the given system. See Fig. 38*a*, where F_1, F_2, and F_3 are given forces to be compounded; P_1 and Q_1 are components of F_1; P_2 and Q_2 are components of F_2; etc. P_2 and Q_1 are made collinear and equal, and hence balance, as do P_3 and Q_2 also. The resultant of P_1 and Q_3 is the resultant of the given system.

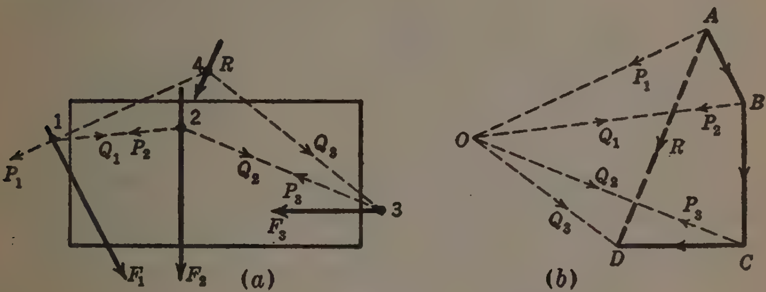


FIG. 38.

The system of components is arrived at as follows: At any point 1 on its line of action, F_1 is resolved into arbitrary components P_1 and Q_1. At 2, where Q_1 intersects F_2, F_2 is resolved into P_2 and Q_2, P_2 being made equal and opposite to Q_1 (see Art. 15). At 3, where Q_2 intersects F_3, F_3 is resolved into P_3 and Q_3, P_3 being made equal and opposite to Q_2. (You should now make this resolution of three assumed forces, say, 50, 80, and 70 lb, by means of three separate triangles, and finally arrive at the magnitude, line of action, and sense of the desired resultant. Having done this, you will more fully appreciate the following.)

The bare construction of this second method is as follows: A force polygon *ABCD* (Fig. 38*b*) for the given forces is drawn first. Then a convenient point *O* is chosen for a common vertex of the triangles of resolution *OAB*, *OBC*, and *OCD*. (You see that P_2 and Q_1 are equal and opposite, likewise P_3 and Q_2.) The lines of action of all components are drawn next, in the manner already explained. *AD* is the magnitude and direction of the resultant *R* of P_1 and Q_3 (and of the given forces). The line of action of *R* passes through 4, the intersection of the lines of action of P_1 and Q_3. (You see that the magnitude and direction of *R* are found just as in the first method.)

When employing this construction it is advantageous to use *Bow's notation*, in which each force vector is denoted by capital letters, one at each end, and the corresponding line of action is denoted by the same lower-case letters, one on each side. The notation is illustrated in Fig. 39, which represents

the solution of the example above. Also, there is a convenient terminology (not associated with Bow): O is the *pole* of the force polygon; lines emanating from O are *rays;* the lines in the space diagram parallel to the rays are *strings;* the collection of strings is the *string* or *funicular polygon* (also called

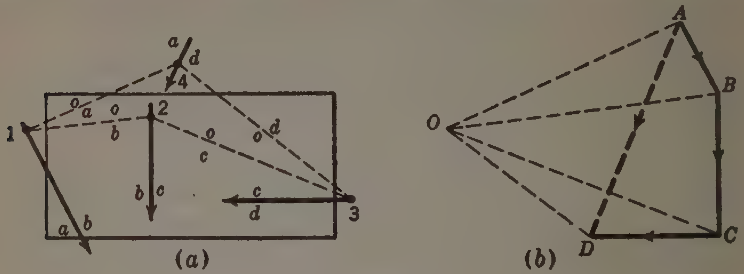


FIG. 39.

equilibrium polygon, especially when the given forces are balanced or in equilibrium). The rays are sometimes referred to by number, OA being the first, OB the second, etc.; likewise the strings.

Special case; the force polygon closes. The resultant is a couple consisting of the first component P_1 and the last component Q_n of the system of components into which the given system is resolved. Thus, you see again, there are many couples equivalent to a given system, and therefore equivalent to each other. Their moments are equal.

If the first and last strings happen to coincide, the string polygon is said to be *closed;* the moment of the couple is zero, and the resultant of the system is nil.

29. Coplanar Parallel System; Algebraic Composition. If the forces are given sign, those in either direction positive and those in the other negative, then the algebraic sum of the forces equals the vector sum; hence the algebraic sum gives the magnitude and direction of the resultant, the sign of the sum indicating the sense of the resultant.

The line of action of the resultant is determined by means of the law of moments (Art. 19), which requires that the moment of the resultant about any point in the plane of the forces be equal to the moment of the given system about that point. For example, consider a system of vertical forces for which $\Sigma F = -20$ lb, and $\Sigma M_0 = -50$ ft-lb, where O (Fig. 40) is the chosen origin of moments. The sign of ΣF shows that the resultant R of the forces acts downward, and the sign of ΣM_0 that the moment of R about O is clockwise; therefore, the line of action of R is to the right of the origin. The arm is $50 \div 20 = 2\frac{1}{2}$ ft, as indicated.

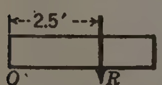


FIG. 40

Special case; the algebraic sum of the forces of the given (parallel) system equals zero. The system is not necessarily in equilibrium; the resultant is,

in general, a couple. Let R' denote the resultant of all but one, say F_n, of the given forces. R' and F_n are equal, parallel, and opposite, and, in general, not collinear. They constitute the resultant couple.

The moment of this resultant couple could be calculated by first determining the line of action of R', from which one could ascertain the arm of the couple, etc. The moment can be found more simply, however; for, according to the law of moments, the moment equals the moment ΣM of the given system about any chosen point in the plane of the forces. The couple is adequately described by stating its moment, with sign.

PROPERTIES OF TWO UNEQUAL PARALLEL FORCES. 1. The resultant of two parallel forces lies nearer the larger force. If the two forces are alike in sense the resultant lies between them and acts in their direction. If they are unlike in sense, the resultant is outside them and acts in the direction of the larger. See Fig. 41*a*, where R is the resultant of P and Q, P being larger than Q.

2. The resultant of two parallel forces divides any secant in the plane of the forces into segments whose lengths are inversely proportional to the forces. Thus for either case $AC : BC : : Q : P$.

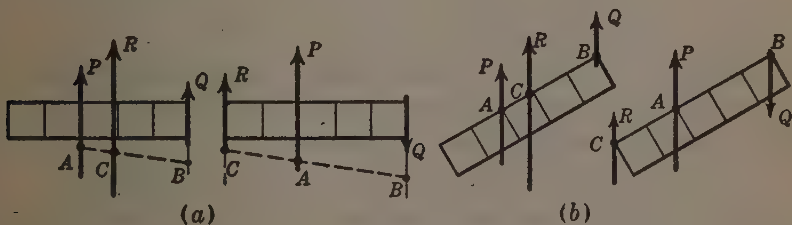


FIG. 41.

3. The resultant of two parallel forces having fixed points of application A and B on a body (Fig. 41*b*) passes through a fixed point C on the line AB even if the angle between AB and the forces is changed. That is, the point C does not shift in AB if the body is turned about into any new position and the forces remain parallel. Therefore, the point C is called the *center* of the parallel forces.

Query. What can you say about the resultant of a coplanar parallel system in the following cases? (i) $\Sigma M_a = 0$, where a is any point in the plane of the forces; (ii) $\Sigma M_a = \Sigma M_b = 0$, where b is another point in the plane.

30. Coplanar Nonconcurrent Nonparallel System; Algebraic Composition. It is shown in Art. 28 that the resultant is in general a single force, and that, in a certain special case (described below), it is a couple. As pointed out, the magnitude and direction of the resultant are just the same as though

the forces were concurrent. Therefore, from Art. 27, $R_x = \Sigma F_x$, $R_y = \Sigma F_y$,

$$R = (R_x^2 + R_y^2)^{\frac{1}{2}} \quad (1) \qquad \text{and} \qquad \theta = \tan^{-1}(R_y \div R_x) \qquad (2)$$

where θ is the acute angle between R and the x axis.

The line of action of R is located by means of the law of moments, which requires that the moment of R be equal to the moment of the given system of forces about any point in the plane of the system (Art. 19). Therefore ΣM_c, the moment of the system for a convenient center C, is calculated. Then, since $Ra = |\Sigma M_c|$,* where a is the arm of R with respect to C,

$$a = |\Sigma M_c \div R| \qquad (3)$$

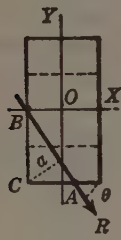


FIG. 42.

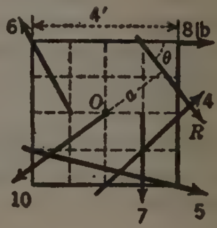


FIG. 43.

The position of R relative to C, above or below, to right or left, can be ascertained from the sense of its moment, which must agree with the sign of ΣM_c. Consider a numerical example, where, say,

$$R_x = +20 \text{ lb} \qquad R_y = -30 \text{ lb} \qquad \Sigma M_c = -40 \text{ ft-lb}$$

The signs of R_x and R_y show that R acts to the right and downward. The sign of ΣM_c shows that the moment of R about C is clockwise; hence R is on the right-hand side of C (Fig. 42). From Eqs. 1, 2, and 3, respectively, $R = 36$ lb, $\theta = 56.3°$, and $a = 1.11$ ft.

It may be desirable to locate the line of action of R by its intercepts on lines through C parallel to the x and y axes, or by either intercept and θ. The approximate position of the line as in the figure having been decided upon,

$$CA = |\Sigma M_c \div R_y| \qquad \text{and} \qquad CB = |\Sigma M_c \div R_x|$$

(These expressions for intercepts follow, as you should see, from the law of moments.) For the numerical example above, $CA = 1\frac{1}{3}$ and $CB = 2$ ft.

EXAMPLE. We find the resultant of the six forces indicated in Fig. 43. (Their magnitudes and directions respectively are the same as those of Fig. 33, where the forces are concurrent.) Hence here, also, R acts to the right and downward;

$$\theta = 50.3° \qquad \text{and} \qquad R = 7.82 \text{ lb}$$

* The symbol || means that the sign of the quantity enclosed should be disregarded.

The moments of the forces about O are recorded in the adjoining table; $\Sigma M_0 = -15.43$ ft-lb, which is also the moment of R about O. Hence the line of action of R is to the right of O, and the arm of R is

$$a = 15.43 \div 7.82 = 1.97 \text{ ft}$$

F	M_0
8	−16
4	+5.66
6	−5.36
10	0
7	−7.00
5	+7.27
	−15.43 ft-lb

Special case. $\Sigma F_x = \Sigma F_y = 0$. The system is not necessarily in equilibrium; the resultant is, in general, a couple. The moment of this couple, according to the law of moments, is equal to the moment ΣM, of the system under discussion, about any point in the plane. This couple is adequately described by stating its moment, with sign.

FORCE AND COUPLE. A force F and couple C in the same plane (see Fig. 44a) constitute a special coplanar nonconcurrent force system. *The resultant of*

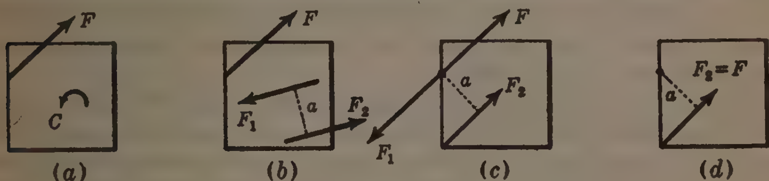


FIG. 44.

F and C is a single force, equal to F and having the same direction as F, and so located as to have, about any point on F, a moment equal to C. This can readily be proved as follows: Let the forces F_1 and F_2 of C be made equal to F, the arm becoming a so that $Fa = C$ (Fig. 44b); then turn and shift the couple so that one of the forces (here F_1) will balance F, leaving the other force (here F_2) of the couple (Fig. 44c). This force is the resultant of the given force F and couple (Fig. 44d). You see that the resultant is equal to and has the same direction as F and that the moment of the resultant about any point on F is the same as the moment of the couple.

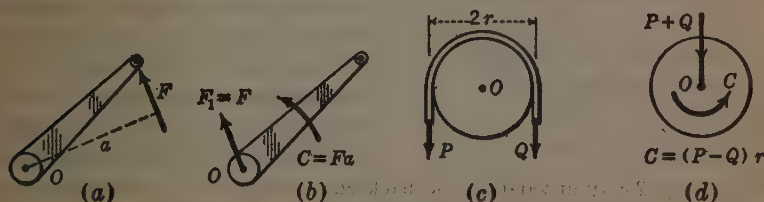


FIG. 45.

It follows from the above that, conversely, a given force F can be resolved into a force acting through any chosen point O and a couple; the force will be equal to and have the same direction as F; the couple will be in the plane

of F and O and will have a moment equal to the moment of F about O. See Fig. 45a, where F is represented as a push on the handle of a crank. Figure 45b indicates components of F consisting of a force F_1 acting through O and a couple C. Figure 45c represents a belt pulley and a portion of the belt; P and Q denote belt pulls. Resolving each pull into a force at O and a couple, and then combining the two forces at O and also the two couples, leads to the result indicated in Fig. 45d.

Query. What can you say about the resultant of a coplanar nonconcurrent nonparallel system in the following cases? (i) $\Sigma F_x = 0$; (ii) $\Sigma M_a = 0$, a being a point in the plane of the forces; (iii) $\Sigma M_a = \Sigma M_b = 0$, b being another point in the plane; (iv) $\Sigma F_x = \Sigma M_a = 0$; (v) $\Sigma F_x = \Sigma F_y = \Sigma M_a = 0$.

31. Noncoplanar Parallel System; Composition. It is shown in Art. 29 that the resultant of any two given parallel forces is in general a force parallel to the two forces, and that its magnitude and sense are given by the algebraic sum of the two. It follows that the resultant of any number of parallel forces is parallel to the forces, and that its magnitude and sense are given by the algebraic sum of the forces.

The line of action of the resultant may be located conveniently by means of the arms of the resultant with respect to two rectangular axes, each perpendicular to the forces. Such arms can be computed readily from the law of moments, which requires that the moment of the resultant about any axis equals the algebraic sum of the moments of the forces about the same axis. For example, consider vertical forces acting on a 3-by-4-ft horizontal platform (Fig. 46), and suppose that the forces are such that

$$\Sigma F_y = -50 \text{ lb} \qquad \Sigma M_x = +120 \text{ ft-lb} \qquad \Sigma M_z = -180 \text{ ft-lb}$$

(The positive directions for moments are indicated on the figure.) The resultant R is a force of 50 lb and acts down. In order to have a positive moment of 120 ft-lb about the x axis, the (downward) resultant must act in front of the x axis and at a distance therefrom of $120 \div 50 = 2.4$ ft. In order to have a negative moment of 180 ft-lb about the z axis the resultant must act to the right of the z axis and at a distance therefrom of $180 \div 50 = 3.6$ ft. Therefore, the resultant of the system is a downward force of 50 lb acting 2.4 ft in front of the back edge, and 3.6 ft to the right of the left edge, of the platform. It is represented in the figure by R.

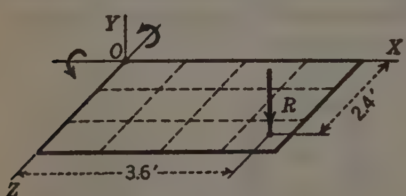


FIG. 46.

Special case; the algebraic sum of the forces is zero. The system is not necessarily in equilibrium; the resultant is, in general, a couple. For the resultant of all the forces save one will be parallel, equal, and opposite to that force, and will constitute with it a couple unless collinear with it, in which

event the resultant really is zero. The magnitude of the forces, the arm, and the plane of the resultant couple will depend on which one of the given forces is not compounded, but all resultant couples that may be determined will be found to lie in parallel planes and to have equal moments, and so are equivalent.

Query. What can you say about the resultant of a noncoplanar parallel system in the following cases? (i) $\Sigma M_x = 0$, x being an axis perpendicular to the forces; (ii) $\Sigma M_x = \Sigma M_y = \Sigma F = 0$, y being another axis perpendicular to the forces.

32. Noncoplanar Nonconcurrent Nonparallel System; Composition. In general, this kind of system can be reduced, as demonstrated in the second paragraph, to a force acting through any chosen point (of the body to which the system is applied) and a couple. How to determine them is explained in the third and fourth paragraphs. This force-couple system is commonly regarded as the resultant of the given system, but the force and couple can always be reduced to two forces (see special cases below), and the two-force system is really the resultant of the given system, according to the definition of resultant (Art. 11).

As shown in Art. 30, any force can be replaced by a force acting through any chosen point and a couple. Imagine all forces F_1, F_2, etc., of the given system replaced in this way, one single point O (of the body on which the given system acts) being chosen for all forces. The new system consists of a set of concurrent forces at O and a set of couples. The resultant of the concurrent forces is a single force acting through O (Art. 27), and the resultant of the couples is a single couple (Art. 25). This force and couple are equivalent to the given system, of course. We call them R and C and show next how to determine them.

The forces G_1, G_2, etc., of the concurrent system respectively are equal to and have the same directions as the forces F_1, F_2, etc., of the given system. Hence, for rectangular axes through O, $\Sigma G_x = \Sigma F_x$, etc.; and so R can be found from the given forces just as though they were concurrent at the chosen point O. And you see that R is independent of the position of O.

Let C_x, C_y, and C_z denote components of C in planes perpendicular to the x, y, and z axes, respectively, and ΣM_x, ΣM_y, and ΣM_z the moments of the given system about these axes, respectively. Since R, C_x, C_y, and C_z together are equivalent to the given system, their combined moment about the x axis, say, is equal to the moment of the given system about that axis. Obviously R has no moment about the x axis; C_y has no moment about the x axis, for the moments of the two forces comprising C_y are equal and opposite; similarly, C_z has no moment about the x axis. Hence (C_x)* $= \Sigma M_x$; similarly, $(C_y) = \Sigma M_y$ and $(C_z) = \Sigma M_z$.

* To distinguish between a couple and its magnitude or moment, we here write (C) for the moment of a couple C.

From these and Fig. 47, it follows that

$$(C) = [(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2]^{\frac{1}{2}}$$

and that the direction-cosines of the vector of C are

$$\frac{\Sigma M_x}{(C)} \quad \frac{\Sigma M_y}{(C)} \quad \frac{\Sigma M_z}{(C)}$$

You see that C depends on the position of O.

Special cases. (i) If R is not parallel to the plane of C, R and C can be compounded into two noncoplanar forces. For, let C be moved in its plane until one of its forces, say P, intersects R. These two forces can be compounded into a single force, say Q. Q and the other force P' of the couple are not coplanar, hence cannot be compounded. The system $P'Q$ is the resultant of the given system. (ii) If R is parallel to the plane of C, then R and C can be compounded into a single force. For the couple may be shifted about in its plane until its forces P and P' become parallel to R, and then it is clear that the resultant of P, P', and R is a single force. (iii) If $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$, the resultant of the system is a couple, given by the above equations.

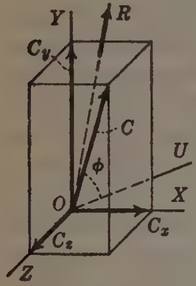


FIG. 47.

Query. What can you say about the resultant of a noncoplanar nonconcurrent nonparallel system of forces in the following cases? (i) $\Sigma F_y = \Sigma F_y = 0$; (ii) $\Sigma M_x = 0$; (iii) $\Sigma M_x = \Sigma M_y = 0$; (iv) $\Sigma F_x = \Sigma F_y = \Sigma F_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0$.

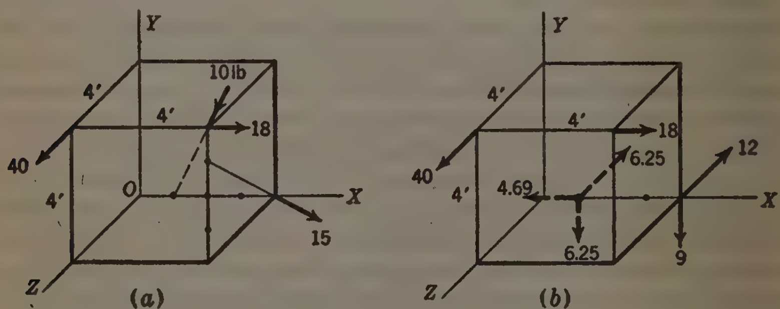


FIG. 48.

EXAMPLE 1. Determine the resultant of the four forces acting on the 4-ft cube in Fig. 48a. The system, except for being nonconcurrent, is like the one of Fig. 35.

Solution. We regard the required resultant as consisting of a force R and a couple C, choosing O as the point through which R shall act. Since, as explained above, R may be found from the given forces just as though they were concurrent, the results of the example

relating to Fig. 35 may be used here. The force $R = 29.7$ lb; acts to the right, down, and forward; and makes angles of 63.4, 59.1, and 42.9° with the x, y, and z axes, respectively.

For the purpose of finding ΣM_x, ΣM_y, and ΣM_z, the axial components of the given forces computed in Art. 27 are recorded in the adjoining table and are represented in Fig. 48*b*. (The 10- and 15-lb forces are shown resolved at the points where they intersect the x axis for easy calculation of their moments.)

F	F_x	F_y	F_z	M_x	M_y	M_z
18	18.00	0.00	0.00	0.00	72.00	−72.00
40	0.00	0.00	40.00	160.00	0.00	0.00
15	0.00	−9.00	−12.00	0.00	48.00	−36.00
10	−4.69	−6.25	−6.25	0.00	6.25	−6.25
	+13.31	−15.25	+21.75	+160.00	+126.25	−114.25

The moment of the resultant couple C is

$$(160.00^2 + 126.25^2 + 114.25^2)^{\frac{1}{2}} = 233.8 \text{ ft-lb}$$

The angles that the vector representing C makes with the x, y, and z axes, respectively, are

$$\cos^{-1}\frac{160.00}{233.8} = 46.8^\circ \qquad \cos^{-1}\frac{126.25}{233.8} = 57.3^\circ \qquad \cos^{-1}\frac{114.25}{233.8} = 60.7^\circ$$

Thus the four given forces have been reduced to a force R acting through O, and a couple C acting in any plane perpendicular to the vector that represents it.

EXAMPLE 2. Figure 49*a* represents a two-throw crankshaft. The angle between the cranks is 90°, and the distance between their center lines is b. Suppose that each crank is subjected to a pull F acting along its center line as indicated. Their resultant R–C referred to O, midway between A and B, is required.

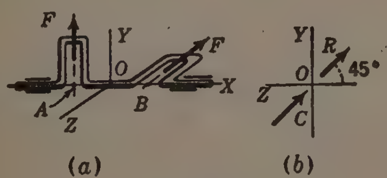


FIG. 49.

Solution. It is plain from the figure that $\Sigma F_x = 0$, $\Sigma F_y = +F$, and $\Sigma F_z = -F$. Hence $R = \sqrt{2}F$; it acts in the yz plane as indicated in Fig. 49*b*. It is plain also that $\Sigma M_x = 0$, $\Sigma M_y = +(F \times \frac{1}{2}b)$, and $\Sigma M_z = -(F \times \frac{1}{2}b)$. Hence $C = \frac{1}{2}\sqrt{2}Fb$; its vector is marked C. (We suggest that you find the resultant R–C referred to A, and the resultant R–C referred to B.)

If the shaft *were rotating*, then each crank would have a centrifugal force like F. The resultant of these forces would be strictly comparable to the resultant R–C just found, but the line of action of this R and the plane of this C would rotate with the shaft.

MOMENT OF A SYSTEM OF FORCES ABOUT VARIOUS LINES THROUGH A CHOSEN POINT O. Each moment can be represented by a vector laid off on the line

about which the moment is taken; see Fig. 50, which corresponds to Fig. 47. As already shown, $\Sigma M_x = (C_x)$; $\Sigma M_u = (C_u)$, where C_u denotes the u component of C. The maximum moment of the system $\Sigma M_m = (C)$; hence

$$\Sigma M_m = [(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2]^{\frac{1}{2}}$$

(The subscript m suggests not only maximum but also the line to which ΣM_m refers if that line is called the m line or axis.)

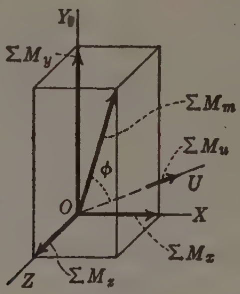


FIG. 50.

The figure suggests two ways to find ΣM_u for a given system: (i) Choose convenient x, y, and z axes; calculate ΣM_x, ΣM_y, and ΣM_z and then find ΣM_m; the u component of ΣM_m is ΣM_u. (ii) Having found ΣM_x, ΣM_y, and ΣM_z, find their u components; the algebraic sum of these components is ΣM_u.

CHAPTER III

FORCES IN EQUILIBRIUM

A. Principles of Equilibrium

33. Preliminary. Most engineering problems that involve statics require the finding or determination of one or more unknown forces from given data, loads, dimensions, etc. The solution of any such problem usually involves calculations that pertain to forces in equilibrium, and in order to make these calculations it is necessary first to distinguish a force system that is in equilibrium and that includes the unknown forces in question, and then to apply **conditions of equilibrium** that are suitable to this system.

How systems of forces in equilibrium can be distinguished, clearly and correctly, is explained in Arts. 34 and 35; conditions of equilibrium are set forth and discussed in Arts. 36 to 43; and the way in which they are applied is explained and illustrated in succeeding articles. It is absolutely essential that you should understand the definitions and methods that are given in the next two articles; if they are really mastered you should have little difficulty in solving most of the problems you will encounter in statics.

34. External and Internal Forces. It is convenient to distinguish between forces as external or internal with reference to any particular body: a force is said to be **external** to a body if it is exerted on that body by some other body, and **internal** if the force is exerted on a part of that body by some other part of the same body. (Note that the word "body" is used here and in mechanics generally in a broad sense to denote any definite portion of matter, whether simple and rigid like a stone or log, or complex like a bridge or locomotive, or fluid like the steam in a boiler or the water in a pond. In accordance with this definition, we call any part of a stone, bridge, etc., a body if that part is of special interest and under separate consideration.)

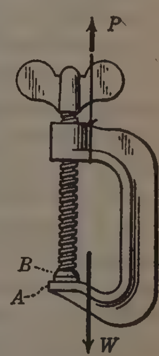


FIG. 51.

As an example of the distinction between external and internal forces, consider the screw clamp in Fig. 51. It is suspended by means of a string from a support above, and the screw is turned down hard so that parts A and B are tightly pressed together. With reference to the entire clamp, the upward pull P of the string and the downward pull W of the earth (weight of clamp) are external forces; the upward push Q of A on B and the equal downward push Q' of B on A are internal forces. With reference to the

screw considered alone, Q is an external force; and with reference to the frame alone, Q' is an external force.

With reference to any given body all the external forces, collectively considered, constitute what is called **the external system,** and all the internal forces, **the internal system.**

Though all forces occur in pairs the forces of which are equal, opposite, and collinear ("action and reaction," see Art. 5), there is an important distinction between the internal and the external systems of forces in respect to this duality, namely, both forces of each pair occur in the internal system but only one force of each pair in the external. For example, with reference to the whole clamp, both Q and Q' belong to the internal system whereas the reactions corresponding to the forces P and W do not belong to the external system, because they are not exerted on the clamp, but on the string and the earth, respectively.

All the forces that act on a body at rest constitute a system in equilibrium. This system includes both the external and the internal forces, but since the internal system, consisting of pairs of equal, opposite, and collinear forces, is balanced, it follows that the external system also is balanced. Hence it may be stated that *the external system of forces acting on any body at rest is in equilibrium.*

35. The Free-Body Diagram. Since the external system of forces acting on any body at rest is in equilibrium, the logical way to distinguish or identify a system of forces that is in equilibrium is, first, to select *a body* that is known to be at rest, and then to identify the system of external forces which acts on this body. A device or scheme which is of great aid in doing this is the **free-body diagram** — a sketch showing (i) *the selected body by itself entirely isolated from other bodies, which really may be in contact with it,* and (ii) *all the external forces acting on the selected body.* This is a complete description, but the following remarks may aid you to make such a diagram correctly.*

The fbd must show the body in question and no other body; it must show all the external forces that act on the body in question and no other forces. Those external forces that are known should be fully represented; those that are not known should be represented as fully as available information permits. Care should be taken to guard against showing forces exerted *by*, instead of *on*, the body, and to avoid showing internal forces.

It is to be remembered that an external force is one exerted on the body in question by something else, in general either by the earth through gravitation or by some other body through contact. It follows that the number of external forces is in general equal to one plus the number of contacts between the given body and other bodies. (It is here assumed that there is no force without contact except gravity.) The logical procedure is to represent first the known forces completely, indicating the magnitude, line of action, and sense of each, and then the unknown forces as fully as possible.

Consideration of the nature of the contact at a place where an unknown force

* The term "free-body diagram" is abbreviated in this book to fbd.

is applied often suggests information concerning that force. Thus, if the bodies merely bear against each other, the force that either exerts on the other must be normal to the surfaces of contact if the bodies are smooth but can be more or less inclined if they are rough.* If the bodies are actually connected, as by a pin, then the force can have any direction. If the direction of a force is not known, the force may be shown as acting at an unknown angle with the horizontal (or other reference line), or it may be represented by unknown rectangular components. More detailed directions for constructing a fbd are given in connection with the examples that follow.

Six examples of the construction of a fbd follow. In each, at least one of the forces is a supporting force or reaction, the nature of which should be carefully noted, since many such forces occur in subsequent problems.

EXAMPLE 1. A cylinder rests in a trough formed by two smooth inclined planes (Fig. 52*a*). It is required to construct the fbd for the cylinder.

A separate sketch of the cylinder alone is made. The weight of the cylinder, which acts vertically downward through its center, is represented by W. The places where the cylinder is in contact with other bodies are noted; these are at A and B. At A the plane exerts a force upon the cylinder; since the surfaces in contact are smooth, this force is a push normal to the plane; it is represented by P. At B the plane exerts a force upon the cylinder; since the surfaces in contact are smooth, this force also is a push normal to the plane; it is represented by Q. Figure 52*b* is the complete fbd for the cylinder.

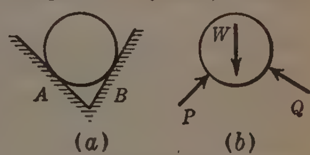


FIG. 52.

EXAMPLE 2. A and B (Fig. 53*a*) are so rough that no slipping occurs when A is subjected to the force P. A is shown free in Fig. 53*b*. The external forces on A are gravity (100 lb),

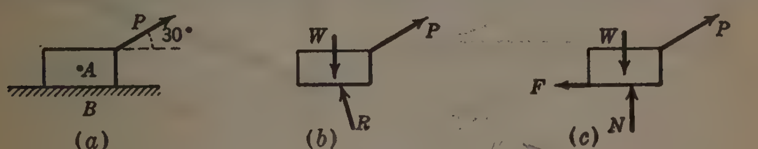


FIG. 53.

the applied force P (40 lb), and the reaction R exerted by B. Since A does not move to the right, R is obviously inclined to the left, and is a push. The magnitude and direction of R are unknown.

The fbd might be made as in Fig. 53*c*, where R is represented by its two components, one along the contact and one normal to it; both are unknown. (The first diagram is preferable for a graphical solution, and the second for an algebraic one.)

* The components of the force along and perpendicular to the surface of contact are called *friction* and *normal pressure*, respectively. It is the friction that tends to prevent the sliding of one body on another, and experience teaches that friction is usually large between rough bodies and small between smooth ones. If the surfaces in contact were perfectly smooth the friction would be nil, and the forces exerted would consist simply of the normal pressures. Of course no actual surfaces are perfectly smooth, but some may, for practical purposes, be so regarded, and for brevity these will be called smooth and those whose resistance to sliding is to be taken into account will be called rough. (See Chapter V for a fuller discussion of friction.)

EXAMPLE 3. The bar AB (Fig. 54a) is supported at A by a smooth pin and at C by a post with smooth rounded top. The bar is shown free in Fig. 54b. The external forces on it are the weight of the bar (100 lb), the pull of the rope, and the reactions at C and A. Since the contact at C is smooth, the reaction P is normal to the tangent plane of contact, that is the undersurface of the bar; and, since the contact between pin and bar is smooth, the pressure between them is, at every point of the contact, normal to the surface there. The resultant Q of the pressures at all points, therefore, acts through the center (or axis) of the pin. But the direction of Q is unknown, and so it is represented as acting at an unknown angle θ. Thus there are three unknowns in this system of forces, namely P, Q, and θ.

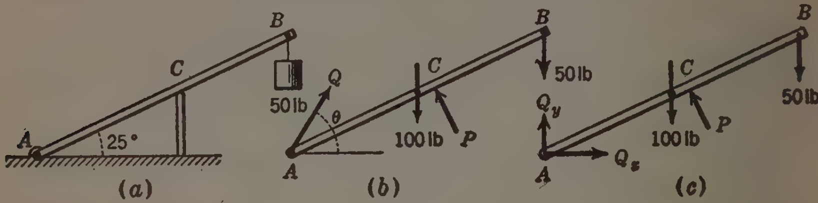


FIG. 54.

Instead of representing Q as a single force, we can represent it equally well by its unknown horizontal and vertical components Q_x and Q_y as in Fig. 54c. (The first representation is preferable for a graphical solution, and the second for an algebraic one.)

EXAMPLE 4. The simple wall crane represented in Fig. 55a consists of a beam AB and a tie rod CD. It is pinned at A and C to a wall. The beam is shown free in Fig. 55b. The external forces are the weight of beam 1000 lb, the load 2000 lb, the reaction P of the tie rod,

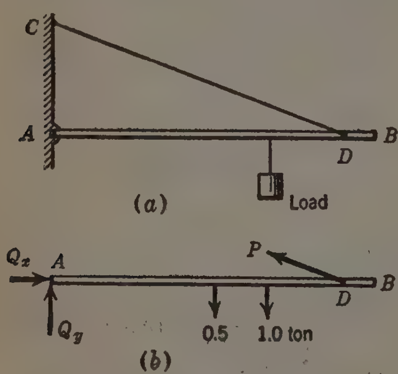


FIG. 55.

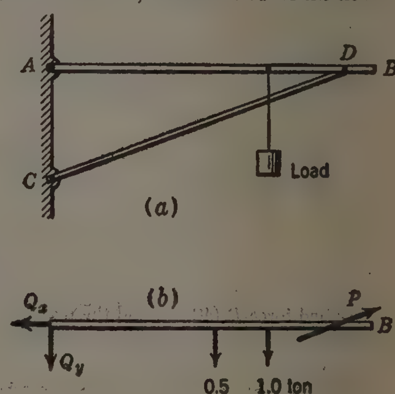


FIG. 56.

and the reaction Q of the pin A represented by its horizontal and vertical components. Assuming that the weight of the rod is negligible, we therefore regard the rod as subjected to only two external forces, one at each end. Since these two forces are in equilibrium, they are collinear and act along the axis of the rod (Art. 10). *Moreover*, the forces which the rod exerts on the beam and wall also act along the axis of the rod; P is so shown. The external system for the beam has three unknowns, P, Q_x, and Q_y.

EXAMPLE 5. A second crane is represented in Fig. 56a. It is just like the first except that a strut replaces the tie rod. Figure 56b is a fbd for the beam; P is shown acting along the strut on the supposition that the weight of the strut is negligible. Also, P is obviously a push; hence Q_x is shown acting toward the left.

EXAMPLE 6. If the weight of the strut of Ex. 5 is not negligible, then the strut is a three-force piece, and the forces are either parallel or concurrent (Fig. 57), according to circumstances; see Art. 36. If the member is not vertical, then neither of the pin pressures P and Q acts along the axis of the strut; for, if either or both did, they could not balance W. Therefore the reactions which such a strut exerts on the members to which it is pinned do not act along its axis.*

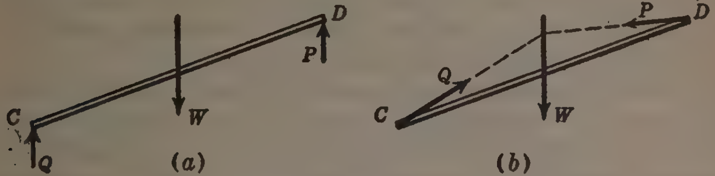


FIG. 57.

36. Conditions of Equilibrium. By this title is meant the conditions which a system of forces in equilibrium satisfies. An obvious condition is this: *The resultant of the system is nil.* It is *a general condition* of equilibrium which holds for any kind of system. Though not always directly applicable it leads readily to two other conditions which are very useful, namely:

A. *The algebraic sum of the rectangular components of all the forces along any line equals zero,* and

B. *The algebraic sum of the moments of all the forces about any line (for coplanar forces, about any point in the plane) equals zero.*

By means of A and B, any number of equations can be written for any system in equilibrium. Thus, for a coplanar concurrent system, A gives $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_u = 0$, etc., where x, y, u, etc., are axes of resolution in the plane of the forces; and B gives $\Sigma M_a = 0$, $\Sigma M_b = 0$, $\Sigma M_c = 0$, etc., where a, b, c, etc., are origins of moments in the plane. Of such equilibrium equations, only a certain number are independent; any additional ones follow from these, of necessity, and so, although valid, are useless.

The independent equations of equilibrium are such as are necessary and sufficient to insure a zero, or vanishing, resultant. For example, the resultant of a coplanar concurrent system, if any, is given by $R = [(\Sigma F_x)^2 + (\Sigma F_y)^2]^{\frac{1}{2}}$ (see Art. 27). R cannot be zero unless $\Sigma F_x = 0$ and $\Sigma F_y = 0$, and, if each sum does equal zero, then R must equal zero. Hence, $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are independent equations of equilibrium for this particular kind of force system. If u is any other axis, obviously $\Sigma F_u = 0$, but this equation is not independent of $\Sigma F_x = 0$ and $\Sigma F_y = 0$; it follows from them.

* It is often convenient to designate a body according to the number of external forces that act on it, as a one-force body (or member, or piece), a two-force body, a three-force body, etc. Thus the tie rod of Ex. 4 is a two-force body; the cylinder of Ex. 1 is a three-force body; the bar of Ex. 3 is a four-force body but looked upon as five-force body in Fig. 54c. This designation of bodies according to the number of external forces that act on them suggests the application of the special conditions of equilibrium given in Art. 36.

Since only the independent equations of equilibrium for a particular kind of system are useful, it is important to know what these equations are. They are given in Arts. 38 to 43, which deal at length with equilibrium problems involving the various kinds of force systems.

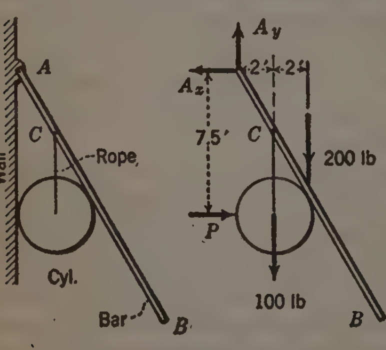
SOME SPECIAL CONDITIONS OF EQUILIBRIUM. Certain conditions of equilibrium depending on the *number* of forces in the system will be found useful. As already noted (in Art. 10) a *single force cannot be in equilibrium,* and *two forces in equilibrium are necessarily collinear, equal, and opposite.*

Three forces in equilibrium are coplanar and either concurrent or parallel. Proof: Since two of the forces, say F_1 and F_2, balance the third force F_3, the resultant of F_1 and F_2 is a single force R collinear with F_3; and since F_1 and F_2 are coplanar with R they are coplanar with F_3. If F_1 and F_2 are concurrent, then R is concurrent with them, and hence F_3 also. If F_1 and F_2 are parallel, then R is parallel to them, and hence F_3 also.

If four coplanar forces are in equilibrium, then the resultant R of any two of the forces balances the other two. Hence, (i) if any two are concurrent and the other two also, then R passes through the two points of concurrence; (ii) if any two are concurrent and the other two parallel, then the resultant R of the first pair acts through the point of concurrence and is parallel to the second pair. These facts are useful in graphical analysis of four-force systems.

B. Applications of Principles of Equilibrium

37. Common Type of Problem and Solution. A given body (structure, machine, appliance, or part thereof) is at rest while other bodies exert forces on it. Some of these forces are wholly known; some are partly or wholly unknown. Some or all of these unknown forces are to be determined.



(a) (b)

FIG. 58.

A program for determining these unknown forces is described below. It will be understood more readily if preceded by an example. Suppose that it is desired to find the pressure between the wall and cylinder represented in Fig. 58*a*. The cylinder is suspended from the bar AB which is pinned at A to the wall. The weights and all necessary dimensions are supposed given.

Figure 58*b* is a fbd for the cylinder-rope-bar body. The external system of forces has three unknowns, P, A_x, and A_y. The system is a coplanar nonconcurrent nonparallel one, for which there are three independent equations of equilibrium (see Art. 40). We choose, say,

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \text{and} \qquad \Sigma M = 0$$

The third equation, with center of moments at A, written out for the system is

$$-(100 \times 2) - (200 \times 4) + (P \times 7\tfrac{1}{2}) = 0$$

from which $P = 133$ lb. (If A_x and A_y were desired, they could be found now from the first two equilibrium equations written out for the system.)

PROGRAM. 1. *Select a body*, the given one or a portion of it, so that the external system of forces exerted on that body includes one or more of the forces to be determined.

2. *Make a fbd for the selected body.* This is the crucial part of the solution; indeed, the importance of this diagram can scarcely be overstated. It serves at once to define the problem, to show the kind of force system involved, to present the available data, and to indicate the quantities to be determined; also it aids in the next step.

3. *Ascertain whether the system is solvable.* If not, repeat steps 1 and 2 until a solvable system is found. Usually one trial, or at most two, suffices. Solvability depends upon the relative number of unknowns and independent conditions of equilibrium available. In an algebraic solution, not all the unknowns can be found if their number exceeds the number of independent equilibrium equations. Under such circumstances, solution by graphical methods is also impossible, for in graphical solutions one really makes use of the same conditions as in algebraic solutions, although the statement of these conditions takes a different form. When counting the number of unknowns, for comparison with the number of equations of equilibrium, you should regard the unknown magnitude and sense of a force as one unknown, for they are determined jointly.

4. *Apply appropriate conditions of equilibrium* to the system of external forces of the diagram, and so determine the unknowns. "Appropriate conditions" are such as pertain to the particular kind of system at hand (coplanar concurrent, coplanar parallel, etc.); see Arts. 38 to 43. "Applying" algebraic conditions means writing out or setting up the equations of equilibrium, and solving them simultaneously for the unknowns they contain. "Applying" graphical conditions means drawing the closed force polygon and any other diagrams that may be needed. Such construction determines the unknowns.

38. Coplanar Concurrent System in Equilibrium (Fig. 59). There are two independent equations of equilibrium; they can be expressed in three forms, namely:

$$\Sigma F_x = \Sigma F_y = 0 \qquad (1)$$
$$\Sigma F_x = \Sigma M_a = 0 \qquad (2)$$
$$\Sigma M_b = \Sigma M_c = 0 \qquad (3)$$

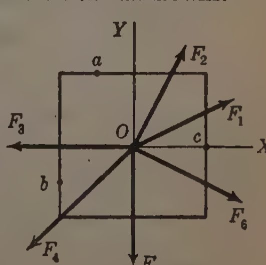


FIG. 59.

Axes x and y may be taken anywhere but conveniently through the point of concurrence O. To

insure the independence of Eq. 2 the moment origin a should not be on the y axis; and, for Eq. 3, the line joining the origins b and c should not pass through O. (See Fig. 59.)

The graphical condition of equilibrium, for a concurrent system, is that the force polygon for the forces closes. If the system is a three-force system, its (closed) force polygon is a triangle, from which it follows readily that the forces are respectively proportional to the sines of the angles between the other two. Or

$$\frac{F_1}{\sin \alpha_1} = \frac{F_2}{\sin \alpha_2} = \frac{F_3}{\sin \alpha_3}$$

where α_1, α_2, and α_3, respectively, denote the angles, acute or obtuse, between F_2 and F_3, F_3 and F_1, and F_1 and F_2. The relation expressed above is known as *Lami's theorem.*

The examples following are rather simple. Although you may be able to solve them unaided, in your own way, we strongly urge you to learn and adopt at once the professional way (free-body-diagram and conditions-of-equilibrium). By so doing, you will become prepared to handle problems too difficult for you now.

EXAMPLE 1. The pressures P and Q of Ex. 1, Art. 35, are required for this numerical problem. $W = 100$ lb, and the inclinations of the planes A and B, respectively, to the horizontal are 50° and 75°.

Solutions, following the program of Art. 37. The fbd already explained in the example referred to is reproduced in Fig. 60a but with additions. The external system has two unknowns, P and Q; it is coplanar concurrent and so has two equations of equilibrium. Hence the system is solvable.

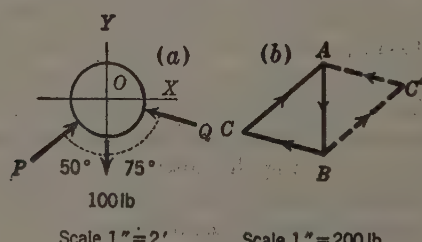


FIG. 60.

Graphical solution. The graphical condition of equilibrium is that the force polygon for the system closes, that is, that the vector sum of the forces is zero. To make the polygon, AB (Fig. 60b) is drawn first to represent W, next from A (or B) a line is drawn parallel to P, and then from B (or A) a line parallel to Q. The arrowhead on AB points downward, of course; it fixes the other arrowheads since all must be confluent (point the same way around). The vectors BC and CA represent Q and P, respectively, in magnitude and direction. They scale 93 and 118 lb, respectively. The senses of P and Q just found from the vector polygon agree with the senses here known in advance.

Solution by equilibrium equations. With axes as indicated in the figure,

$$\Sigma F_x = P\ (\cos 40) - Q\ (\cos 15) = 0$$

and $$\Sigma F_y = P\ (\sin 40) + Q\ (\sin 15) - 100 = 0$$

or $P(0.766) - Q(0.966) = 0$ and $P(0.643) + Q(0.259) = 100$. Solving simultaneously: From the first $P = 1.262Q$, and from the second $P = 155.5 - 0.403Q$; so

$$1.262Q = 155.5 - 0.403Q$$

whence $Q = 93.5$ lb. This value substituted for Q in either equation gives $P = 118$ lb.

Simultaneous solution of the equilibrium equations could be avoided by choosing either coordinate axis along or parallel to one of the unknown forces, either P or Q. Write the equations for axes so taken, and then note the number of unknowns in each.

Solution by Lami's theorem. Applied to Fig. 60*a*, the theorem gives

$$\frac{P}{\sin 75} = \frac{Q}{\sin 50} = \frac{100}{\sin 55}$$

These give, without simultaneous solution,

$$P = \frac{100}{\sin 55} \sin 75 = 118 \qquad \text{and} \qquad Q = \frac{100}{\sin 55} \sin 50 = 93.5 \text{ lb*}$$

EXAMPLE 2. Cylinders A and B (Fig. 61) rest as shown on an inclined plane, and A against a vertical plane. The diameters of the cylinders respectively are 6 in. and 24 in.; their weights are 50 lb and 100 lb. The pressure between A and the vertical plane is required. (Lines AB, AC, and BC are necessary for some trigonometric calculations. AC and BC respectively are parallel and perpendicular to the inclined plane.)

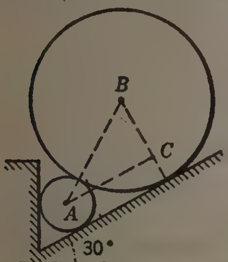


FIG. 61.

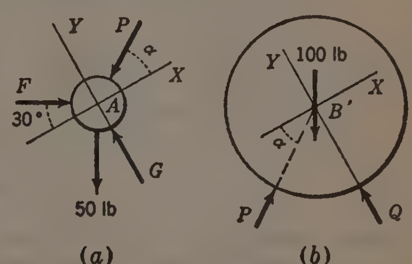


FIG. 62.

Solution by equilibrium equations. The pressure F of the wall on A is an external force for A, and so the fbd for A (Fig. 62*a*) brings the desired force F into the picture. The four external forces are gravity 50 lb, and the three contact forces F, G, and P. These last three are shown radial or normal to the contact surfaces, respectively, because the surfaces are smooth. The system has three unknowns but only two independent equilibrium equations; hence it is not solvable. If G or P could be found, from some other system, then we could return to the first system and solve it for F.

* You are urged always to record the results of a solution such as those of the foregoing example on the fbd, underscoring or otherwise marking the results so as to distinguish them from the given data. When this is done, the fbd shows at a glance the values obtained for the unknown quantities. This is not done in the examples in this book because it is desired to place emphasis on the given data and to avoid any possibility of confusion between the quantities that are originally known and the quantities that are solved for.

Now the pressure of A on B is equal and opposite to P, and this pressure can be found from Fig. 62b, the fbd for B. The external forces are gravity 100 lb, P, and the contact force Q normal to the inclined plane. This system is solvable. Solution gives $P = 62.5$ lb. (To find P, one needs the sine and cosine of the angle α, equal to BAC, Fig. 61. You see that $AB = 15$ in. and $BC = 9$ in.; hence $AC = 12$ in., and so $\sin\alpha = 0.6$ and $\cos\alpha = 0.8$.) And now supplying the value of P in Fig. 62a you see that

$$\Sigma F_x = F\ (\cos 30) - 62.5\ (\cos\alpha) - 50\ (\sin 30) = 0$$
$$\Sigma F_y = -F\ (\sin 30) - 62.5\ (\sin\alpha) - 50\ (\cos 30) + G = 0$$

Since the first equation contains only one unknown, F, it yields the value of that force, thus

$$F(0.866) - 62.5(0.8) - 50(0.5) = 0 \quad \text{or} \quad F = 86.6 \text{ lb}$$

A graphical solution could be made. Is a solution by Lami's theorem possible?

EXAMPLE 3. The bent bar B (Fig. 63a) is pinned at A to a wall. The cylinder C is suspended between the wall and the bar, both smooth, by means of a cord CD. (Actually the cord would extend from D to a yoke or bail fastened to pins at the ends of the axis of the cylinder.) The cylinder weighs 200 lb. All forces acting on the bar due to the weight of the cylinder are required. (The weight of the bar is to be disregarded.)

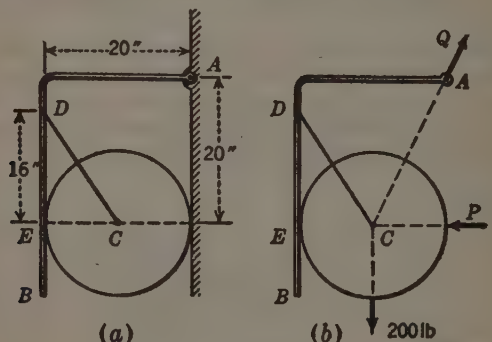


FIG. 63.

Solution. Figure 63b is a fbd for "the whole thing," the assemblage of all the parts. Generally it is advisable to begin a solution with such a diagram. There are three external forces, gravity 200 lb, and the contact forces P and Q, exerted by the wall and pin, respectively. P is normal to the contact surface and obviously acts toward the left. Since the three forces are in equilibrium, they are concurrent or parallel (see Art. 36). Obviously they are concurrent, and so Q acts not only through A but also through the center C, as indicated. The system, having two unknowns and two equilibrium equations, is solvable for P and Q.

Query. Assuming that P and Q have been found, how would you find the pulls of the cord on the bar and cylinder and the pressure between bar and cylinder? Suppose that the above-outlined solution has not been made. Plan a new solution, starting with a fbd of the cylinder, say. Is the external system solvable? If not, make a fbd of the bar. Is the external system solvable? What in your opinion is the best order of fbd's?

When you make fbd's and plan solutions you are applying mechanics; it is this process that is new to you and that you must learn. When you write and solve equilibrium equations you are applying mathematics, which should be "old stuff" to you, but which must not be slighted on this account.

Checking answers. Sometimes, after a problem has been solved, it is possible to make an additional fbd in which the external system includes some or all of the forces that have been found. If it should contain fewer unknowns than the number of independent equilibrium equations available, then more or less checking is possible. To illustrate, Fig. 64 is an additional fbd for the solution of Ex. 2. The external system is coplanar nonconcurrent nonparallel, for which there are three equilibrium equations (see Art. 40); it has two unknowns. Hence, checking is possible. Thus, for the x axis indicated,

$$\Sigma F_x = 86.6\,(\cos 30°) - 50\,(\sin 30°) - 100\,(\sin 30°)$$

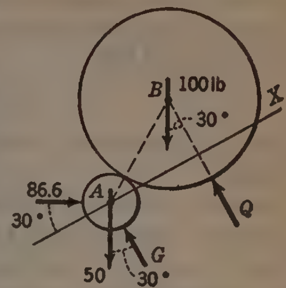


FIG. 64.

Since this sum is zero, the system, which includes an answer (previously found force), does satisfy an equilibrium equation; therefore the answer is correct.

Suppose that Ex. 2 had called for *all* the forces acting on each cylinder. Q would have been found, from Fig. 62*b*, to be 49.1 lb, and G, from Fig. 62*a*, to be 124.1. Figure 64 would then have *no* unknowns. If the system satisfied $\Sigma F_x = \Sigma F_y = \Sigma M = 0$, we could conclude that the answers for F, G, and Q were correct; and, since $P = 62.5$ lb was used to find F, G, and Q, we would conclude that the answer for P was correct.

Determination of sense. In the preceding examples the senses of forces to be found were apparent. Problems arise, however, in which the senses are not

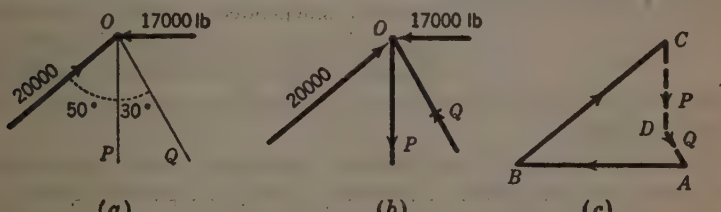


FIG. 65.

apparent. For example, suppose that O (Fig. 65*a*) is a bridge pin that connects four members of a truss of the bridge, that these exert the only forces on the pin, and that two of the forces are completely known but only the lines of action of the other two, P and Q, all as shown; the senses of P and Q are not apparent. Suppose that the magnitudes and senses of P and Q are required. Apparently, the system has four unknowns, two magnitudes and two senses. Really it has (see Art. 37) only two, for the magnitude and sense of each force, P or Q, can be found simultaneously, as shown below.

In an algebraic solution, the senses of P and Q must be guessed or assumed prior to writing equilibrium equations and adhered to in a solution of the equa-

tions. The correct sense of either is indicated by the sign of its computed value; *a positive sign indicates that the sense assumed is correct, and a negative sign that the sense assumed is wrong.* Thus we guess the senses of P and Q (see Fig. 65*b*), write the equilibrium equations $\Sigma F_x = \Sigma F_y = 0$, solve them simultaneously, and find

$$P = +9950 \quad \text{and} \quad Q = -3360 \text{ lb}$$

Hence, the arrowhead on P is correct and that on Q is wrong. P is a pull, Q also.

The arrowhead on Q is crossed by a short line to indicate that it was found to be wrong. Such a crossed arrowhead tells just what was done in respect to the sense of the force. *You should not make a correction by inserting a correct arrowhead over the incorrect one or in its place.*

In a graphical solution it is not necessary to assume senses or arrowheads for P and Q in advance. Figure 65*c* is a force polygon for the four forces. The arrowheads on AB and BC are known in advance and fix the other two, since all must point the same way around. Thus P and Q are found to be pulls, as in the algebraic solution.

39. Coplanar Parallel System in Equilibrium (Fig. 66). There are two independent equations of equilibrium; they can be expressed in two forms, namely

$$\Sigma F\updownarrow = \Sigma M = 0 \tag{1}$$

$$\Sigma M_a = \Sigma M_b = 0 \tag{2}$$

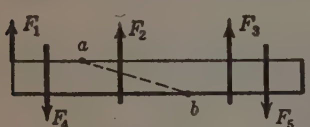


FIG. 66.

To insure independence of Eq. 2 the line joining a and b must not be parallel to the forces (Fig. 66). The two graphical conditions of equilibrium are: (i) the force polygon for the forces closes, (ii) the string polygon also closes.

It is stated and proved in Art. 36 that if three forces are in equilibrium they are coplanar and concurrent or parallel. If they are parallel, then, as you readily see, the middle force is opposite to the other two in direction, and the moments of any two of the forces about a point in the other force are equal in magnitude and opposite in sense or sign.

A very common type of problem on the equilibrium of coplanar parallel forces is to find the reactions (supporting forces) on a loaded horizontal beam resting on two supports. The best practice is to use Eq. 2, taking the moment origins on the lines of action of the two reactions. Each such equation contains only one unknown and furnishes that unknown directly. Finally, a check is made by calculating ΣF for all the forces; the sum should be zero.

EXAMPLE. The horizontal beam ABC (Fig. 67) is supported at B and C. It bears three loads as shown and its own weight, $W = 800$ lb. The supporting forces or reactions R_1 and R_2 are required.

Solution. To calculate the moment of the weight of the beam, or of any other distributed load, the weight may be regarded as concentrated at the center of gravity of the beam or load. Therefore we ask you to represent a downward force of 800 lb at the midpoint of AC. Then the figure becomes a fbd. Note that the senses of R_1 and R_2 are assumed upward. The appropriate equations of equilibrium are

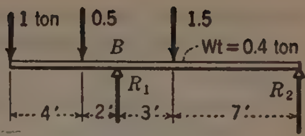


FIG. 67.

$$\Sigma M_B = (2000 \times 6) + (1000 \times 2) - (3000 \times 3) - (800 \times 2) + (R_2 \times 10) = 0$$
$$\Sigma M_C = (2000 \times 16) + (1000 \times 12) + (3000 \times 7) + (800 \times 8) - (R_1 \times 10) = 0$$

The first equation gives $R_2 = -340$ lb, and the second gives $R_1 = +7140$ lb. The negative sign shows that the assumed sense of R_2 is wrong, that is R_2 acts down; the positive sign shows that the assumed sense of R_1 is right, that is R_1 acts up. As a check on the solution, the condition $\Sigma F = 0$ is applied; thus,

$$\Sigma F = +7140 - 340 - 2000 - 1000 - 3000 - 800$$

which *does* equal zero.

40. Coplanar Nonconcurrent Nonparallel Systems in Equilibrium (Fig. 68). There are three independent equations of equilibrium; they can be expressed in three forms, namely:

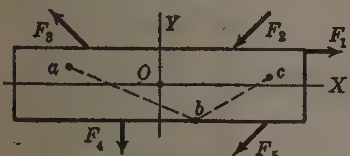


FIG. 68.

$$\Sigma F_x = \Sigma F_y = \Sigma M_a = 0 \qquad (1)$$
$$\Sigma F_x = \Sigma M_a = \Sigma M_b = 0 \qquad (2)$$
$$\Sigma M_a = \Sigma M_b = \Sigma M_c = 0 \qquad (3)$$

The x and y axes and the moment origin in Eq. 1 may be anywhere in the plane of the forces. For Eq. 2 the line joining the origins a and b must not be perpendicular to the x axis, to insure independence in the equations. And, for Eq. 3, the three origins a, b, and c must not be in a straight line. For graphical solutions we explain, below, a good plan that employs only one condition of equilibrium, namely, the force polygon closes.

Two common types of problems are represented in Figs. 69 and 70. In each, the system of forces is in equilibrium; forces F_1, F_2, F_3, etc., are *wholly* known, and the others only partially. But the two types differ as follows:

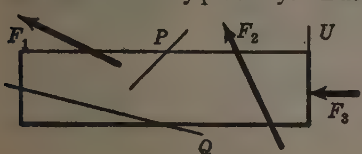


FIG. 69.

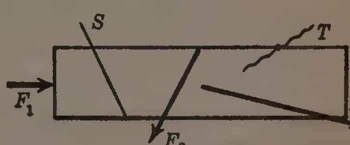


FIG. 70.

In type 1, the lines of action of P, Q, and U are known. The magnitudes and senses of these forces are required. (If P, Q, and U are concurrent, the problem is indeterminate.)

In type 2, the line of action of S and a point in the line of action of T are known. The magnitude and sense of S and the magnitude and direction of T are required.

Each problem really has three unknowns. In type 1, the magnitude and sense of each force (P, Q, U) constitute one unknown. In type 2, the magnitude and sense of S are one unknown; the magnitude and direction of T are two unknowns. Either problem can be readily converted into the other type; sometimes it is advantageous to do so (see Ex. 3, below).

ALGEBRAIC SOLUTION OF THESE TYPICAL PROBLEMS. If you will write Eq. 1, 2, or 3 for a given numerical problem, taking axes and moment origins at random, you will find, in general, that each equation contains more than one unknown; and hence the equations must be solved simultaneously to find the unknowns. But it is possible to choose axes and moment origins so that each equation will contain only one unknown and hence give that unknown directly. Any axis or moment origin which leads to an equation with only one unknown will be called a *good axis* or *good origin*. The intersection of any two of the unknown forces P, Q, and U is a good origin; so is the known point of T.

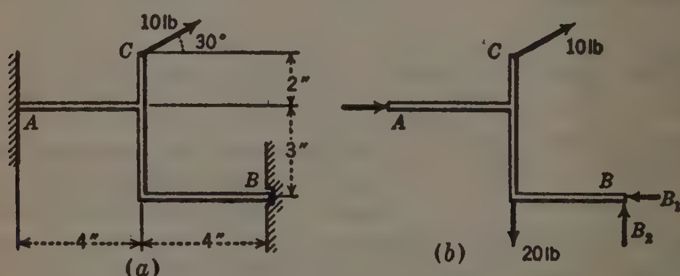


FIG. 71.

EXAMPLE 1. ABC (Fig. 71a) is an irregular bar supported by two walls. It rests against the smooth face of one wall and in a shallow notch of the other. The bar weighs 20 lb and is subjected to a force of 10 lb as indicated. The supporting forces or reactions are required.

Solution. Figure 71b is a fbd of the bar. There are five external forces, namely, the weight 20 lb, the applied force 10 lb, and the reactions at A and B, called A, B_1, and B_2. The indicated senses of the reactions were guessed. The problem is of type 1.

You see that a vertical line is a good y axis;

$$\Sigma F_y = +B_2 - 20 + 10 \sin 30 = 0$$

Hence $B_2 = +15$ lb, and the guessed sense of B_2 is correct. B, the intersection of B_1 and B_2, is a good moment origin;

$$\Sigma M_B = +(20 \times 4) - (10 \cos 30 \times 5) - (10 \sin 30 \times 4) - (A \times 3) = 0$$

Hence, $A = +5.57$ lb, and the guessed sense of A is correct. Finally

$$\Sigma F_x = +5.57 + 10 \cos 30 - B_1 = 0$$

Hence, $B_1 = +14.23$ lb, and the guessed sense of B_1 is correct.

EXAMPLE 2. AB (Fig. 72a) is a beam supported by three struts, AC, BD, and BE. The beam weighs 500 lb; it supports a load of 2000 lb at F. The struts are pinned to the beam and the floor as indicated; they are short and their weights are negligible. Required, the reaction of each strut on the beam.

Solution. Each strut is a two-force piece, and so the force it exerts on the beam acts along the axis of the strut. See the fbd in Fig. 72b, where these strut reactions are denoted by A, B_1, and B_2. This problem is of type 1. B is a good moment origin.

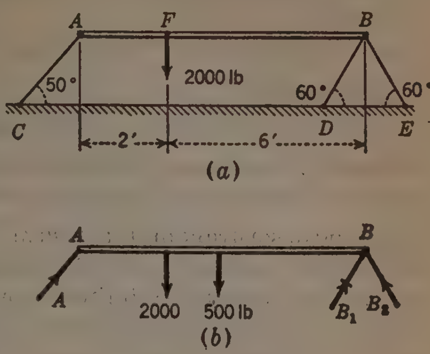


FIG. 72.

$$\Sigma M_B = 0 \text{ gives } A = +2285 \text{ lb}$$

The intersection of A and B_1 (or B_2) is a good origin, but not practicable, since its use requires finding its location and the arm of B_1 (or B_2) with respect to the located origin. Therefore we resort to

$$\Sigma F_x = +(2285 \cos 50) + (B_1 \cos 60) - (B_2 \cos 60) = 0$$

and $$\Sigma F_y = +(2285 \sin 50) - 2500 + (B_1 \sin 60) + (B_2 \sin 60) = 0$$

These, solved simultaneously, give $B_1 = -1040$ and $B_2 = +1900$ lb. The arrow on B_1 was guessed wrong; it is shown corrected in the usual way.

EXAMPLE 3. Suppose that the bar of Ex. 1 does not rest in a notch at B but is pinned there to the wall, and that the force applied at C is a *push* of 10 lb. The reactions at A and B are required.

Solutions. Figure 73a is a fbd for the bar. The external forces are 20 lb, 10 lb, and the reactions at A and B. The reaction at B is unknown in magnitude and direction. This problem is of type 2. Since B is a good moment origin, as in Ex. 1, we write

$$\Sigma M_B = +(20 \times 4) + (10 \cos 30 \times 5) + (10 \sin 30 \times 4) - A \times 3 = 0$$

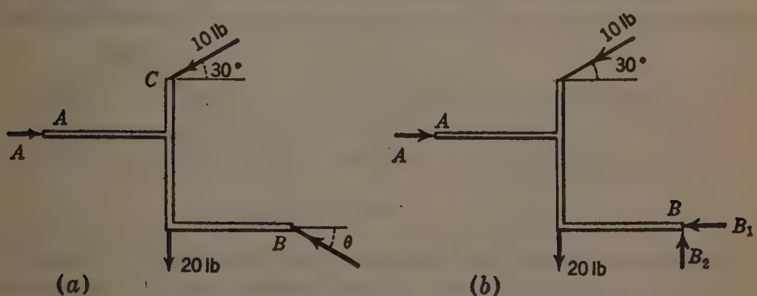


FIG. 73.

whence $A = +47.8$ lb. The arrowhead on A is correct. There is no other good moment origin, nor is there a good x or y axis. For x horizontal and y vertical,

$$\Sigma F_x = +47.8 - (10 \cos 30) - (B \cos \theta) = 0$$
$$\Sigma F_y = -20 - (10 \sin 30) + (B \sin \theta) = 0$$

These, solved simultaneously, give $\theta = 32.6°$ and $B = 46.4$ lb.

Another solution first converts the problem to type 1 by substituting for the unknown force B its unknown horizontal and vertical components, say B_1 and B_2. Figure 73*b* is the fbd; the arrowheads on B_1 and B_2 are guessed or assumed. There are three unknowns as in the first solution.

$$\Sigma F_y = 0 \text{ gives } B_2 = +25; \quad \Sigma M_B = 0 \text{ gives } A = +47.8; \quad \Sigma F_x = 0 \text{ gives } B_1 = +39.1$$

All arrows are correct.

$$B = (25^2 + 39.1^2)^{\frac{1}{2}} = 46.4 \text{ lb} \qquad \theta = \tan^{-1}(25 \div 39.1) = 32.6°$$

In this example, there is little choice between the two solutions. *Generally, it is advisable to convert a type 2 problem to type 1 for an algebraic solution.*

GRAPHICAL SOLUTION OF THE TYPICAL PROBLEMS. The essential feature of the good plan mentioned in the first paragraph of this article consists in simplifying the given system by substituting for the wholly known forces their resultant R. In a type 1 problem, the new system consists of four forces, R, P, Q, and U. In a type 2 problem it consists of three forces R, S, and T.

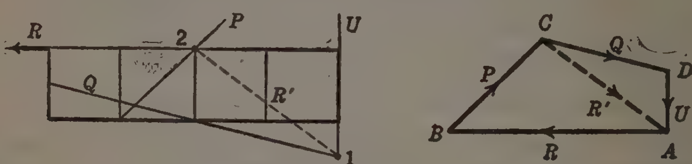


FIG. 74.

The four-force system will, in general, be such that no two of the forces are parallel (Fig. 74). Let R' denote the resultant of any two, say Q and U, of the unknown forces (P, Q, and U). R' passes through point 1, and, since R, P, and R' are in equilibrium, R' passes through point 2. The force polygon $ABCA$ for R, P, and R' can now be drawn; it determines P (and R'). The force polygon $ABCDA$ for R, P, Q, and U can be drawn also; it determines Q and U.

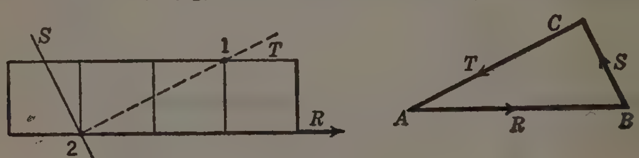


FIG. 75.

The three-force system will in general be concurrent (Fig. 75); hence T acts not only through the given point 1 but also through point 2. The force polygon for the three forces can be drawn; it is $ABCA$. S and T are determined now.

41. Noncoplanar Concurrent System in Equilibrium. There are three independent equations of equilibrium; the convenient form is

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$$

The graphical condition of equilibrium, the force polygon closes, is difficult to apply since the polygon is not a plane figure. This remark applies also to other noncoplanar systems (Arts. 42 and 43), and so their graphical conditions of equilibrium are not even stated.

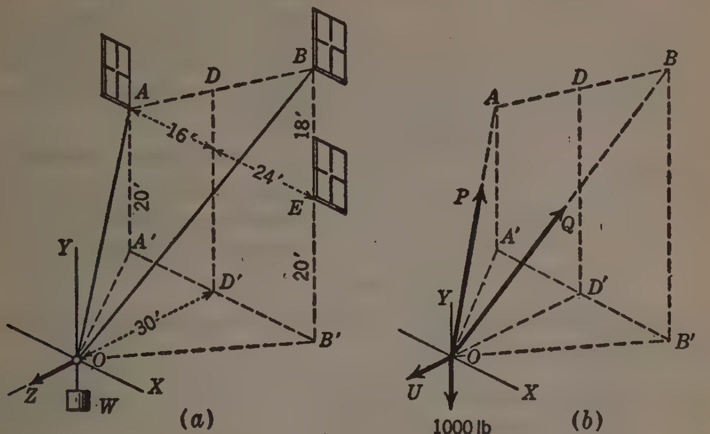


FIG. 76.

EXAMPLE. A heavy body W (Fig. 76a) weighing 1000 lb is suspended from a ring over the center of a street 60 ft wide; the ring is supported by three ropes OA, OB, and OC, A and B being points on the face of a building as shown, and C a point on the face of a building (not shown) on the opposite side of the street; OC is perpendicular to the faces of the buildings. It is required to determine the tension in each rope.

Solutions. 1. The ring O is shown free in Fig. 76b. The external forces are the downward pull, 1000 lb, and the pulls P, Q, and U of the ropes as indicated. Let α_1, α_2, and α_3 denote the direction angles of OA, and β_1, β_2, and β_3 those of OB. The equilibrium equations of the system of forces are

$$\Sigma F_x = -P\cos\alpha_1 + Q\cos\beta_1 = 0$$
$$\Sigma F_y = +P\cos\alpha_2 + Q\cos\beta_2 - 1000 = 0$$
$$\Sigma F_z = -P\cos\alpha_3 - Q\cos\beta_3 + U = 0$$

The direction-cosines of P are, respectively,

$$\frac{D'A'}{OA} = \frac{16}{39.4} \qquad \frac{A'A}{OA} = \frac{20}{39.4} \qquad \frac{OD'}{OA} = \frac{30}{39.4}$$

and the direction-cosines of Q are, respectively,

$$\frac{D'B'}{OB} = \frac{24}{54.0} \qquad \frac{B'B}{OB} = \frac{38}{54.0} \qquad \frac{OD'}{OB} = \frac{30}{54.0}$$

Substituting these cosines in the equilibrium equations and solving the equations gives

$$P = 868 \qquad Q = 794 \qquad U = 1100 \text{ lb}$$

2. Evidently the ropes OA and OB could be replaced by a single rope fastened to the building at any point vertically above D'. Suppose such replacing rope to be fastened at D,

which is in the plane of OA and OB, and let T denote the tension in this rope. Figure 77a is the new fbd, from which you find readily that

$$U = 1100 \quad \text{and} \quad T = 1490 \text{ lb}$$

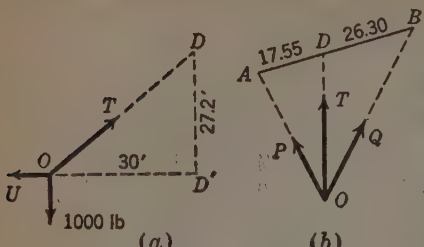


FIG. 77.

Now T can be resolved into components along the ropes OA and OB; these components are P and Q. A graphical resolution can be made readily from a layout of these ropes. See Fig. 77b, not to scale; the indicated lengths were calculated from Fig. 76a.

42. Noncoplanar Parallel System in Equilibrium. There are three independent equations of equilibrium; the convenient form is

$$\Sigma F = \Sigma M_1 = \Sigma M_2 = 0$$

The moment axes 1 and 2 must not be parallel; axes perpendicular to the forces are most convenient.

EXAMPLE. The 12-in. square plate (Fig. 78) weighs 100 lb. It is suspended in a horizontal position by means of three vertical cords fastened to it at A, B, and C. The pulls of these cords are required.

Solution. With indicated notation for the required pulls and moment axes, the equilibrium equations are

$$\Sigma F = P_1 + P_2 + P_3 - 100 = 0$$
$$\Sigma M_x = -(P_2 \times 4) - (P_3 \times 12) + (100 \times 6) = 0$$
$$\Sigma M_z = (P_1 \times 8) + (P_3 \times 6) - (100 \times 6) = 0$$

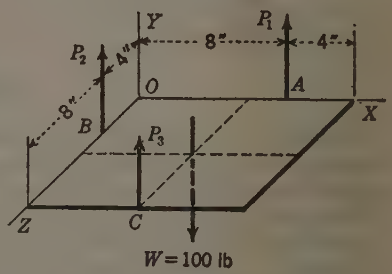


FIG. 78.

Solved simultaneously, the equations give

$$P_1 = 40.9 \qquad P_2 = 13.6 \qquad \text{and} \qquad P_3 = 45.5 \text{ lb}$$

43. Noncoplanar Nonconcurrent Nonparallel System in Equilibrium. There are six independent equations of equilibrium. The convenient set is

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0$$

PROJECTED SYSTEMS. If every force in the given system were represented by a vector, and all these vectors were projected on three rectangular coordinate planes, the three sets of projections would represent three coplanar force systems, each of which would be in equilibrium. This is a property of projected systems. (Proof is omitted.)

In general, three equilibrium equations may be written for each projected system. Among the nine, however, we find exact duplicates: two $\Sigma F_x = 0$, two $\Sigma F_y = 0$, and two $\Sigma F_z = 0$. Therefore we can write only six independent equations for the three projected systems.

When you wish to write the equilibrium equations for a complicated system, with many forces, awkward geometrical relations, etc., it may be advisable to arrive at the desired equations through the device of projected systems. This device does not shorten the work, but it breaks up a complicated problem into three simpler ones (see Ex. 2 below and especially Ex. 2, Art. 57).

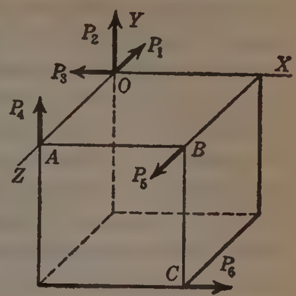


FIG. 79.

EXAMPLE 1. The 12-in. cube (Fig. 79) weighs 100 lb. It is supported in the position shown (four edges vertical) by six cords fastened to it at O, A, B, and C. The pull of each cord is parallel to an edge as shown. These forces, $P_1 \cdots P_6$, are required.

Solution. With indicated notation, the equilibrium equations are

$$\Sigma F_x = P_6 - P_3 = 0 \qquad \Sigma M_x = -(P_4 \times 12) + (100 \times 6) = 0$$
$$\Sigma F_y = P_2 + P_4 - 100 = 0 \qquad \Sigma M_y = -(P_5 \times 12) + (P_6 \times 12) = 0$$
$$\Sigma F_z = P_5 - P_1 = 0 \qquad \Sigma M_z = (P_6 \times 12) - (100 \times 6) = 0$$

These equations give 50 lb for each of the six pulls P.

EXAMPLE 2. Figure 80*a* is an end view of a four-throw crankshaft. Cranks 1, 2, 3, and 4 are in that order along the shaft. The center-to-center distance between adjacent cranks is b.

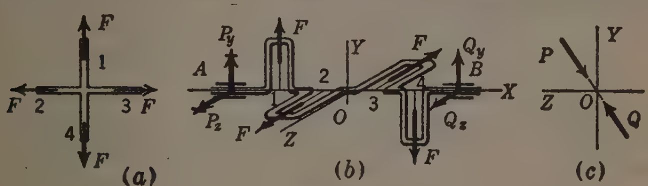


FIG. 80.

The shaft has two end bearings whose distance from the adjacent cranks is b, center to center. Suppose that each crank is subjected to a pull F acting along its center line as indicated. The reactions, due to these pulls, of the bearings on the shaft are required.

Solution. Figure 80*b* is a fbd of the shaft; the desired reactions P and Q, at A and B, respectively, are represented by x and y components. Equilibrium moment equations for axes coinciding respectively with P_z, P_y, Q_z, and Q_y give

$$Q_y = 0.60F \qquad Q_z = 0.20F \qquad P_y = -0.60F \qquad P_z = -0.20F$$

Hence $P = 0.63F$ and $Q = 0.63F$; they are parallel and opposite (see Fig. 80*c*) and constitute a couple.

(If the shaft were rotating, the cranks would have centrifugal pulls acting like the pulls F, and the reactions of the bearings would be strictly comparable to those found above. Then the lines of action of the reactions would rotate with the shaft.)

44. Statically Indeterminate Problems. Some equilibrium problems cannot be solved by principles of statics alone. If one such pertains to a structure, the structure also is said to be statically indeterminate. A loaded beam resting on more than two supports is statically indeterminate, for the reactions of the

supports cannot be found by principles of statics alone. If you doubt this statement, try to find the reactions in this simple example: the beam rests on three supports, one at each end and one at midlength; there are two equal loads, one at the middle of each span. Disregard the weight of the beam. After you have given up, see the answers at the end of this article.

A beam on more than two supports is, as far as equilibrium is concerned, oversupported. Only two supports are necessary; the others are *superfluous* or *redundant*. A general characteristic of statically indeterminate structures is that they have redundant supports or members or both. (See Art. 45 for additional comments on this subject.)

Statically indeterminate problems are beyond the scope of this book. Various types are dealt with in strength of materials, and still others in structural engineering and machine design. All methods of solving them are based, in part, on elastic properties of the parts of the structure or machine under consideration. The following simple example will give you some idea of one method.

Three cords supporting a small ring (Fig. 81a) are alike, having been taken from a single coil or spool. They are carefully tied to hooks A, B, and C, in the

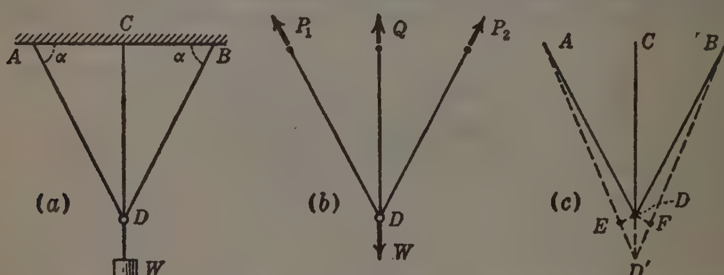


FIG. 81.

same vertical plane, and to the ring D so that they are just taut before the heavy body W is hung from the ring. Obviously, the structure consisting of cords and ring is oversupported; the hook C is unnecessary, or if C is used A and B are unnecessary. Apparently the structure is statically indeterminate.

We try to find the three reactions P_1, P_2, and Q (Fig. 81b) at the upper end of the structure. These and W constitute the external system, which is coplanar concurrent. It has only two independent equilibrium equations, as

$$\Sigma F_x = -P_1 \cos \alpha + P_2 \cos \alpha = 0 \qquad (1)$$
$$\Sigma F_y = P_1 \sin \alpha + P_2 \sin \alpha + Q - W = 0 \qquad (2)$$

and so the unknowns cannot be found from statics alone. Equation 1 shows that $P_1 = P_2$, which is obvious from the symmetry of the set-up. These equal pulls being denoted by P, Eq. 2 becomes $2P \sin \alpha + Q = W$.

This last equation and another one involving P or Q or both, but no other unknown, would furnish P and Q. One such can be found by a consideration

of the stretches of the three upper cords. They are represented, much exaggerated, in Fig. 81*c*; DD' is the stretch of the middle cord, and ED' and FD' are the stretches of the outer ones. The straight lines DE and DF are practically perpendicular to AD and BD, respectively. We assume that the cords in stretching obey Hooke's law; then the stretches are, respectively, proportional to the pulls and the lengths of the cords. That is

$$\frac{FD'}{DD'} = \frac{P \times AD}{Q \times CD} \quad \text{or} \quad \frac{P}{Q} = \frac{FD'}{DD'}\frac{CD}{AD} \tag{3}$$

The third ratio in this equation is equal to $\sin \alpha$, and the second is practically equal to $\sin \alpha$; hence $P/Q = \sin^2 \alpha$. This equation and $2P \sin \alpha + Q = W$ give

$$P = \frac{\sin^2 \alpha}{1 + 2 \sin^3 \alpha} W \quad \text{and} \quad Q = \frac{1}{1 + 2 \sin^3 \alpha} W$$

Answers to the beam problem of the first paragraph: end reactions $\frac{5}{16}W$ and middle reaction $\frac{22}{16}W$, where W denotes each load. These answers are correct only for the common type of beam (homogeneous, straight, and uniform in cross section), resting on supports at the same level. They were obtained by methods explained in books on strength of materials.

45. Anchorage; Degrees of Freedom. It is often necessary to know how a structure should be anchored or supported in order that it will stay in position in spite of any loads imposed upon it, or how an instrument or tool should be mounted in order that it will function properly, being left free to move in the manner desired but constrained against undesired motion. The problem of determining the amount of constraint necessary is essentially a problem in equilibrium, but its solution requires an understanding of the various degrees of freedom which a body may have. A rigid body is said to have as many degrees of freedom as there are ways in which it can move, either by sliding or turning. We now explain this further by means of easily visualized examples.

1. A body immovably clamped, as in a vise, has zero degrees of freedom.

2. A square rod in a closely fitting straight square tunnel has one degree of freedom; it can slide along the tunnel. A flywheel mounted in fixed bearings with side stops that prevent sliding also has one degree of freedom; it can rotate about the axis of the bearings.

3. A circular rod in a closely fitting straight circular tunnel has two degrees of freedom; it can slide along the tunnel and can also rotate about its own longitudinal axis.

4. A disk of uniform thickness fitting closely between two horizontal surfaces has three degrees of freedom; it can move east-west and north-south, and can also rotate about any vertical axis.

5. A cylindrical rod under the conditions described in paragraph 4 has four degrees of freedom; in addition to moving in the three ways available to the disk, it can rotate about its own longitudinal axis.

6. A sphere under the conditions described in paragraph 4 has five degrees of freedom; it can slide north-south and east-west, and it can rotate about a vertical axis, a north-south axis, and an east-west axis.

7. An airplane in flight has six degrees of freedom; it can move parallel to its own longitudinal axis, to the transverse axis of the wings, and to the vertical axis, and it can also rotate about each of these axes. This condition represents complete freedom, or complete absence of constraint.

NECESSARY CONSTRAINT. We now discuss the amount of constraint necessary to insure complete anchorage and the amount necessary to limit possible motion to the particular kind desired. It is usually easiest to do this by a sort of commonsense reasoning, since the constraint afforded by any given mechanical arrangement is usually obvious. But the final arrangement decided upon should always be checked by formal application of the appropriate equilibrium equations, in the way illustrated below.

It was shown in Art. 43 that the general conditions for equilibrium of any force system can be expressed by the six equations: $\Sigma F_x = \Sigma F_y = \Sigma F_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0$, where the subscripts refer to rectangular axes. If a rigid body is so supported that no matter what loads are applied to it the supports can supply the reactions necessary to satisfy these six equations, then the body is fixed; if one or more of the equations cannot be satisfied, then the body has one or more degrees of freedom and can move accordingly.

We first consider the minimum amount of constraint necessary to fix a rigid body and the arrangement of supports that will provide this constraint. Suppose the body to be represented by the block of Fig. 82. If one point of the body, as A, is fixed by a universal joint, the reaction there is a force that can have any direction and that can, together with any external forces that may be applied, satisfy the three equations $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$. This means that the body cannot slide, or move as a whole, in any direction. But, as it can still rotate about any axis through A, it has three degrees of freedom. If a second point B is partly anchored by a pinned bar 1 (two-force member) the axis of which does not pass through A, point B can move only in a direction normal to the plane of this bar and A; a second bar 2 not in this plane would prevent any motion of B. The only way in which the body could now move would be by rotating about the line AB. A third bar 3, attached at some other point as C, and neither parallel to nor intersecting line AB, would prevent such rotation. Obviously the reaction at A supplied by the universal joint could be equally well furnished by three noncoplanar bars concurrent there; therefore, complete constraint or anchorage of a body can be effected by six hinged bars arranged as here described. Furthermore, the six independent equations of equilibrium available make it possible to calculate the six unknown forces exerted by the

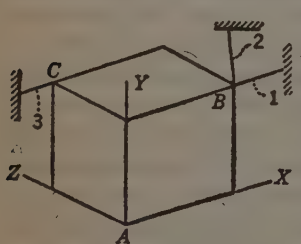


FIG. 82.

bars to balance any given system of applied loads. Additional bars are superfluous or redundant; they introduce more unknowns than there are independent equations, and the problem of finding the reactions becomes statically indeterminate.

The arrangement of the six bars described is not the only effective one, but any chosen arrangement must satisfy certain requirements in order to be effective. For example, four of the bars must not be concurrent. To illustrate: suppose bar 2 of Fig. 82 to be attached at A, making four bars concurrent there; then a line passes through A and intersects both the other two (non-parallel) bars 1 and 3, and the body could rotate about that line. (That there is such a line can readily be perceived if you imagine a long, slender rod hinged at A and then turned, while being kept against bar 1, until it comes in contact with bar 3.) Also, if bars 1 and 3 are made parallel, they cannot prevent rotation about a line through A parallel to them. (You should study other arrangements of the six bars that *would* give complete constraint.)

46. Clamps and Slides. It is sometimes necessary to design a mounting which will fix an instrument while in use but permit it to be removed and replaced in exactly the same position. Thus it may be desired to fix a microscope stand on a flat table top without permanently attaching it to the table. Suppose the stand to be provided with three rigid legs, each having a spherical end. If one leg is placed in a V-shaped groove, another leg in a socket having the form of an inverted three-sided pyramid and lying in line with the groove, and the third leg on the flat table top, there will be six points of contact between stand and table, and the stand will be in a fixed position as long as contact is maintained at all six points. The stand is not really anchored, because each of the supporting forces can have only one sense; the effect of gravity, or of an additional downward force supplied, say, by a spring is needed to make it unnecessary for any of the pressures to act as a pull, and thus to make constraint complete. An arrangement of this kind has been called a *geometrical clamp.**

Suppose that it is desired to mount the stand so that it can be made to slide in one direction only. This can be accomplished by letting two legs rest in the V-shaped groove and the third rest on the table top; the stand now has the one degree of freedom desired. An arrangement of this kind is called a *geometrical slide.**

Again, suppose that it is desired to mount the stand so that it can be rotated about a fixed vertical axis. This can be accomplished by letting one leg rest in the socket described and the other two on the table top; the stand can now be rotated about the vertical axis of the socket.

For each of the three arrangements just described and for each of the examples given at the beginning of Art. 45 you should identify the constraining forces or couples and satisfy yourself that they are just adequate to insure the degree of constraint required.

* Kelvin and Tait, *Treatise on Natural Philosophy.*

CHAPTER IV

SIMPLE FRAMEWORKS

A. Simple Frameworks; Truss Type

47. Description. A framework consisting of straight members in the same plane, and so connected as to form a triangle or series of triangles, is called a **truss.** Such a framework is rigid under loads applied at the joints and in the plane of the members, and it will be seen presently that in the usual type of construction the members are subject only to longitudinal tension or compression as a result of such loading. Figure 83 represents a simple truss of a common type.

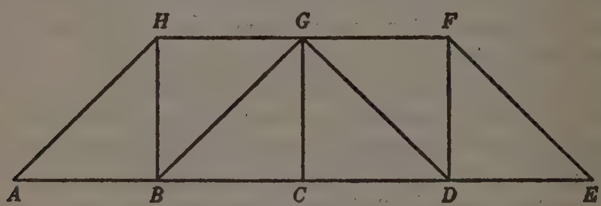


FIG. 83.

In order to make the axes of all members lie in one plane, and the truss be symmetrical with respect to that plane, some of the members must be made in parts or with forked ends. For example, see Fig. 84, which shows plan and elevation of joint D of the truss. Here four members are pinned together, one vertical V (double), one diagonal D (single), and two horizontals H_1 and H_2 (each double).

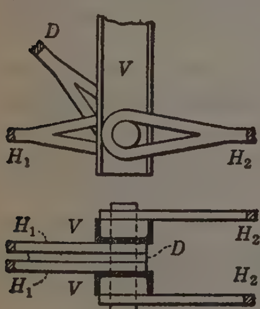


FIG. 84.

Trusses are used in roof construction, for bridges, and in general wherever loads must be carried over a wide span or at points remote from support. They are commonly constructed of wood, of steel, or of a combination of wood and steel. Wooden members are usually rectangular and are nailed or bolted together, or joined with specially designed metal connectors. Steel members are generally made up of "structural shapes" (angles or channels) and plates and are connected by pins as shown in Fig. 84, or riveted together.

Figure 85 represents a small bridge of two trusses and the following parts: five transverse floor beams supported at AA', BB', etc., by the trusses; many longitudinal stringers resting on the floor beams; and the floor (not shown)

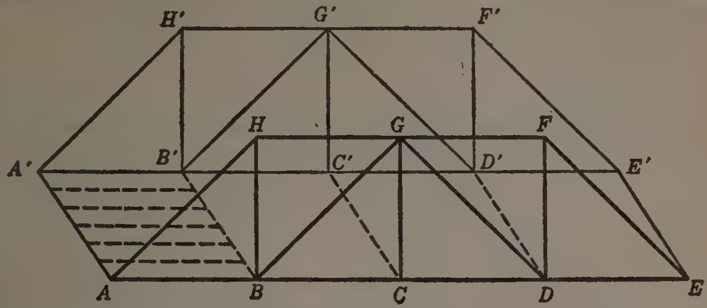


Fig. 85.

resting on the stringers. There is usually some transverse bracing, not shown in Fig. 85; its function is to resist wind effect and to keep the cross section of the bridge rectangular. In most large buildings, trusses are used to support the roof system, the parts of which (purlins, rafters, roof boards, etc.) correspond closely to the parts of the floor system of the bridge described.

Truss loads. The principal loads on trusses are their own weights, the weight of floor or roof system, the weight of snow, wind pressure, and for bridges the weight of traffic. In the simpler structures, at least, all weight loads come on the trusses at the joints. The weight of each truss member is actually applied at its center of gravity but is transmitted to the pins at its ends, and so is considered as coming on the truss at the joints.

48. Cases Here Considered. "Truss analysis" is a separate large field of study in engineering. The subject is touched upon in this book because its simplest problems afford suitable practice for beginners in statics. Incidentally, this practice gives a glimpse of this field of study.

We can consider only simple trusses and loadings described as follows: (i) the members of each truss are connected to one another at their ends only; (ii) they are connected by smooth pins perpendicular to the plane of the truss; (iii) the loads and reactions are applied to the pins only, and in such manner that the line of action of each cuts the axis of the pin to which it is applied; and (iv) the weight of each member, unless neglected, is replaced by two components (each one-half the weight) at the ends of the member. It follows from these assumptions that each member is subjected to only two forces (the pin pressures at its ends), and that these two forces, being in equilibrium, are equal and opposite and act along the axis of the member.

Many existing trusses are not simple in the sense defined above; a truss with riveted joints is an example. If small and relatively unimportant, such trusses were doubtless analyzed and designed as though they were simple. An impor-

tant nonsimple truss to be designed is first analyzed as though it were simple, and then the analysis is supplemented in reference to the special conditions that make it nonsimple.

49. Stress in a Member. As just explained in the preceding article, the forces exerted on any member are equal pulls (Fig. 86*a*) or equal pushes (Fig. 86*b*) directed along the axis of the member.

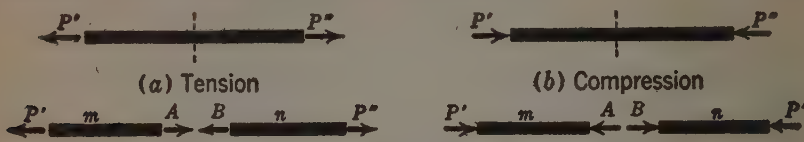


FIG. 86.

Any two parts of a member, as m and n, on opposite sides of a transverse section exert forces upon each other which are equal, opposite, and collinear (Art. 34). Let A denote the force exerted on m by n, and B the force exerted on n by m. Since m and n are at rest under the action of two forces only, it follows that A balances P' and B balances P''. Therefore, (i) A and B each equals the end pull or push; (ii) A and B act along the axis of the member; and (iii) A and B are pulls if the end forces are pulls and they are pushes if the end forces are pushes. It should be noted that results i, ii, and iii are independent of the relative lengths of m and n; that is, they hold for any transverse section.

By **stress in a member** of a truss we mean either of the forces (A or B) which two parts of the member on opposite sides of any transverse section exert upon each other. *Amount or magnitude of the stress* means the magnitude of either of the forces mentioned. Thus, to determine the amount of the stress in any particular member, one has only to determine in some way the magnitude of A or B. *Kind of stress* refers to the pull-or-push aspect of the stress. If A and B are pulls, then the stress is said to be *tensile;* if they are pushes, the stress is said to be *compressive.* To determine the kind of stress in a member, one has only to determine in some way whether A (or B) is a pull or push.

50. Determination of Stress; General Method. An explanation of the general method by which the stress in a truss member is determined will be more readily understood if preceded by simple illustrations.

1. Suppose that it is desired to find the stress in member BC of the truss shown in Fig. 87*a* due to the indicated loads. The truss is pinned at A to a shelf on a wall, and held at B by a horizontal link so that the supporting force there is horizontal. Imagine the members BC, BE, and AE cut so that the portion of the truss to the right of the line 1–1 is separated from the remainder of the structure. (This cutting off of a part of a truss in order clearly to isolate it is sometimes called " passing a section "; here we pass the section 1–1.) The portion of the truss thus isolated, cross ruled to emphasize the identity of

the body considered (Fig. 87b), is in equilibrium under the external forces that act on it. These external forces are the loads and the forces exerted on the cut

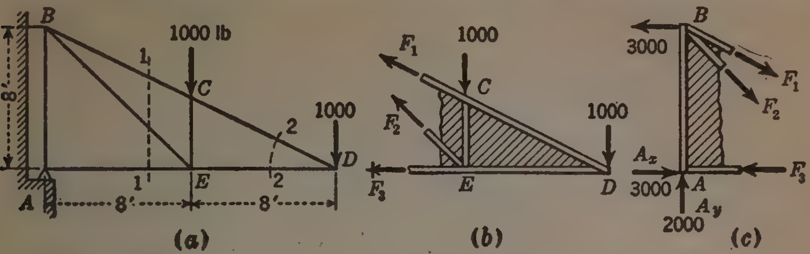


FIG. 87.

ends of the portions of BC, BE, and AE; they are the stresses in the respective members and so are axial. Let it be assumed* that all these unknown stresses are tensions, and let them be accordingly represented by the pulls F_1, F_2, and F_3; the figure then becomes a fbd for the isolated portion of the truss. The system of external forces acting is readily solved for F_1, thus:

$$\Sigma M_E = -(1000)(8) + (F_1)(3.58) = 0 \quad \text{whence } F_1 = +2230 \text{ lb}$$

The positive sign indicates that the assumed sense of F_1 is correct; therefore F_1 is a pull and the stress in BC is 2230 lb tension.

If the stresses in BE and AE also are desired they may readily be found by solving for F_2 and F_3, using the conditions

$$\Sigma F_x = 0 \qquad \text{and} \qquad \Sigma F_y = 0$$

These equations give $F_2 = +1410$ lb and $F_3 = -3000$ lb. The signs indicate that the assumed senses of F_2 and F_3 are respectively right and wrong. (The arrowhead for F_3 is now corrected as shown.) Therefore F_2 is a pull and F_3 a push, and so the stresses in BE and AE are respectively 1410 lb tension and 3000 lb compression.

In a similar manner the stresses in the three members BC, BE, and AE may be found from a fbd of the part of the truss to the left of section 1–1 (Fig. 87c). First, however, it is necessary to determine the values of the external supporting forces, A_x, A_y, and B, from a fbd of the entire truss (not shown). Solution for these three forces by the methods of Art. 40 gives $A_x = B = 3000$ lb, and $A_y = 2000$ lb. The force acting on the cut end of member AE may now be readily found, thus:

$$\Sigma M_B = (3000)(8) - (F_3)(8) = 0 \quad \text{whence } F_3 = 3000 \text{ lb}$$

Since the force F_3 is a push, the stress in AE is 3000 lb compression. The two

* The signs of the computed forces whose senses were assumed are interpreted as heretofore. See Art. 38, from which we quote, "*A positive sign indicates that the assumed sense is correct, and a negative sign that the assumed sense is wrong.*"

remaining forces F_1 and F_2 would usually be found by solving any two of the following equations: $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_B = 0$, $\Sigma M_A = 0$, and $\Sigma M_D = 0$. (Which two would you use?) The stresses in BE and BC are found to be 1410 lb tension and 2230 lb tension, respectively, equal to those previously found from Fig. 87*b*.

2. Suppose that it is required to find the stress in CD (Fig. 88). Obviously many sections may be passed that cut member CD, but not all will be satisfactory for the purpose of finding directly the stress in CD. Four sections, each cutting CD, are shown in Fig. 88. Figures 89*a*, *b*, and *c*, respectively, are fbd's of the portions of the truss to the right of sections 1, 2, and 3; and Fig. 89*d* is the fbd of the portion encircled by section 4. In each case the external forces are stresses (denoted by F's) and loads, if any. It is evident, by symmetry,

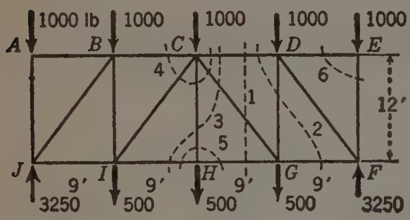


FIG. 88.

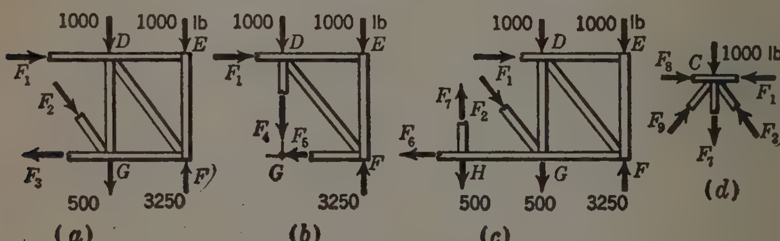


FIG. 89.

that the vertical reactions at F and J are each equal to 3250 lb. The forces F_1, F_2, and F_3 are readily found from Fig. 89*a*; thus

$$\Sigma M_G = +(3250)(9) - (1000)(9) - (12)(F_1) = 0 \quad \text{whence } F_1 = +1690 \text{ lb}$$
$$\Sigma F_y = -0.8F_2 - 1000 - 1000 - 500 + 3250 = 0 \quad \text{whence } F_2 = +938 \text{ lb}$$
$$\Sigma M_C = -(1000)(9) - (1000)(18) - (500)(9) - (12)(F_3) + (3250)(18) = 0$$
$$\text{whence } F_3 = +2250 \text{ lb}$$

The force F_1, and if desired the forces F_4 and F_5, may be found from Fig. 89*b*. $\Sigma M_G = 0$, $\Sigma F_y = 0$, and $\Sigma F_x = 0$, respectively, give $F_1 = +1690$, $F_4 = +1250$, and $F_5 = +1690$ lb.

It is not possible to obtain F_1 or any other indicated stress directly from Fig. 89*c* because there are four unknowns and only three independent equilibrium equations. If, however, the fbd of the portion encircled by section 5 were drawn, F_7 could be found. Then, with F_7 known, (*c*) could be solved for the three remaining unknowns. The force F_1 cannot be found directly from (*d*)

because the number of unknowns exceeds the number of independent equilibrium equations.

Queries. Draw fbd's of the portions of the truss encircled by sections 5 and 6. From these diagrams what can you say about the stress in members IH and HG; in members DE and EF?

From the above examples, it is seen that the problem of determining the stress in a truss member is solved essentially like any other problem in equilibrium, namely, by choosing a body for consideration, drawing the fbd, and applying appropriate conditions of equilibrium to the external forces. The body chosen for consideration is a portion of the truss cut off or isolated from the remainder by passing a section as in the examples given, and it must be such a portion that (i) the desired stress is one of the external forces acting, and (ii) the system of external forces acting is solvable for the desired stress.

The system of external forces acting on such an isolated portion of a truss consists of the reaction and loads (if any) on that part and the stresses exerted on that part by the other (adjoining) part. The loads will be regarded as completely given or known; the reactions can be computed by methods already explained. The lines of action of the stresses coincide with the axes of the respective members as proved in Art. 49; their magnitudes and "kinds" are generally unknown at the outset. In an algebraic solution of such an external system, it is necessary to assume senses (arrowheads) for the stresses. For the sake of practice you are advised to judge the kind of stress by inspection of the entire truss diagram or the part under consideration and to place arrowheads in the fbd accordingly (see Art. 52). For tension, the stress is a pull, and the arrow points away from the part; for compression the stress is a push, and the arrow points toward the part.

51. Stress Analysis of a Truss. By stress analysis of a truss is meant the determination of the amount and kind of stress in each member, caused by stated loads on the truss. The analysis might be accomplished by determining the stresses in any random order, but it is advantageous to proceed after some definite plan. Two similar plans are explained presently. Very often short cuts or special methods would be employed by a person experienced in truss analysis. For these the student is referred to special works on the subject.

1. JOINT-TO-JOINT PLAN. By joint of a truss we mean a pin and adjoining short parts of the members it connects. A joint is isolated by passing a section around the pin and cutting all members meeting there. Joint D of Fig. 87a is isolated by section 2–2 and is represented by the double lines of Fig. 90. The external forces acting on a joint consist of the load or reaction (if any) and the stresses (pushes or pulls). We call these the *forces at the joint*. The forces at joint D are the 1000-lb load, the stress F_4 (tension), and the stress F_5 (compression).

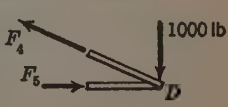


FIG. 90.

The forces at a joint are concurrent and coplanar. The system is solvable

only if it has at least one known force and not more than two unknowns. A joint at which the forces meet this requirement will be called a *solvable joint.* To analyze a truss on this plan (joint to joint), proceed as follows: (i) Look for a solvable joint. At the outset, there may be none, in which event determine one or all of the reactions of the supports of the truss; that done, a solvable joint can usually be found. (ii) Solve the system of forces at that joint for the unknowns. (iii) Repeat this procedure again and again until all or as many as possible of the stresses have been determined. (iv) If you cannot repeat in the manner just explained at all the joints, determine by the general method (Art. 50) one of the remaining unknown stresses. Then it will usually be possible to proceed " by joints " to the complete analysis. The computed stresses should be recorded on the truss diagram, the kind of stress being indicated by *t* for tension or *c* for compression.

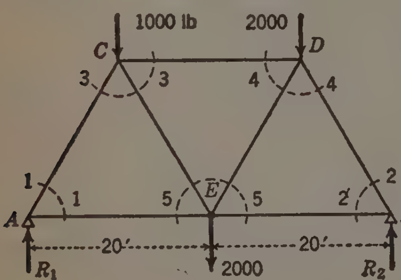


FIG. 91.

EXAMPLE 1. The truss of Fig. 91 is supported at its ends; the acute angles between members are 60°. It carries three loads as indicated. A stress analysis is required.

Solution. Since there is no solvable joint, the reactions at the ends are determined first; they are found to be $R_1 = 2250$ and $R_2 = 2750$ lb, respectively. Now joints A and B are solvable. Figure 92*a* is a fbd for joint A, the forces there being R_1 and the two unknown stresses F_1 and F_2. Apparently F_1 is tension and F_2 compression, and they are so indicated.

$$\Sigma F_y = 2250 - F_2 \sin 60 = 0 \quad \text{whence } F_2 = +2600 \text{ lb}$$

The positive sign shows that the sense of F_2 was correctly assumed; F_2 is a push; the stress in AC is a compression.

$$\Sigma F_x = -2600 \cos 60 + F_1 = 0 \quad \text{whence } F_1 = +1300 \text{ lb}$$

The positive sign shows that the sense of F_1 was correctly assumed; F_1 is a pull; the stress in AE is a tension.

Figure 92*b* is a fbd for joint B; F_3 and F_4 are shown as tension and compression, respectively,

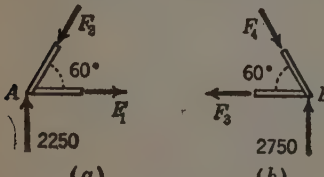


FIG. 92.

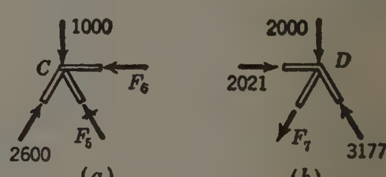


FIG. 93.

in view of the findings for F_1 and F_2. $\Sigma F_y = 0$ gives $F_4 = +3177$ lb; the kind of stress was correctly assumed. $\Sigma F_x = 0$ gives $F_3 = +1588$ lb; the kind of stress was correctly assumed.

Now all remaining joints are solvable. Joint C is considered next; Fig. 93*a* is the fbd. The stress in AC having been found to be 2600 lb compression, the force on the stub of that member is a push, as shown. The stresses F_5 and F_6 are assumed to be compressive.

$\Sigma F_y = 0$ gives $F_5 = -1444$ lb; the kind of stress was incorrectly assumed. $\Sigma F_x = 0$ gives $F_6 = +2022$ lb; the kind of stress was correctly assumed.

Joint D is considered next; Fig. 93*b* is the fbd. Since the known stresses are compressive, they are shown as pushes on the joint; the unknown stress F_7 is assumed to be tensile, in view of the kind of stress in CE. Either $\Sigma F_x = 0$ or $\Sigma F_y = 0$ furnishes $F_7 = +866$ lb; the stress *is* tensile.

Checking. All stresses have now been found, without having used ΣF_y for joint D, or ΣF_x or ΣF_y for joint E. The unused equilibrium equations may be employed for checks on the calculations already made. ΣF_y for the system on joint D and ΣF_x and ΣF_y for the system on joint E *should* equal zero. Actually, each sum is practically zero.

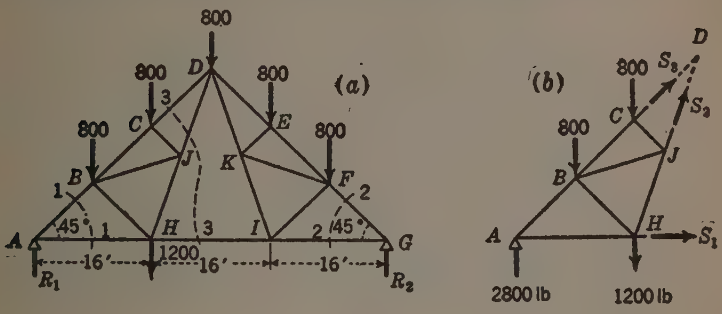


FIG. 94.

EXAMPLE 2. The truss shown in Fig. 94*a* is supported at each end and sustains six vertical loads as shown. The stress in each member is required.

Solution. At the outset there is no solvable joint. Hence, the end reactions are determined first. They are found to be $R_1 = 2800$ lb and $R_2 = 2400$ lb.

Now the end joints, A and G, are solvable. Solutions give, for the stress in AB, 3960 lb compression; for the stress in AH, 2800 lb tension; for the stress in GF, 3400 lb compression; and for the stress in GI, 2400 lb tension.

There is now no other solvable joint, and it is impossible to proceed further by the joint method. Obviously, if the stress in almost any one of the remaining members could be ascertained somehow, the solution by the joint method could be resumed. Thus if the stress in HI were known it would be possible to solve for the unknown stresses at joints H and I.

To determine the stress in HI the general method of Art. 50 is employed. Section 3 is passed, cutting HI and two other members as shown. The fbd for that part of the truss to the left of this section is shown in Fig. 94*b*; the external forces are the reaction at A, the three loads, and the stresses on the stub members, assumed tensile. $\Sigma M_D = 0$ gives $S_1 = +1600$ lb tension.

(Make a copy of Fig. 94*a*; then indicate thereon sections 4, 5, 6, etc., around the remaining joints in such order that each in turn becomes solvable as the analysis proceeds.

EXAMPLE 3. The truss represented in Fig. 95 is pinned to supports at its ends. (If the truss merely rested on the supports, the ends would slide outward and the truss would collapse.) The stresses S_1, S_2, etc., are required.

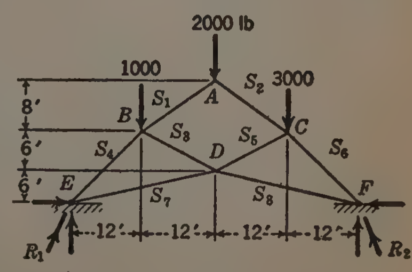


FIG. 95.

Solution, indicated only. Joint A is solvable. (If the vertical components of the end reactions be determined, from a fbd of the truss, then A is still the only solvable joint.) Suppose that S_1 and S_2 have been found; then joints B and C are solvable. Suppose that S_3, S_4, S_5, and S_6 have been found; then joint D is solvable. Suppose that S_7 and S_8 have been found; then joints E and F are solvable for the reactions there. Finally, for checking, the vertical components of the reactions might be recalculated from a fbd of the whole truss.

2. PIN-TO-PIN PLAN. In this plan the *pins* are considered in succession, and the forces exerted on each of them by the members it joins are found. From these forces, the stresses in the members are readily inferred. The calculations are identical with those made in the joint-to-joint plan; the only difference in the plans is in the selection of the body, the pin instead of the joint.

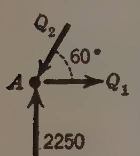


FIG. 96.

To illustrate this method we make a partial solution of Ex. 1. See Fig. 91, and recall that the left reaction is 2250 lb. Figure 96 is a fbd for the pin A. Q_1 and Q_2, respectively, denote the forces exerted on A by members AE and AC. The equilibrium equations for finding Q_1 and Q_2 are just like the preceding equations for finding F_1 and F_2, except that Q_1 replaces F_1 and Q_2 replaces F_2. Having found $Q_1 = +1300$ lb and $Q_2 = +2600$ lb, you know that Q_1 is a pull and Q_2 is a push, hence that *the members exerting these forces* are, respectively, in tension (1300 lb) and compression (2600 lb).

To save time, an experienced computer would not draw the pin free but would use the pin " as is " in the truss diagram, and there represent the forces acting on it (see Fig. 97). After he had calculated Q_1 and Q_2 he would record their values on the members that exert these forces, and place arrowheads on the other ends of AE and AC as shown to indicate the forces these members respectively exert *on pins E and C*. He would use these arrowheads when considering pins E and C for the forces exerted on them.

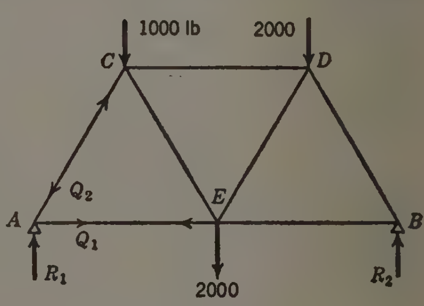


FIG. 97.

If you prefer the pin-to-pin plan, and wish to dispense with detached fbd's, do not fail to keep in mind that you are dealing with forces exerted on the pins, and note that *the two arrowheads on a tension member point toward each other, and the two on a compression member point away from each other.* Thus the arrowheads serve to indicate the kind of stress. This device is quite commonly employed.

52. Solution by Inspection. After a certain amount of practice in the analysis of trusses you will find that sometimes the stress in members can be determined simply by inspection, and that often the kind of stress and the

relative magnitude of different stresses can be so determined. Three illustrations follow.

1. The truss of Fig. 98 supports five loads P. Passing section 1–1 and considering the left portion of the truss, you will see that there are only two vertical forces acting on it, namely, the reaction $2.5P$ up and the stress in BL, which must then be equal and opposite to the reaction. Therefore the stress in BL is $2.5P$ tension. In the same way on passing section 2–2 the stress in CK is seen to be $1.5P$ tension, while the stresses in CD and KL are seen to be equal and of opposite kinds, since they are the only horizontal forces acting on that part of the truss to the left of 2–2. If the joint at A is isolated it is seen that the vertical component of the stress in AB balances the upward reaction and that therefore the stress in AB is compression and is greater than $2.5P$.

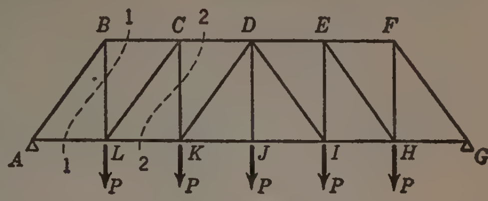


FIG. 98.

2. For the truss of Fig. 99a, which supports two loads P, it is evident that at joint A the vertical component of the stress in AB balances P, and that therefore this stress is tension and is greater than P. At joint B it is evident that the stresses in AB and BC are the only forces with vertical components, that these vertical components must therefore be equal and opposite, and that, as the members have the same inclination to the horizontal, their stresses must be equal in magnitude and opposite in kind.

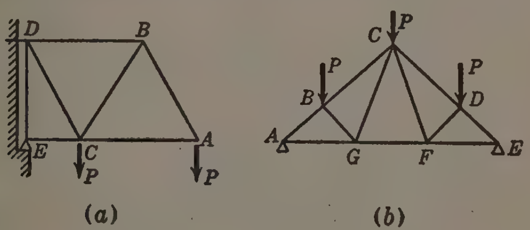


FIG. 99.

3. For the truss of Fig. 99b, which supports three loads P, you may imagine the joint B isolated and forces summed up along BG (perpendicular to ABC). Evidently the stress in BG is compression and, being equal to the component of P perpendicular to ABC, is less than P. Again, if the joint at G is isolated and forces are summed up along the vertical, the stress in GC is seen to be tension, an upward pull balancing the downward push of BG. Since the vertical components of the stresses in these two members are equal, the more nearly horizontal member will have the greater stress.

Such analysis by inspection develops an understanding of truss action and is

of great value as a means of checking the results of a formal solution. You are advised to practice it, comparing the results so obtained with those found by a complete analysis.

53. Graphical Stress Analysis. The general schemes of choosing free bodies, etc., described in Arts. 50 and 51 apply here. Now, however, the systems of forces of the fbd's are to be solved graphically; and that is possible if they can be solved algebraically. In the joint or pin plan, the force systems would be solved by drawing a polygon for each. When making a polygon, it is advantageous, as you will see in Art. 54, to represent the forces in the order, clockwise or counterclockwise, in which they appear about the fbd. A polygon so drawn is called a *polygon for the joint or pin.*

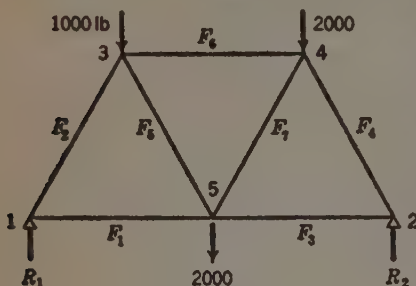


FIG. 100.

For illustration, we take Ex. 1 of Art. 51; Fig. 100 is like Fig. 91 but with different notation; Fig 101*a* is like Fig. 92*a* but without arrowheads on F_1 and F_2. *ABCA* (Fig. 101*b*) is a polygon for joint 1. The arrowhead on *AB*, which is known in advance, *fixes* the arrowheads on F_1 and F_2 as indicated. Hence, F_1 is a pull on the joint, and the stress in member 1–5 is tensile; F_2 is a push on the joint, and the stress in member 1–3 is compressive. In a similar way, polygons for joints 2, 3, and 4 or 5, respectively, determine F_3 and F_4, F_5 and F_6, and F_7.

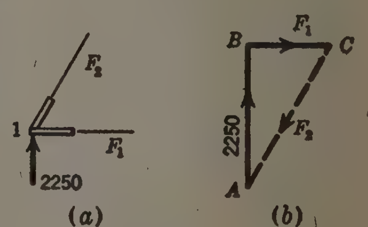


FIG. 101.

You should make these additional polygons from a good copy of the truss diagram, all on a suitable sheet of paper. (We suggest a sheet not smaller than 8 by 10 in. and scales about 8 ft and 1000 lb to the inch.) Record the amount and kind of each stress you find on the corresponding member of the truss. The amounts will not differ more than 3 per cent from the exact amounts found in Art. 51 if your drawing is correct and accurate.

A check is afforded, you should note, by the last polygon you draw, for, when you come up to joint 4 or 5 you know all the forces there except F_7; three sides of the force polygon are already known or determined. The fourth or closing side *should* be parallel to member 4–5. If you do not find it so, or nearly so, then there is a gross error somewhere or your drawing is inaccurate.

You saw in Ex. 2, Art. 51, that the joint plan failed at a certain stage of the analysis, and that the cause of the failure was removed by finding a stress or force belonging to a nonconcurrent system. That system *could* be solved

graphically, of course, and so the entire solution could be made graphically if desired.

54. The Stress Diagram. When a separate force polygon is drawn for each joint of the truss, as in the preceding article, each stress is represented twice. It is possible so to combine the force polygons for all the joints that each external force (load or reaction) and each stress is represented only once, thus reducing labor. Such a combination of force polygons is called a **stress diagram.** To construct a stress diagram for a truss it is necessary to follow a systematic procedure, for which directions with explanatory comments will now be given.

1. Make a scale drawing of the truss, and represent thereon all external forces. Number the joints; each joint can then be designated by its number and each member by the numbers denoting the joints it connects. This drawing we call the truss diagram (Fig. 102*a*).

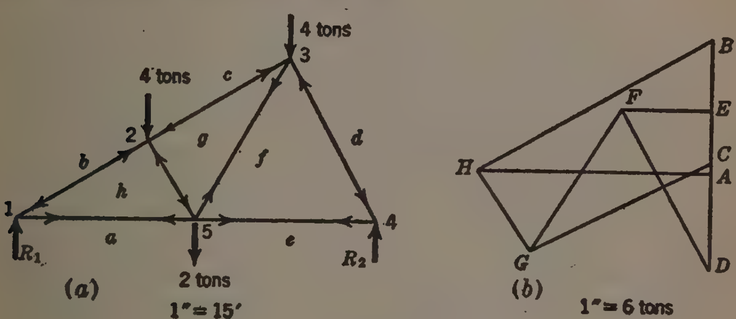


FIG. 102.

2. Mark each triangular space in the truss diagram and each space between consecutive lines of action of external forces with a lower-case letter. Each stress can now be designated by the two letters on opposite sides of the corresponding member, and each external force by the two letters on opposite sides of its line of action, in accordance with Bow's notation. (See Art. 28.) Each member also can be designated by the letters that denote the stress in it.

3. Construct a force polygon for all the external forces (loads and reactions), taking these forces in clockwise* order around the truss. (In Fig. 102*a* the order would be reaction *AB*, load *BC*, load *CD*, etc., the force polygon being started with any one of these forces.) Using capital letters, mark the tail of each vector with the letter that appears on the counterclockwise side of the corresponding line of action, and mark the tip with the letter that appears on

* The order may be either clockwise or counterclockwise, as long as it is the same for all polygons. In the directions here given, the word "counterclockwise" may be substituted throughout for the word "clockwise."

the clockwise side. (In Fig. 102*b*, *AB* is the vector for the reaction at 1, whose line of action is *ab*. This force polygon for the external forces is, of course, always a closed one, and when all the forces are vertical, as here, it is a straight line. We urge you to sketch the force polygon just described on a separate sheet of paper, and also to sketch the force polygons for the various joints which are described below. In so doing you will see the stress diagram develop or grow, and will understand it more clearly.)

4. Select any solvable joint (where there are at least one known force and not more than two unknown stresses), and construct the force polygon for that joint, taking the forces in clockwise order around the joint, known forces first. (In Fig. 102*a*, joint 1 was selected and the force polygon was started with vector *AB*. The next force in clockwise order is the stress in member 1–2, represented by vector *BH*, and the next is the stress in member 1–5, represented by the vector *HA*, which closes the polygon. The sense of *BH* is down and left, hence member 1–2 *pushes* on the joint and is in *compression;* the sense of *HA* is to the right, hence member 1–5 *pulls* on the joint and is in *tension.*) Put an arrowhead on each member near the joint to show the direction of the force the member exerts thereon, and another arrowhead, pointing in the opposite direction, near the other end of the member to show the direction of the force it exerts on the joint there. (This is a convenient way to record the kind of stress; arrowheads pointing towards the joints a member connects indicate compression, and arrowheads pointing away from the joints indicate tension, as explained in Art. 51.)

5. Select any other solvable joint and repeat the process described under paragraph 4, continuing in this way until all the stresses have been determined. Always take the forces in clockwise order around the joint, and always start with the known forces. (In Fig. 102*a*, joint 2 was taken next, and the polygon was started with vector *HB* and continued with *BC*, *CG*, and *GH* in sequence. You will find that, by " going around " this polygon, following the vectors, it is easy to ascertain that the sense of each force is as indicated by the arrowheads shown at joint 2. Joint 3 was taken next; the polygon was started with vector *GC* and continued with *CD*, *DF*, and *FG*. Joint 5 was taken next. There are five forces, namely, the load and four stresses, one of which, that in *fe*, was unknown at this stage. The polygon for the four known forces is *EAHGF*; *FE* closed the polygon for the five forces and represents the stress in *fe*. To represent this stress correctly, *FE* must be parallel to *fe*, and, had it not been so parallel, we would say that the stress diagram had failed to check or *close*. Like the polygon for the external forces on the whole truss, the stress diagram should close, and to some extent the accuracy of the solution can be judged by how nearly it does so. The drawing of line *FE* completed the solution and incidentally the polygon for joint 4.)

6. Scale the vectors in the stress diagram to determine the magnitude of the stresses, and record the values on the corresponding members in the truss dia-

gram. (It is advantageous to postpone the scaling of the vectors until after completion of the stress diagram.)

It should be noted that in a solution such as that outlined above the order in which the joints are selected is immaterial; in Fig. 102*a*, joint 4 could have been taken first, then either 3 or 1, then either 2 or 5. But it is essential that at each joint the forces be taken in clockwise order.

As explained in Art. 51, it is sometimes impossible to analyze a truss completely by the joint-to-joint plan. When this plan is not possible, the graphical method must be supplemented by the general method of Art. 50. When one or more of the unknown stresses have thus been found, the graphical solution can usually be resumed.

In some exceptional trusses the reactions cannot be determined in advance but must be found from the stresses. If such a truss is statically determinate it can be solved graphically, the stress diagram being built up on the incomplete polygon for the known external forces and each reaction being finally determined from the stresses that balance it. The truss of Fig. 95 is an example. The algebraic solution is given in Ex. 3 of Art. 51; for a graphical solution the joints would be taken in the same order: A, B, C, D. The polygon for joint E would then give the reaction R_1, and the polygon for joint F would give the reaction R_2. You should carry out the solution as described.

B. Simple Frameworks; Crane Type

55. Description. Like the trusses which have been discussed, the frames here considered are plane and symmetrical with respect to the plane of the frame. Again, like the trusses, they are assumed to be made of members connected by pins; thus each pin pressure lies in the plane of the frame, and its line of action cuts the axis of the pin. Unlike the trusses, these frames may include members which are pinned to others at more than two points, and the loads are applied anywhere, not necessarily at a pin or joint. The result of these conditions is that some or all the members are subject not merely to tension or compression, but also to bending stress, and if such members are cut the forces on the cut ends are not simple axial pulls or pushes. Because of this we do not attempt to determine the stresses in these members but limit the solution to a determination of the forces (pin pressures, reactions of supports, etc.) that act on them. For the same reason, we do not pass sections or cut members as in truss analysis. But, when a two-force member does occur in a crane, we make use of the fact that the forces on its ends are axial, and we do determine the stress in it.

Many existing cranes have riveted joints. If relatively small and unimportant, they were analyzed and designed as though pin connected. An important riveted crane to be designed would be first analyzed as though pin connected and then studied in reference to effects due to stiffness of its riveted joints.

56. Force Analysis of a Crane. A force analysis consists in the determination of every force (magnitude and direction) acting on each part or member, due to the weight of the crane or to the applied loads or both. Solution is effected by regarding the whole crane or parts of it as bodies in equilibrium and applying the appropriate conditions of equilibrium to the external forces acting thereon. There is opportunity for the exercise of judgment in deciding on the sequence in which the parts are to be considered. Usually it will be found advantageous to consider first the crane as a whole, proceeding then to a consideration of single members or groups of members.

It must be kept in mind that the members of a crane are not, in general, two-force members as in a truss, and that therefore the pressure of a pin on a member does not, in general, act along the axis of the member. Usually, its direction is unknown, and you will find it convenient in an algebraic solution to represent the pressure by rectangular components with senses assumed. Remember that at a pinned joint the forces exerted on the members there connected are exerted *by the pin*, and not by the members on each other. At a joint where only two members are pinned together it is permissible to regard the forces as exerted directly by the members on each other, because the pin simply transmits the pressure from one to the other. But at a joint where more than two members are pinned together the force on each member should be regarded as exerted by the pin, and the pin itself should be considered a body in equilibrium under the action of the forces exerted on it by the members it connects.

After a crane has been analyzed, the results of the analysis should be presented by showing, on a new fbd of each individual member, the forces that act on that member, with their values.

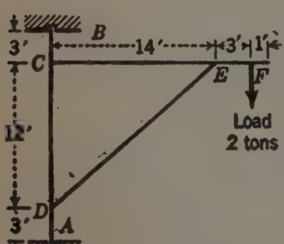


FIG. 103.

EXAMPLE 1. The crane represented in Fig. 103 is supported at A and B in sockets in the floor and ceiling. A force analysis of the crane for the load indicated is required.

Solution. The crane as a whole is first considered. Figure 104*a* is the fbd; the external forces are the load, the reaction B (horizontal) at the ceiling, and the reaction at the floor represented by its horizontal and vertical component A_x and A_y. (The senses of the reactions are assumed.) This system is solvable for all unknowns. $\Sigma M_A = 0$ gives $B = 1.89$, $\Sigma F_y = 0$ gives $A_y = 2.0$, and by inspection it is seen that $A_x = 1.89$ tons.

Of the several parts of the crane, either the post or the boom should be considered next, because no force acting on the brace is completely known. Since the brace is a two-force member, the forces on its ends (obviously pushes) act along the member, and the forces it exerts on the post and boom act in that same line. Thus the external forces on the boom (Fig. 104*b*) are the push at E, the load 2 tons, and the pressure at C represented by its horizontal and vertical components (senses assumed). This system is solvable for all unknowns. For that purpose, the angle α and the arm CF are utilized. $\alpha = \tan^{-1} \frac{12}{14} = 40.6°$, and $CF = 14 \sin 40.6 = 9.11$ ft. $\Sigma M_C = 0$ gives $E = 3.73$; $\Sigma F_x = 0$ gives $C_x = 2.84$; $\Sigma F_y = 0$ gives $C_y = -0.43$ tons. The negative sign means that C_y acts not as assumed but downward.

Checking. All forces on the post and brace are now known; those on the post are represented in Fig. 105. ΣF_x, ΣF_y, and ΣM for the post are practically zero, and so all's well.

EXAMPLE 2. A force analysis for the crane of Ex. 1, taking into account the weights of the members, is required. The post weighs 0.8, the boom 0.9, and the brace 1.1 tons.

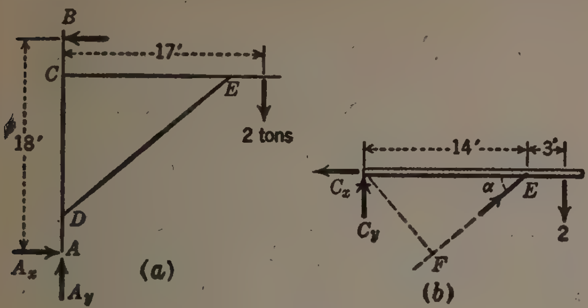


FIG. 104.

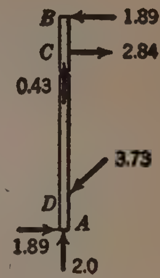


FIG. 105.

Solution. The entire structure is first considered. Figure 106*a* is the fbd. The unknowns are found to be

$$A_x = 2.77 \qquad A_y = 4.8 \qquad \text{and} \qquad B = 2.77 \text{ tons}$$

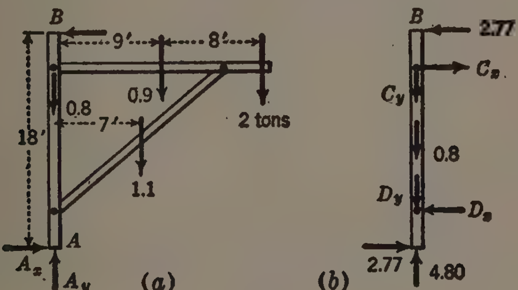


FIG. 106.

The post is considered next. Figure 106*b* is a fbd; the external forces are the weight 0.8 ton, the known forces at A and B, the unknown force at C represented by its components C_x and C_y, and the unknown force at D represented by its components D_x and D_y. Only two of these four unknowns can be determined from this system; thus

$$\Sigma M_D = 0 \text{ gives } C_x = 4.15 \qquad \Sigma F_x = 0 \text{ gives } D_x = 4.15$$
$$\Sigma F_y = 0 \text{ gives } C_y + D_y = 4.00 \text{ tons}$$

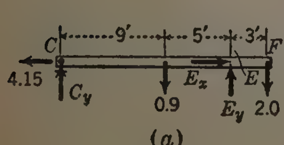


FIG. 107.

C_y is found from the fbd of the boom (Fig. 107*a*). The external forces are the weight 0.9 ton, the load 2, $C_x = 4.15$ (toward the left), C_y, and the force E exerted by the brace, represented by its components E_x and E_y

$\Sigma M_E = 0$ gives $C_y = -0.11$; hence $D_y = 4.11$ tons. $\Sigma M_C = 0$, or $\Sigma F_y = 0$, gives $E_y = 3.01$; obviously $E_x = C_x = 4.15$ tons.

Checking. All forces on the brace are now known; they are represented in Fig. 107*b*. Since ΣF_x, ΣF_y, and ΣM equal zero, the calculations are correct.

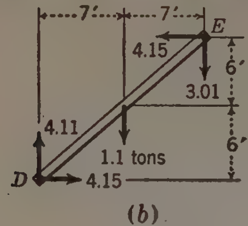


FIG. 107.

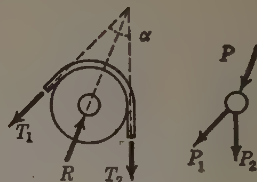


FIG. 108.

57. Force Analysis, Continued. We now consider a crane equipped with hoisting appliances and one involving noncoplanar forces. For simplicity we assume that hoisting ropes are without stiffness and that the pins supporting the pulleys are without friction. Let Fig. 108 represent a piece of such a rope on a pulley supported by such a pin. The tensions T_1 and T_2 act along the axis of the rope, as indicated, and the reaction R of the pin acts through the axis of the pin. Moreover, the three forces are concurrent; and from symmetry it follows that R bisects the angle α between T_1 and T_2, and T_1 equals T_2. Denoting these equal forces by T, you see that

$$R = 2T \cos \tfrac{1}{2}\alpha$$

If $\alpha = 90°$, a common value, $R = 1.414T$. (For a discussion of tackle hoists with stiff ropes and pin friction, see Art. 68.)

The pressure P which the pulley exerts on the pin is equal and opposite to R. P_1 and P_2 denote components of P parallel to T_1 and T_2; obviously, $P_1 = P_2 = T$.

EXAMPLE 1. The crane represented in Fig. 109*a* rests in a shallow socket at A in a floor and against a side of a hole at B in the ceiling of a room. The crane is quite like the one of Ex. 1, Art. 56, except that it is equipped with some simple hoisting gear as indicated. Moreover, the principal dimensions of, and the loadings on, the cranes are the same. The hoisting

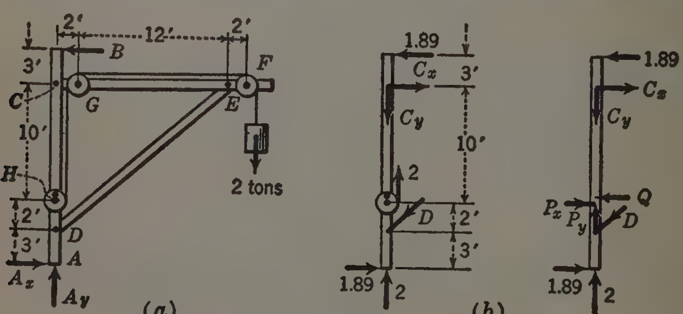


FIG. 109.

drum is operated by means of hand cranks not shown, and it is locked, when the load is suspended, by means of a pin passing through a hole in the post AB and one in the drum. The

center-to-center distance between this pin and the axis of the drum is 6 in., and the diameter of drum and sheaves is 2 ft. (A real hoist with adequate mechanical advantage would be less simple than the one shown and described.) All forces acting on the three main members of the crane, for a load of 2.0 tons, are required.

Solution. Omitting the weights of the members, the external forces on the whole crane are the weight of the load, 2.0 tons (acting through the center of gravity of the load, of course), the reaction at B (horizontal), and the reactions at A (horizontal and vertical). The system is just like the external system for the crane of Ex. 1, Art. 56. Hence the reactions here as there are

$$A_x = 1.89 \qquad A_y = 2.0 \qquad B = 1.89 \text{ tons}$$

Figure 109*b* shows two fbd's. In the first, the body consists of the post, the drum, and the chain wound upon it. The external system consists of the supporting forces at A and B, the 2.0-ton pull on the stub end of the chain, and the forces at C and D. The force at D acts along the brace, since it is a two-force member; the force at C is represented by its components C_x and C_y. The three unknowns can be found from this fbd. The values are

$$C_x = 3.00 \qquad D = 3.95 \qquad C_y = 1.43 \text{ tons}$$

(These forces might have been calculated from the second fbd, for the bare post; the external system consists of A_x, A_y, B, C_x, C_y, D, the force P of the pin H, and the force Q of the locking pin. From a fbd of the drum with its chain it is readily seen that $Q = 4$ horizontal, $P_x = 4$, and $P_y = 2.0$ tons in directions shown. Now C_x, C_y, and D can be calculated.)

Checking. Figure 110 shows two fbd's either of which affords opportunity for checking calculations already made. In the first, the free body consists of the boom, the two sheaves, and part of the chain as shown. The second free body is the boom; the pin pressures F and G are represented by their horizontal and vertical components.

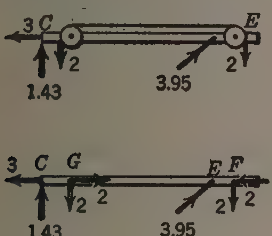


FIG. 110.

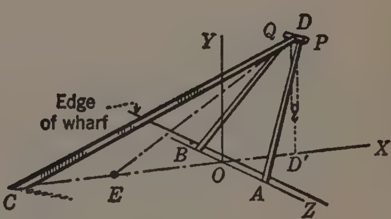


FIG. 111.

EXAMPLE 2. Figure 111 represents a shear leg wharf crane. It consists of two front legs AP and BQ, and a back leg CD, all connected at their tops as by a horizontal pin. The front legs are supported by pins at the floor of the wharf so that the legs can be rotated about the line AB. The back leg is restrained at the floor by a holding-down rail and a long horizontal screw which works in a nut in the lower end C. The purpose of the screw is to move C along OE (horizontal and perpendicular to AB at its midpoint O), thus rotating the front legs about AB and moving the load toward or away from the wharf as desired. The following data are assumed: length of front legs 160 ft, distance between their lower ends 50 ft, distance between their upper ends 10 ft, length of back leg 210 ft, weight of each front leg 44 tons, weight of back leg 53 tons. It is required to determine the pressures on the ends of the legs due to their own weights when the crane is in its position of greatest overhang, $OD' = 64$ ft.

Solution. Chosen coordinate axes are indicated in Fig. 111. The external forces on the crane are the weights of the legs and the supporting forces or reactions at A, B, and C which we

denote by the same letters. (Weight of the pin at D and that of the hoisting tackle are disregarded.) Presumably A and B have x, y, and z components; we denote them in the customary manner, that is, by A_x, A_y, A_z, etc. Obviously, C has no z component.

Figure 112a is a side view or projection of the crane on the xy plane and of the vectors of the external system. This projected system is in equilibrium (see Art. 43), and it may be solved for C_y and $(A_y + B_y)$ in the usual way.

$$\Sigma M_0 = 0 \text{ gives } C_y = 25 \text{ tons}$$
$$\Sigma F_y = 0 \text{ gives } (A_y + B_y) = 166 \text{ tons}$$
$$\Sigma F_x = 0 \text{ gives } (A_x + B_x) = C_x$$

It is obvious from the symmetry of the crane that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$. (These equalities can be arrived at from a consideration of a plan view and a front view of the crane and of the vectors representing the external forces. The plan would show that $A_x = B_x$ and $A_z = B_z$; the elevation would show that $A_y = B_y$ and $A_z = B_z$.) Hence $A_y = B_y = 83$,

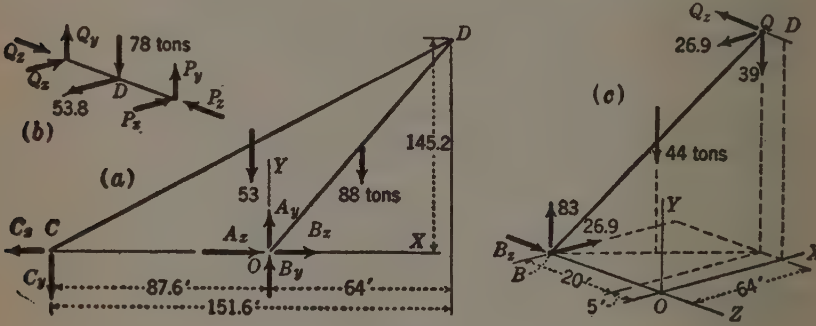


FIG. 112.

and $A_x = B_x = \frac{1}{2}C_x$. A fbd of the back leg would show readily that $D_y = 78$ and $C_x = 53.8$. Hence A_x and $B_x = 26.9$ tons.

Figure 112b is a fbd of the top pin. The external system consists of the known forces D_x and D_y exerted by the back leg, and the unknown forces P and Q exerted by the two front legs. P and Q are represented by their axial components. It follows readily from the figure that

$$P_x = Q_x = 26.9 \qquad P_y = Q_y = 39 \qquad P_z = Q_z$$

The remaining unknowns, A_z or B_z and P_z or Q_z, are found from Fig. 112c, a fbd of front leg BQ (in perspective). The only unknown external forces are B_z and Q_z. The moment of the external system about the vertical through Q is

$$B_z \times 64 - 26.9 \times 20 = 0 \quad \text{whence } B_z = 8.41 \text{ tons}$$

also A_z, Q_z, and $P_z = 8.41$ tons.

58. Statically Indeterminate Trusses and Cranes. Statically indeterminate structures are mentioned and defined in Art. 44. The truss of Fig. 113a is an example. It has one redundant member. If 1–5 or 3–6 were omitted, the truss would still be complete; if both were omitted, the truss would collapse under any unsymmetrical loading.* This incomplete truss could be made to

* We are assuming that the truss is pin connected and that each member can take either tension or compression. Some trusses are made so that one or more members can take only tension or only compression.

serve if its supports were such that they could furnish not only vertical but also horizontal reactions.

The crane of Fig. 113*b* is another example. The post 1–3 is a single rigid member; so is the boom 2–4; they are pinned together at 5. If members 1–2 and 2–3 were omitted, the frame would still be complete and statically determinate. Omission of 3–4 and 1–2, or 3–4 and 2–3, would leave a complete and statically determinate frame, but one less stiff than the original one.

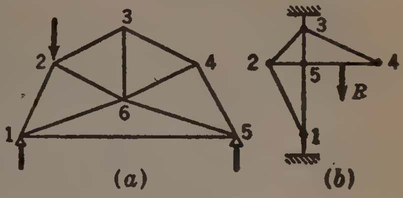


FIG. 113.

How to ascertain whether a given structure is incomplete (unstable), is just complete (stable and statically determinate), or has redundant members (statically indeterminate) is discussed fully in works on structures. Your engineering judgment might enable you to tell; if in doubt you should attempt a stress analysis. If the truss is incomplete, you will sooner or later arrive at a force system that cannot be in equilibrium; if the truss is indeterminate you will reach an impasse, finding it impossible to continue the solution because of too many unknowns.

Addition of a redundant support or member to a statically determinate structure stiffens and usually, but not always, strengthens the structure. Thus, if to a beam on end supports there is added a central support, the beam under the same load will have a greater factor of safety and will deflect less than before; and similarly the structure represented in Fig. 81 is stronger and stiffer than it would be without the cord *CD*, say.

CHAPTER V

FRICTION

A. General

59. Preliminary. A simple example of the resistance which a body encounters when it slides or tends to slide over another is represented in Fig. 114*a*. P is a force applied to A which makes A slide or tend to slide over B; R is the reaction, obviously inclined somewhat as shown, that B exerts on A; W is the weight of A. For purposes of reasoning, especially in less simple problems, it is often convenient to deal with components of R in and normal to the surface on which R acts, as shown (Fig. 114*b*). The first component of the reaction R is called **friction**; the second is called **normal pressure.** (Of course, A exerts on B a force R' equal and opposite to R, and R' has components F' and N' equal and opposite to F and N, respectively.)

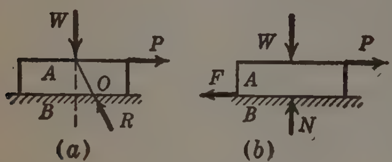


FIG. 114.

In Fig. 114*a*, R is represented as a concentrated force. Really R is distributed over the bottom of A. It may be looked upon as the resultant of the reactions on all elementary parts of the bottom. The point where R pierces the bottom is called the " center " of the pressure R. This point O is to the right of the line of action of W since R, W, and P are concurrent.

Friction between two bodies is called **static friction** if slipping does not occur, and **kinetic friction** if slipping does occur. The amount or magnitude of static friction between two bodies depends on the *degree* of the tendency to slip. For example, suppose that P of Fig. 114*b* is increased gradually from zero to a value that will just move A. During this time the external forces on A are balanced, and therefore F equals P at each instant. Thus, static friction is a passive force, which comes into action only to prevent a slipping that other forces tend to cause, and which is, in any given case, *only* so large as may be necessary to prevent that slipping. It increases as the tendency to slip increases, having its greatest value when slipping impends. Likewise, the inclination of the total reaction to the normal pressure increases as the tendency to slip increases, having its greatest value when slipping impends.

EXAMPLE 1. A and B (Fig. 115*a*) weigh 100 and 200 lb, respectively. The surfaces of A and B in contact are very rough; those of B and C in contact are very smooth, practically frictionless. The applied pull P does not cause slipping. Fbd's of A, of B, and of A and B

are required. Figure 115*b* is a fbd of A and B. The external forces are: the weight of A and B 300 lb, the pull P 40 lb, pull Q of the rope, and the reaction N of C which is wholly normal. It is plain that $Q = 40$ lb and $N = 300$ lb. Figure 115*c* is a fbd of A. The external forces are the weight of A 100 lb, Q 40 lb, and the reaction of B represented by N_1 and F_1. It is plain that $N_1 = 100$ and $F_1 = 40$ lb. Figure 115*d* is a fbd of B. The external forces are the weight of B 200 lb, the applied pull 40 lb, the reaction of C 300 lb, and the reaction of A represented by its components 100 and 40 lb.*

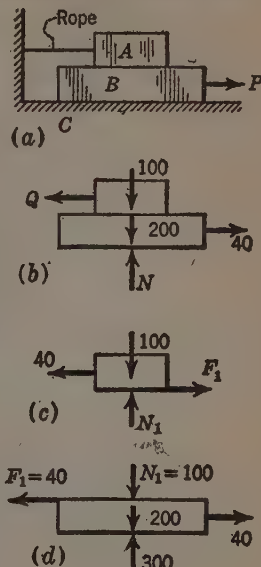


FIG. 115.

It is apparent that the applied pull might have such a magnitude and line of action as to cause the blocks to tip or overturn, but it will be assumed in all examples of this type that neither tipping nor overturning occurs. This assumption renders the use of moment equations unnecessary, and so the positions of the lines of action of the forces need not be further considered.

EXAMPLE 2. A (Fig. 116*a*) does not move; the normal pressure between A and B is required. Obviously, the normal pressure is less than W and $N = W - P \sin \theta$.

If the sense of P is reversed, the normal pressure is greater than W and $N = W + P \sin \theta$.

EXAMPLE 3. A (Fig. 116*b*) does not move; the normal pressure between A and B is required. From an imagined fbd you see, with your mind's eye, that $N = W \cos \alpha - P \sin \theta$. If the sense of P is reversed, $N = W \cos \alpha + P \sin \theta$.

EXAMPLE 4. Refer to Fig. 116*b*. Given $\alpha = 30°$, $W = 100$ lb, $P = 80$ lb, and $\theta = 40°$; A and B are rough so that A does not move. Make a fbd for A, assuming that F acts upward, implying that A tends to move downward. Find F, N, and R.

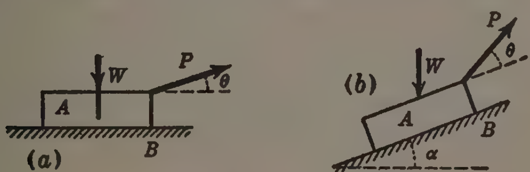


FIG. 116.

Solution. For x and y axes respectively parallel and perpendicular to the inclined plane;

$$\Sigma F_x = +80 \times \cos 40 - 100 \times \sin 30 + F = 0$$
$$\Sigma F_y = +80 \times \sin 40 - 100 \times \cos 30 + N = 0$$

The first equation gives $F = -11.28$ lb; the assumption made is wrong. The second equation gives $N = +35.16$ lb. Hence $R = 36.9$ lb.

60. Impending Slip. The state or condition when slipping impends is important practically because the friction then has a maximum value and the

* In Fig. 115 no attempt was made to represent the lines of action of the normal pressures in their correct positions because the solution of Ex. 1 does not depend on those positions. These circumstances obtain in many examples and problems where conditions $\Sigma F_x = 0$ and $\Sigma F_y = 0$ furnish a solution.

(total) reaction a maximum inclination to the normal. Several special terms which pertain to this particular state or condition will now be defined.

Limiting friction is a name sometimes given to the friction of impending slip. We denote it by F_m because it is a maximum value (Fig. 117*a*). The

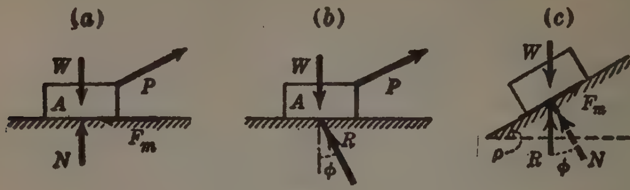


FIG. 117.

coefficient of static friction for two surfaces is the ratio of the limiting friction to the corresponding normal pressure. We denote it by μ; then

$$\mu = F_m \div N \qquad \text{or} \qquad F_m = \mu N \qquad \text{also} \qquad F \not> \mu N \tag{1}$$

The numerical value of the coefficient of static friction between two given surfaces depends upon several different factors, discussed in Art. 61.

The **angle of friction** for two surfaces is the angle between the directions of the normal pressure and the total reaction when slipping impends. We denote it by ϕ (see Fig. 117*b*); then, since R may be looked on as the resultant of F_m and N, $\tan \phi = F_m/N$, and so

$$\mu = \tan \phi \qquad \text{and} \qquad \phi = \tan^{-1}\mu \tag{2}$$

The **angle of repose** for a body on an inclined plane is that inclination of the plane at which slipping impends.* Our symbol for this angle is ρ (Fig. 117*c*). The reaction R of the plane on the body is vertical and is so indicated in the figure. Because slipping impends, the angle between R and the normal is ϕ, and is so indicated. It is obvious from the geometry of the figure that

$$\rho = \phi \tag{3}$$

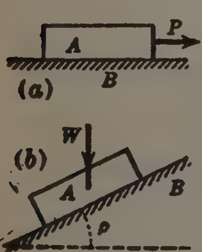


FIG. 118.

61. Coefficient of Static Friction. The coefficient of static friction for two bodies A and B may be found in several ways: (i) Place A on B as in Fig. 118*a*, and measure the horizontal pull P which will just start A; then μ equals P divided by the weight of A. Or (ii) tilt B as in Fig. 118*b* and measure the least angle of tilt at which gravity will start A down; μ equals the tangent of that angle. In either method several repeated determinations are likely to give discordant results because

* This term is also used for the angle of inclination of the side of a heap of granular material (grain, sand, etc.) if the material there is just at the point of sliding or running down. Obviously this angle of repose is akin to the one defined above.

it is difficult to maintain identical test conditions. Accordingly, the range of values of μ should be determined and a fair average value sought.

Many experiments have been made, in the manner just explained, to ascertain the law or laws of friction and the coefficients of friction for particular solids with various conditions of surface. The one important so-called law is this: *the coefficient of static friction for two particular solids is independent of the intensity of the normal pressure N/A, where N = normal pressure and A = area of contact between the two bodies.*

For example, if it were desired to determine the coefficient of friction between a brick and a concrete floor by measuring the horizontal force P required to cause slip, it would make no difference whether the brick rested on edge, on end, or flat on one side. The intensity of normal pressure N/A would be different in each case because A would be different, but P, and therefore $\mu = P/N$, would be found to be the same. Similarly it would not matter whether the brick were used alone or had a load placed on top of it; different values of P would be obtained, but the ratio of P to N would be found to be the same. *You must not take this illustration literally*, for the surfaces of a brick are really not exactly alike, and a very heavy load might cause indentation or abrasion of one or both the surfaces in contact and thus alter the nature of the contact.

Generally speaking, desired results of calculations on static friction need not be exactly correct. They should be on the safe side merely, and to insure this a designer will choose or adopt a coefficient of friction " plenty large " or " plenty small " to calculate respectively (i) the force required to overcome friction as on a wedge or (ii) the friction that may be depended on to hold a load as in a brake.

SAMPLE COEFFICIENTS OF STATIC FRICTION. *Locomotive drivers on steel rails:* in ordinary summer weather, $\frac{1}{4}$; in winter, $\frac{1}{5}$; with sand on dry rails, $\frac{1}{3}$. *Clutches:* metal on metal dry, 0.15; greasy leather on cast iron, 0.20–0.25. *Friction drives:* leather on cast iron, 0.14; wood on metal, $\frac{1}{4}$. *Launching ways:* liberally greased, less than $\frac{1}{16}$.

For detailed information about coefficients you should consult engineering handbooks or books on general machine design, or, if necessary, books and papers dealing with friction in the field of your special interest.

62. Equilibrium When Slip Impends. A common problem is the determining of conditions under which slip will just occur or just not occur. Thus, it may be required to find the force necessary to start a body against friction, to determine the limiting position that a body may have without slipping, or to find the limiting position of a load.

The general plan of solution is to draw the appropriate fbd and, in representing the external forces at contact points where slipping impends, to make use of one or the other of these facts: (i) the total reaction R is inclined at the angle ϕ to the normal, and it acts so as to oppose slip; and (ii) the friction component F_m is equal to μN and acts so as to oppose slip. *Generally* the use of

R is preferable for graphic solutions and the use of F_m and N for algebraic solutions. In either method only one unknown is involved at each point of support — the magnitude of R or the magnitude of N.

EXAMPLE 1. A body A, with a flat base, is supported on a level surface B. A weighs 100 lb, and the coefficient of friction between A and B is 0.7. What applied force P inclined 15° to the horizontal will just start A?

Solutions. Figure 119*a* is a fbd of A, for impending slip. Since A rests, the indicated external system is in equilibrium, and $P \cos 15 = 0.7N$, and $N = 100 - P \sin 15$; hence $P = 61.0$ lb. Figure 119*b* is another fbd. The unknowns of the external system are P and R. The closed force polygon (Fig. 119*c*) gives $P = 60$ lb.

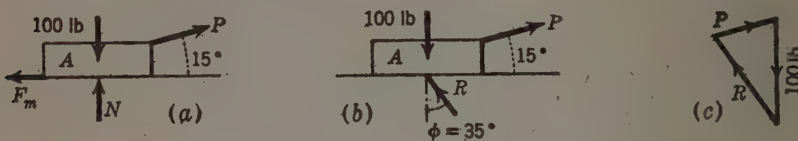


FIG. 119.

EXAMPLE 2. If P of Ex. 1 were horizontal, the starting force would be $0.7 \times 100 = 70$. Note that the starting pull when inclined as in Ex. 1 is less than when horizontal. Perhaps there is an inclination of P for which the starting pull is even less than 61.0 lb. A ready answer to the question here raised is suggested by the force triangle of Fig. 119*c*. Inspection shows that P has its least value if P is perpendicular to R. Such value, scaled, is about 57 lb; calculated from the right-angled triangle, it is $100 \sin 35 = 57.4$ lb. Obviously, P is least when it is perpendicular to R no matter what the value of ϕ is. This general fact is sometimes expressed thus, "The best angle of traction is equal to the angle of friction."

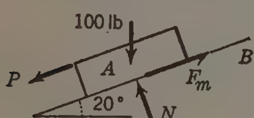


FIG. 120.

EXAMPLE 3. The angle of repose for A (Fig. 120) on the inclined plane B is 30°; A weighs 100 lb. The force P to start A down is required.

Solution. The external forces on A are W, P, and the reaction of the plane represented by N and F_m. $N = 100 \cos 20 = 94$ lb; $\mu = \tan 30$ or 0.58. $F_m = \mu N = 54.5$ lb. Therefore

$$P = 54.5 - 100 \sin 20 = 20.3 \text{ lb}$$

(Calculate the push Q, parallel to the plane, to start A up. Calculate also the friction when $P = 0$; 10 lb downward; 10 lb upward.)

EXAMPLE 4. It is required to calculate the horizontal force P (Fig. 121) that would just start A up the inclined plane. Since slipping impends, the inclination of the reaction R to the normal is the angle of friction for A and B. Lami's theorem gives

$$P = \frac{\sin \text{(angle between } W \text{ and } R)}{\sin \text{(angle between } P \text{ and } R)} W = \frac{\sin (\alpha + \phi)}{\cos (\alpha + \phi)} W$$

from which $P = \tan (\alpha + \phi) \cdot W$. (Suppose that α is less than ρ, so that A would not slip down the plane. Find the horizontal pull Q that would just start A down.)

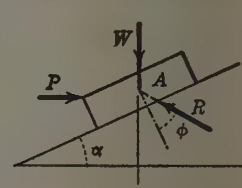


FIG. 121.

63. Test for Rest or Motion. Sometimes a friction problem like the following arises: A body is to be supported in a given position and so that slipping

is possible; it is to be subjected to various given forces. Would the body, in these given circumstances, slip?

To find the answer to this question we suggest this procedure: Assume that the supported body A does not slip, the supporting body B exerting the necessary reaction to keep A at rest. Then determine the inclination of the necessary reaction R, or the magnitudes of its components N and F. Finally ascertain whether B *can* exert such necessary force R or F. For this purpose, recall that the greatest possible inclination of R to N is equal to $\tan^{-1} \mu$, and that the greatest possible F is equal to μN. This suggested procedure pertains to cases of a single contact between A and B with a one-force reaction. In a problem with more than one such contact, the inclination of R and the magnitude of F at each contact are restricted as stated; but the procedure may require modification (see Ex. 3).

EXAMPLE 1. A block weighing 120 lb is placed on a plane inclined at 20° to the horizontal. The coefficient of friction between the block and the plane is $\frac{1}{3}$; a horizontal force $P = 30$ lb is applied to the block, as indicated in Fig. 122a. Does the block slip?

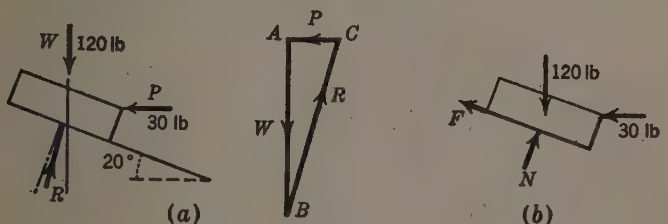


FIG. 122.

Solutions. 1. The forces acting on the block are its own weight W, the applied force P, and the reaction R of the plane (Fig. 122a). For equilibrium, the polygon for these forces closes. $ABCA$ is one such polygon; from it you see that the angle between R and W is $\tan^{-1} \frac{30}{120}$ or 14°. Hence the inclination of R to the normal is 20 − 14 or 6°. Since this inclination is less than ϕ ($\tan^{-1} \frac{1}{3}$ or 18.4°), the plane *can* exert the force R necessary for equilibrium. Therefore the block *does* rest, as assumed.

2. Figure 122b is a fbd of the block. F and N denote necessary reactions of the plane to preserve equilibrium of the block. The arrowhead on F implies that the block tends to slide down. Equilibrium equations for the system of forces when solved give $N = +123.1$ and $F = +12.8$ lb. Since F is less than $\mu N = 41.0$, the plane *can* exert the force F necessary for equilibrium. Therefore the block does rest.

EXAMPLE 2. Suppose that the 30-lb force of Ex. 1 is applied, not as described there, but perpendicularly to the plane of the paper of Fig. 122a and so that its line of action passes through the center of gravity of the block. Ascertain whether the block slips.

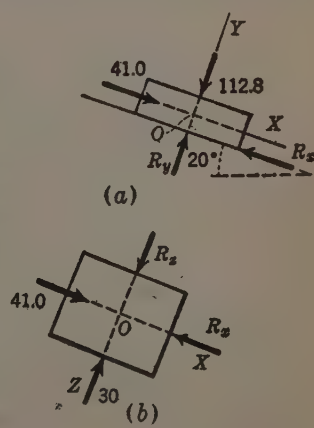


FIG. 123.

Solutions. 1. The center of gravity O of the block is taken as the origin of x, y, z coordinate axes; parallel to the edges of the block. The rectangle of Fig. 123*a* is the xy section of the block, and the square (Fig. 123*b*) is the xz section.

The external system consists of $W = 120$ lb, $P = 30$ lb, and the reaction R of the inclined plane. We resolve each force into its axial components and indicate them in the xy and yz planes. The two indicated systems are in equilibrium (see Art. 43). The x, y, and z components of W are 41.0, 112.8, and 0; the components of P are 0, 0, and 30; the components of R are unknown but denoted by R_x, R_y, and R_z. They are probably not correctly located, but the solution following does not depend on the locations. $\Sigma F_x = 0$ for either system (xy or xz) gives $R_x = 41.0$; $\Sigma F_y = 0$ gives $R_y = 112.8$; $\Sigma F_z = 0$ gives $R_z = 30$ lb. $N = 112.8$; $R = (41^2 + 112.8^2 + 30^2)^{\frac{1}{2}} = 123.6$. The angle between N and $R = \cos^{-1}(N \div R)$ or 24°. Since this is greater than ϕ, which is $\tan^{-1} \frac{1}{3}$, the block slips.

2. Regard the reactions of the inclined plane as consisting of N and F so that the external system is W, P, N, and F. Represent the xy and yz components of this system, and find that

$$F_x = 41.0 \qquad N = 112.8 \qquad F_z = 30 \text{ lb}$$

Hence $F = (41^2 + 30^2)^{\frac{1}{2}} = 50.8$ lb. Since F is greater than μN or 37.6 lb, the plane cannot exert the friction necessary for equilibrium; the block slips.

EXAMPLE 3. A uniform ladder of length l is placed on a floor and against a wall. The angle between the ladder and the horizontal is 40°; the coefficient of friction for ladder and its supports is 0.5. It is required to ascertain whether the ladder remains in this position or slips.

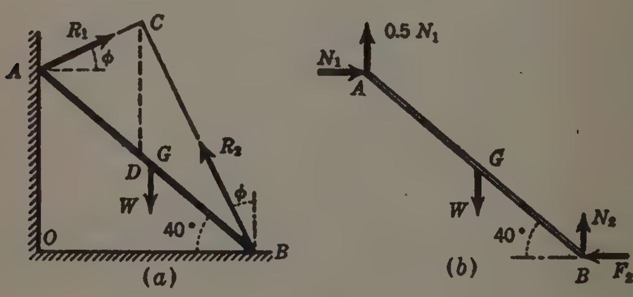


FIG. 124.

Solutions. 1. AB (Fig. 124*a*) represents the ladder; the angle ABO is exactly 40°, and G is the center of gravity. The angles ϕ are exactly $\tan^{-1} 0.5$, or 26.6°; hence AC and BC are the lines of action of the reactions R_1 and R_2, if slipping impends. But W is not concurrent with the indicated R_1 and R_2, and so slipping is not impending. Since G is below D, the ladder does not slip.

2. For an algebraic solution we assume that slipping impends at A and find the necessary reaction at B. Figure 124*b* is the fbd. The reaction at A is represented by its components N_1 and μN_1; the reaction at B is represented by its components N_2 and F_2; N_1, N_2, and F_2 are unknown. The equilibrium equations give $F_2 = 0.373W$ and $N_2 = 0.814W$; since F_2 is less than μN_2 (which is $0.407W$), the floor *can* exert the necessary friction to preserve the assumed state of equilibrium. Similarly, if one assumes that slip impends at B, he will find that the wall can exert the necessary friction to preserve this assumed state of equilibrium.

EXAMPLE 4. A uniform bar of length l with rounded ends rests with one end on a floor and the other against a wall. The coefficients of friction for the ends are equal. Required the inclination of the bar for which slip impends.

Solution. Figure 125 is a fbd of the bar; there are three unknowns, N_1, N_2, and θ. The equilibrium equations are

$$\Sigma F_x = N_1 - \mu N_2 = 0 \qquad \Sigma F_y = \mu N_1 + N_2 - W = 0$$
$$\Sigma M_B = -N_1 \times l \sin\theta - \mu N_1 \times l \cos\theta + W \times \tfrac{1}{2} l \cos\theta = 0$$

Solved simultaneously these equations give

$$\theta = \tan^{-1} \frac{1}{2} \frac{1 - \mu^2}{\mu}$$

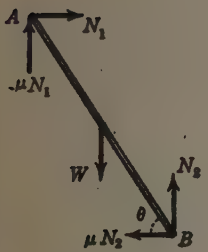


FIG. 125.

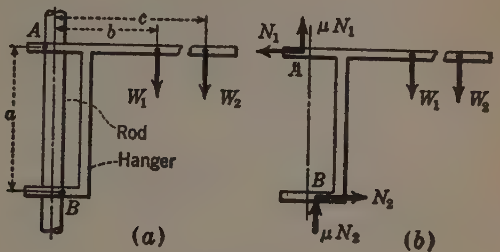


FIG. 126.

EXAMPLE 5. Figure 126*a* represents a simple hanger supported on a vertical rod. The hanger consists of a forked arm with a hole in each part of the fork, permitting it to be slipped over the rod as shown. If properly constructed, the hanger will not slip down the rod on account of its own weight or that of a load unless the load is applied too close to the rod. Determine the position of the suspended load for which slip impends; given μ, a, b, W_1 (weight of hanger), and W_2 (weight of load).

Solution. From the fbd of the hanger, Fig. 126*b*, you get

$$\Sigma F_x = N_2 - N_1 = 0 \qquad \text{or} \qquad N_1 = N_2 = N, \text{ say}$$

$$\Sigma F_y = 2\mu N - W_1 - W_2 = 0 \qquad \text{or} \qquad N = \frac{W_1 + W_2}{2\mu}$$

$$\Sigma M_A = Na - W_1 b - W_2 c = 0$$

or

$$c = \frac{W_1 + W_2}{2\mu W_2} a - \frac{W_1}{W_2} b$$

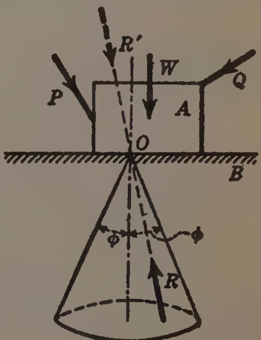


FIG. 127.

CONE OF FRICTION. A (Fig. 127) is a body supposed to be supported by B while subjected to applied forces W, P, Q, etc. Let R' denote the resultant of all forces acting on A but not including the reaction R of B on A, and suppose that R' intersects the surface of contact at some point O. Can B support A, or will A slip? For equilibrium the reaction R must be collinear with R'. Now R can act at O, and it can be inclined to the normal there at any angle less than ϕ (Art. 60); its line of action can therefore lie anywhere within the cone drawn in the figure. Therefore any system of applied forces whose resultant falls inside this cone can

be balanced by the reaction R and will not make A slip, whereas any system whose resultant falls outside this cone cannot be thus balanced and will make A slip. This cone is called the **cone of friction** for A and B; its vertex is at the center of pressure or point of contact O; its axis is the normal at that point; and its apex angle is 2ϕ.

The cone-of-friction criterion for equilibrium is equivalent to conditions already stated but represents a form of statement sometimes preferred.

64. Slipping Occurs; Kinetic Friction. Many familiar examples of kinetic friction may be noted in running machinery, on and in fluids flowing in pipes, etc. Nearly all such occurrences of friction are undesirable as they entail waste of energy. In order to reduce such waste, search for improvements in bearings, lubrication, streamlining, etc., is constant. The vast literature in the field is beyond the scope of this book.

You are reminded that kinetic friction may be *desirable*, as in brakes of automobiles, cars, hoists, etc. The rubbing surfaces of these appliances are, of course, not lubricated.

The coefficient for kinetic friction is analogous to the coefficient for static friction; it is the ratio of the friction to the normal pressure while slipping is in progress.

SAMPLE COEFFICIENTS OF KINETIC FRICTION. *Brakes:* (i) For railway cars in the usual conditions of braking, 0.05–0.30, the coefficient being smaller for the higher speeds. (ii) For hoists, poplar on cast iron, 0.35; steel band on cast iron, 0.18. (iii) For autos, 0.30–0.50 (various kinds of linings). *Auto tires:* new, skidding in line of travel on wet Portland cement concrete and stopping from 10 miles per hour, average coefficient 0.65; stopping from 50 miles per hour, 0.40. Tires smooth, the corresponding coefficients are 0.60 and 0.35. *Sleighs on snow or ice:* wood runners, 0.035; steel runners, 0.02.

B. Some Appliances with Friction

65. Preliminary. Friction plays an important role in the functioning of many mechanical devices that raise or control large loads, such as jacks, wedges, and hoisting tackle. Friction may be *undesirable*, in that it reduces efficiency; or it may be *helpful* in that it makes possible the designing of self-locking mechanisms, which will not reverse or "overrun" under load. Friction is *essential* to the functioning of some appliances, notably drives (including vehicular tires), friction clutches, and brakes. We now discuss several appliances, with special regard to the influence of friction.

66. Wedges. Figure 128*a* represents an appliance for raising or lowering a heavy load by means of relatively small forces. It consists of two wedges, two wedge blocks, and a right and left screw. Turning the screw in one direction makes the wedges approach each other and raises the load; turning it in the other direction separates them and lowers the load. Several such appliances

were used in the construction and placing of each gate of the Panama Canal locks. Each gate was made in parts in the United States, assembled at the canal, and erected on such appliances, by means of which the gate was finally lowered onto its pintle (lower hinge bearing).

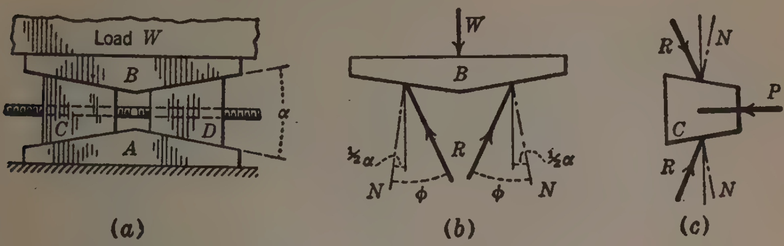


FIG. 128.

EXAMPLE. Referring to the appliance described above, it is required to find the forces, applied to the wedges, that would separate them and so lower the load. The wedge angles are α, and the common angle of friction for all rubbing surfaces is ϕ. Weights of parts are negligible in comparison with the load W.

Solutions. 1. Figure 128*b* is a fbd of B, slip impending. It is assumed that all is symmetrical; slip impends at each side of A, and reactions R are equal. Figure 128*c* is a fbd of C, the required force being denoted by P; it is the resultant of the pressures on all the small parts of C in contact with the screw. The force on the top of C is equal and opposite to the force exerted by C on B; the force on the bottom of C is symmetrical with the force on the top, and so indicated. From these fbd's, respectively, it follows that

$$\Sigma F_y = 2R\cos(\phi - \tfrac{1}{2}\alpha) - W = 0$$

and

$$\Sigma F_x = -P + 2R\sin(\phi - \tfrac{1}{2}\alpha) = 0$$

Solved simultaneously, these equations give

$$P = W\tan(\phi - \tfrac{1}{2}\alpha).$$

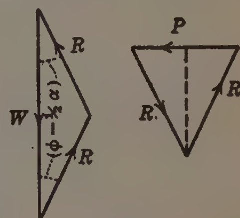


FIG. 129.

This formula can be arrived at readily from the force polygons (Fig. 129) for the fbd's of Fig. 128. From them, respectively,

$$R\cos(\phi - \tfrac{1}{2}\alpha) = \tfrac{1}{2}W \qquad \text{and} \qquad R\sin(\phi - \tfrac{1}{2}\alpha) = \tfrac{1}{2}P$$

Hence, etc.

2. Figure 130 shows fbd's of B and C in which each R of Fig. 128 is replaced by N and μN. $\Sigma F_y = 0$ for B and ΣF_x for C give

$$2N\cos\tfrac{1}{2}\alpha + 2\mu N\sin\tfrac{1}{2}\alpha - W = 0$$
$$2\mu N\cos\tfrac{1}{2}\alpha - 2N\sin\tfrac{1}{2}\alpha - P = 0$$

Simultaneous solution gives

$$P = \frac{\mu\cos\tfrac{1}{2}\alpha - \sin\tfrac{1}{2}\alpha}{\cos\tfrac{1}{2}\alpha + \mu\sin\tfrac{1}{2}\alpha}W$$

Substituting $\tan\phi$ for μ and simplifying gives the result obtained in the first solution.

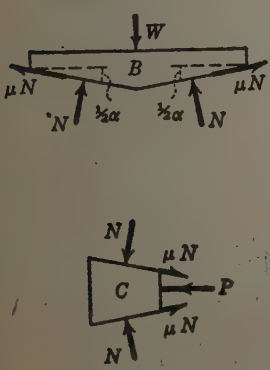


FIG. 130.

A self-locking wedge is a wedge that will not slip out, unaided, under its load. The wedges of Fig. 128 (α less than 2ϕ) are self locking, for it was shown that outward pushes P are required to separate the wedges. (Make fbd's and force polygons for an appliance in which α is greater than 2ϕ, and see that pulls P are required to prevent slip outward.)

67. Screw Jack. Figure 131 represents a common type of screw jack for raising heavy loads. When the jack is in use and the screw is turned, the cap C is held fast by the load and does not turn. Thus friction is developed between cap and head of the screw as well as at the threads. Calculations that relate to raising the load follow; for lowering, only results are given, and you should supply the calculations. The notation is:

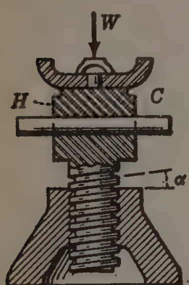


FIG. 131.

α = pitch angle (see figure).
r = mean radius of the thread, assumed "square."
r_0 = mean radius of the annular contact between cap and head.
ϕ = angle of friction at the thread.
μ = coefficient of friction between cap and head.
W = load.

1. *Raising the load.* The moment, about the axis of the screw, of the force or forces applied to the lever bar of the jack must exceed

$$M = \tan(\alpha + \phi) \times Wr + \mu W r_0 \tag{1}$$

To arrive at this formula the moments required to overcome the resistance at the threads and at the cap are calculated separately. We regard the stationary thread of the standard as an inclined plane up which the moving thread of the screw and the load W are to be raised by a horizontal force P applied directly to the thread. The push required thus to raise the load is $W \tan(\phi + \alpha)$; see Ex. 4, Art. 62. Since P acts at the distance r from the screw axis, the equivalent moment is $\tan(\phi + \alpha) \times Wr$. Since the normal pressure between the cap and its seat on the screw is W, the frictional resisting moment there is $\mu W r_0$, to overcome which an equal moment must be applied to the screw. The sum of these two calculated moments is the right-hand member of Eq. 1.

2. *Lowering the load.* Screw jacks are always made with small pitch angles so that, to lower the load, a moment must be applied to the screw in the direction in which the screw must turn for lowering. The moment required is

$$M' = \tan(\phi - \alpha) \times Wr + \mu W r_0 \tag{2}$$

As one would expect, this is less than the moment M for raising.

EXAMPLE. It is required to find the necessary moment applied to the screw of Fig. 128 to lower the load W, given the mean radius r of the thread, the pitch angle α', and the angle of friction ϕ' for the thread. (Primes are used here to avoid confusion with α and ϕ of Fig. 128.)

Solution. Figure 132*a* is a fbd of the two wedges and the screw. One-half the required moment is supposed to be applied at each end of the screw. Obviously the part of the screw between the wedges is under compression, say P. From Art. 66,

$$P = W \tan (\phi - \tfrac{1}{2}\alpha)$$

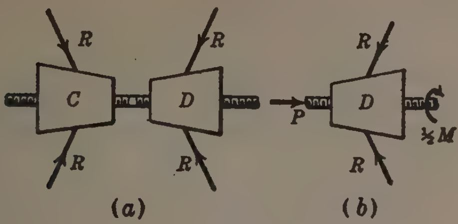


FIG. 132.

Figure 132*b* is a fbd of the wedge D and half the screw. Turned clockwise 90° before you, it resembles in principle the jack of Fig. 131 without cap; D corresponds to the pedestal of the jack, and P corresponds to the load W on the jack; the moment is applied to the lower end of the screw. Figure 132*b* thus turned may be looked upon as a jack. When separating the wedges, this jack is overcoming P (raising the load) with respect to the pedestal D, and so the applied moment is given by Eq. 1 without the last term, that is

$$\tfrac{1}{2}M = \tan (\alpha' + \phi') \times Pr$$
$$M = 2 \tan (\alpha' + \phi') \times \tan (\phi - \tfrac{1}{2}\alpha) \times Wr$$

If the entire moment is applied at one end of the screw, this equation still holds. To verify this statement, suppose that M is applied at the left end of the screw. One half of M is transmitted to the wedge D by the part of the screw between the wedges. The fbd of D and its half of the screw would look just like Fig. 132*b* except that the moment $\tfrac{1}{2}M$ would be shown at the left.

68. Tackle. Figure 133 represents a common hoisting appliance called a tackle. It comprises two "blocks," each consisting of a housing containing one or more pulleys or sheaves, and some rope or chain as indicated. W stands for the weight of the load, and P for the applied pull.

FIG. 133.

By **mechanical advantage** of a tackle is meant the ratio W/P. By **efficiency** of a tackle is meant the ratio of its mechanical advantage to the mechanical advantage if the tackle were frictionless.

Some pulley terms and notations are indicated in Fig. 134. The T's denote tensions in the off-pulley parts of the rope. Obviously the tension in a leading part is greater than the tension in the following part. The ratio of the former to the latter is called **pulley coefficient.** Denoting it by C for Fig. 134*a*, and by C' for Fig. 134*b*,

$$C = \frac{T_l}{T_f} \quad \text{and} \quad C' = \frac{T'_l}{T'_f}$$

Values of C and C' depend on the kinds of bearing, on the rope, and on the diameters of rope and pulley or sheave. A few sample values* are given in the

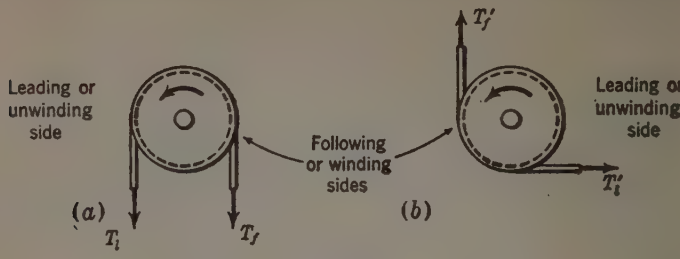


FIG. 134.

table below. The sheave was "metaline" bushed, turning on a 2-in. pin; sheave diameter 16 in.; the rope was of wire.

Rope diameter, inches	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
Coefficient C	1.041	1.045	1.048	1.052
Coefficient C'	1.029	1.031	1.034	1.036

Values of C' were computed from the values of C by means of $C' - 1 = 0.7(C - 1)$, a relation deduced from tests made by the American Bridge Company.

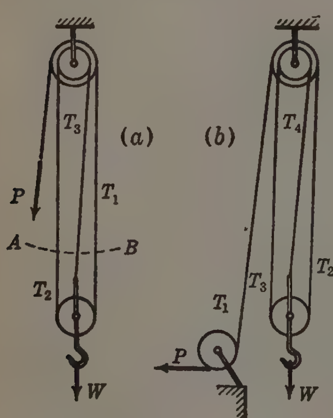


FIG. 135

EXAMPLE 1. For convenience the pulleys in the upper block of Fig. 135a are shown unequal in diameter. The coefficient for all pulleys is 1.09. The mechanical advantage and the efficiency of the tackle are required.

Solution. T_1, T_2, and T_3 denote the tensions in the parts as indicated. Since $1/C = 0.92$,

$$T_1 = 0.92P \qquad T_2 = 0.92T_1 = 0.85P$$
$$T_3 = 0.92T_2 = 0.78P$$

From a fbd of the part of the tackle below the line AB, it is plain that $W = T_1 + T_2 + T_3$; hence $W = 2.55P$. The mechanical advantage, then, is 2.55. If the tackle were without friction the mechanical advantage would be 3; hence the efficiency of the tackle is $2.55 \div 3$ or 0.85.

EXAMPLE 2. The coefficient of the pulley in the snatch block of Fig. 135b is 1.06; the coefficient for the pulleys in the main blocks is 1.09. The mechanical advantage and the efficiency of the tackle are required. Since $1/C' = 0.94$ and $1/C = 0.92$,

$$T_1 = 0.94P \qquad T_2 = 0.92T_1 = 0.86P \qquad T_3 = 0.92T_2 = 0.79P$$
$$T_4 = 0.92T_3 = 0.73P \qquad W = T_2 + T_3 + T_4 = 2.38P$$

Hence, the mechanical advantage is 2.38, and the efficiency is $2.38 \div 3$ or 0.79.

* From tests made by the American Bridge Company in 1914.

69. Coil and Belt Frictions. Perhaps you have seen a deck hand hold a boat against the current of a river by means of a hawser wrapped a few times about a snubbing post, or a cowboy stop a running steer that he has lassoed by wrapping the rope once or twice about the horn of his saddle. The pull exerted by either man on the end of the rope he holds is obviously very much less than the pull exerted at the other end. The relation between the two pulls when slipping impends is shown below to be

$$Q = Pe^{\mu\alpha} \tag{1}$$

where P = smaller pull, Q = larger pull.

α = angle of wrap, in radians; see Fig. 136*a*.

μ = coefficient of friction.

e = Napierian base (2.718).

The ratio Q/P increases rapidly with increase of α and μ. Thus for one-half, one, and two turns of the rope, respectively:

$$\begin{array}{llll} \text{if } \mu = \frac{1}{4}, & Q/P = 2.2 & 4.8 & 23 \\ \text{if } \mu = \frac{1}{2}, & Q/P = 4.8 & 23 & 535 \end{array}$$

Equation 1 may be derived as follows: Figure 136*b* is the fbd for the rope AB. The reaction of the post on this rope is represented by normal and frictional components, which obviously act as shown, opposing the larger pull Q. Let T denote the tension at any point C and θ the angle AOC. Figure 136*c* is a fbd, enlarged for clearness, of a small portion of the rope at C. The external forces are T, $T + dT$, dN, and dF. It is plain that

$$dF = dT \qquad \text{and} \qquad dN = 2T \sin \tfrac{1}{2}d\theta = T d\theta$$

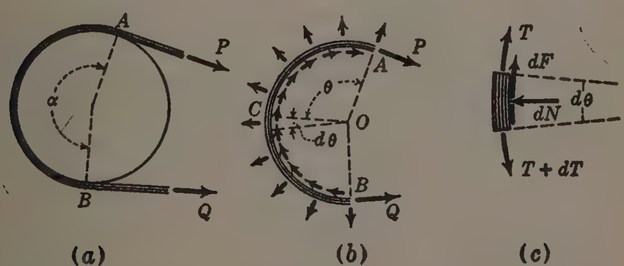


FIG. 136.

Division of these equations gives

$$\frac{dF}{dN} = \frac{dT}{T d\theta} \qquad \text{or} \qquad \frac{dF}{dN} d\theta = \frac{dT}{T}$$

When slip impends, $dF/dN = \mu$; hence $dT/T = \mu d\theta$. Integration gives

$$\Big[\log_e T\Big]_P^Q = \Big[\mu\theta\Big]_0^\alpha \qquad \text{or} \qquad \log_e \frac{Q}{P} = \mu\alpha$$

which can be written in the form of Eq. 1. (Equation 1 holds even if the post is not circular in cross section.)

Belts. When transmitting power a belt is obviously under unequal tensions on the tight and slack portions. For flat belts and impending slip, Eq. 1 holds if the belt speed is rather low. The effect of speed is to lessen the normal pressure between belt and pulley and hence reduce the available friction and the ratio Q/P.*

70. Rollers. Rollers are used often for moving loads in the manner indicated in Fig. 137. The total pressures at the top and bottom of each roller may be taken as equal, since the weights of the rollers are negligible in comparison with these pressures. Each pressure is marked R in Fig. 138, and the horizontal and vertical components R_h and R_v of each R are indicated

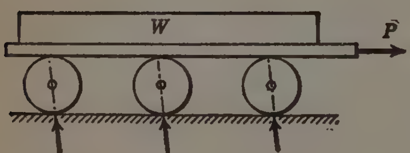


FIG. 137.

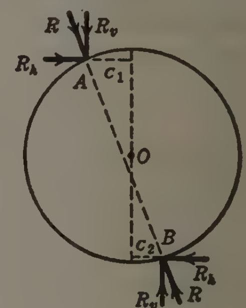


FIG. 138.

also. The horizontal components are called **rolling resistances.** The distances from the points of application A and B of the reactions R to the vertical diameter if rolling impends are called **coefficients of rolling resistance.** They are usually denoted by c and expressed in inches (see end of this article).

If the coefficients for all contacts in the set-up of Fig. 137 are equal, the force required to start rolling is

$$P = \frac{c}{\frac{1}{2}D} W \tag{1}$$

According to this formula, P does not depend on the number of rollers.

The formula can be proved as follows: The two components R_h for a roller comprise a couple, as do also the two components R_v. Their moments are equal; approximately $R_h \times D = R_v \times 2c$. Adding such equations for all rollers gives $D\Sigma R_h = 2c\Sigma R_v$. From a fbd of the load, it is plain that $\Sigma F_h = P$ and $\Sigma R_v = W$; hence $D \times P = 2c \times W$, and this leads to Eq. 1.

The laws of rolling resistance are not well established; probably they are

* For an extended discussion of belt friction, see *Machine Design*, by Hyland and Kommers, or any other standard book on that subject.

not simple. The coefficient certainly depends on the kind of roller and roadway; it may depend on the diameter and the length of the roller and on the load. Relatively few coefficients have been published.

SOME SAMPLE COEFFICIENTS OF ROLLING RESISTANCE. *Lignum vitae on oak:* rollers 2 to 6 in. long, $c = 0.017 - 0.020$ in. *Elm on oak:* rollers 6 to 13 in. long, $c = 0.03$. *Iron on iron:* railway wheels, 20- and 40-in. diameter, $c = 0.018 - 0.020$.

Some tests* indicate that $c = K\sqrt{\frac{1}{2}D}$, where D is diameter in inches and K a coefficient depending on the material. For *cast iron on cast iron*, $K = 0.0063$ in.$^{\frac{1}{2}}$; for *steel on cast iron*, $K = 0.0073$; for *wrought iron on cast iron*, $K = 0.0120$. For the above rollers, respectively, on *steel* plates the values of K are about 13 per cent less than those just given, and on *wrought-iron* plates about 13 per cent more.

* *Trans. Am. Soc. Civil Eng.*, Vol. 32, p. 99, 1894.

CHAPTER VI

CENTER OF GRAVITY AND CENTROID

71. Center of Gravity of a Rigid Body. We regard any body as consisting of a vast number of elementary particles, and the earth's attraction on this body as consisting of forces applied to these particles, one to each particle. These forces, the weights of particles, are parallel; and their points of application are fixed in the body. And as proved presently, *the line of action of the resultant of these forces always passes through some one definite point of the body, or of its extension, no matter how the body is turned.* This unique point is called the **center of gravity** of the body.

Let A, B, C, etc. (Fig. 139), be particles of the body, and let W_1, W_2, W_3, etc., be their weights, respectively. Any two such forces have a " center " through which their resultant always passes as the body is turned about (see Art. 29). Let b be the center of W_1 and W_2. It may be looked upon as the fixed point of application of the resultant R_1 of W_1 and W_2. Similarly R_1 and W_3 (or W_1, W_2, and W_3) have a center c. Since three of the weights have a center, so do four; indeed, all the weights have a center, which is the center of gravity of the body.

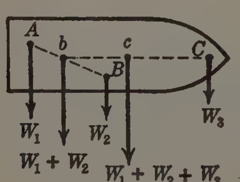


FIG. 139.

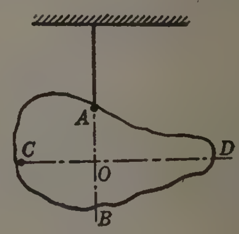


FIG. 140.

72. Center of Gravity by Experiment. 1. *Suspending.* Any single force that supports a body, being in equilibrium with the weight of the body, is collinear therewith, and so passes through the center of gravity of the body. This suggests the following way in which the position of the center of gravity of a given body may be determined by experiment. Determine the lines of action of two or more forces that, singly, will support the body. Each of these lines of action passes through the center of gravity; therefore, the center of gravity is at their intersection.

For example, suppose that it is desired to determine the position of the center of gravity of a thin plate of irregular form. The plate is suspended by a cord attached to it at any point A (Fig. 140); the axis of the string is the

line of action of the supporting force, and this line is marked on the plate as indicated by AB. The plate is then suspended by a cord attached at any other point C, and the line of action of the supporting force CD is marked. The center of gravity of the plate is at O, the point of intersection of AB and CD.

2. *Balancing.* Again, if a body is balanced on a horizontal knife-edge, the center of gravity is known to be in the vertical plane containing this knife-edge. If in some way the position of this plane in the body can be marked, one may, by successively balancing the body in different positions, determine three such planes, and, by their intersections, locate the center of gravity.

3. *Weighing.* The weight W of the body is determined. Then the body is placed on a knife-edge or a two-point support B (Fig. 141) and on a single point-support A which rests upon a platform scale; the reaction R of the point support is weighed, and the distance a from the support A to the vertical plane containing the knife-edge is measured. Finally the distance from the center of gravity to the plane is calculated from $x = Ra/W$, where W = weight of the body. In this way the distances of the center of gravity from other such planes could be determined and the center of gravity of the body thus definitely located.

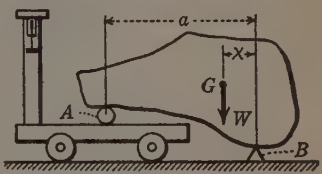


Fig. 141.

4. *Inspection.* The notion of balance, brought out by the above discussion, suggests that for certain forms of homogeneous bodies the position of the center of gravity can be wholly or partly determined by inspection. One perceives instinctively that any homogeneous symmetrical solid will balance, as it were, on a plane of symmetry, and that therefore any plane or axis of symmetry must contain the center of gravity. Thus the center of gravity of a right parallelepiped, right circular cylinder, or sphere is seen to be at the geometrical center; the center of gravity of a right circular cone is seen to be on the axis and nearer the base than the apex, etc. You should perceive, too, that a thin triangular plate would balance in a horizontal position if supported along a median. Thus, you know that the center of gravity is just over the median, and hence just over the intersection of two such medians.

73. Center of Gravity by Calculation. By a composite body we mean one that is naturally regarded as consisting of parts. Thus most flywheels are composite since they consist of a rim, a hub, and generally several spokes. If the parts of such a body are so shaped that the weight and center of gravity of each part are easily determined, then the center of gravity of the entire body can be readily found, for the law of moments (Art. 19) permits calculation of the arm of the resultant of all the weights with respect to each one of a set of coordinate axes, and these arms locate the center of gravity. (See the first solution of Ex. 1, below, for a detailed explanation.)

Although the law of moments is the basis for practically all calculations for locating centers of gravity, an experienced computer does not, when making such calculation, *think* in terms of moments of forces. He carries out the necessary calculations more or less mechanically, following the procedure illustrated in the second solution of Ex. 1, below.

A composite body that has a hole, notch, or other void may be thought of as originally without voids and then modified by removal of material to form the voids. In a calculation for the center of gravity of the actual body you should treat the weight of the material supposed to be taken away as negative. (See the solution of Ex. 4, below, for a detailed explanation.)

EXAMPLE 1. It is required to locate the center of gravity of a composite body made up of three homogeneous parallelepipeds A, B, and C (Fig. 142), having dimensions and weights as indicated.

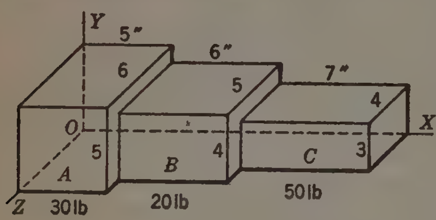


FIG. 142.

First solution. As customary, we denote coordinates of a center of gravity by $\bar{x}$, $\bar{y}$, and $\bar{z}$. Then the law of moments (Art. 19) gives for the z axis

$$100 \times \bar{x} = 30 \times 2.5 + 20(5 + 3) + 50(5 + 6 + 3.5)$$

from which $\bar{x} = 9.6$ in. For the x axis, the law gives

$$100 \times \bar{z} = 30 \times 3 + 20 \times 2.5 + 50 \times 2$$

from which $\bar{z} = 2.4$ in. To determine $\bar{y}$, turn the open book counterclockwise till the x axis is vertical. Then for the z axis

$$100 \times \bar{y} = 30 \times 2.5 + 20 \times 2 + 50 \times 1.5$$

from which $\bar{y} = 1.9$ in. (Instead of turning the book as explained, you may imagine the earth to be far away to the right or left of the book; then gravity acts parallel to the x axis, and, taking moments about the z axis, you get the same equation.)

Simplified solution. The necessary calculations are facilitated by arranging the terms as in the adjoining table. The heading of each column tells what the numbers in the column mean. The numbers in columns 3, 4, and 5 were taken directly from Fig. 142; the numbers in 6, 7, and 8 were calculated. The four numbers in the last line are sums of the numbers immediately above. These four numbers yield the desired coordinates: thus

$$\bar{x} = \frac{960}{100} = 9.6 \qquad \bar{y} = \frac{190}{100} = 1.9 \qquad \bar{z} = \frac{240}{100} = 2.4 \text{ in.}$$

1	2	3	4	5	6	7	8
Part	W	x	y	z	Wx	Wy	Wz
A	30	2.5	2.5	3.0	75	75	90
B	20	8.0	2.0	2.5	160	40	50
C	50	14.5	1.5	2.0	725	75	100
	100				960	190	240

Obviously the products under Wx are simply the moments of the weights of the parts about the z axis, and the products under Wz are the moments about the x axis. Similarly the products under Wy are moments about the z axis when the body is turned as explained in the first solution, and so the second solution is really identical with the first but can be carried out without *thinking* of the products Wx, etc., as moments.

After solving a problem of this kind, it is often a good plan to represent the center of gravity at its determined position in a sketch of the body drawn approximately to scale. You can then nearly always tell whether the position found is a reasonable one by imagining the body to be balanced on a point there, or supported by a wire attached there. In this way a gross error in calculations can generally be detected.

EXAMPLE 2. A slender uniform wire 43 in. long is bent into the form represented by the heavy line in Fig. 143. It is required to determine the position of the center of gravity of this bent wire.

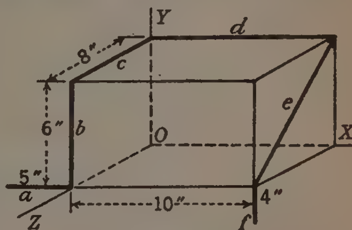


FIG. 143.

Solution. We take w to denote the weight of 1 in. of wire, and choose coordinate axes, as indicated. The weights of the parts are in column 2 of the adjoining table; the coordinates of the centers of gravity are in columns 3, 4, and 5; and the moments of their weights are in columns 6, 7, and 8. (The moments Wx and Wz pertain to the z and x axes, respectively; moments Wy pertain to the z axis when the wire and axes are turned so that the x axis is vertical.)

1	2	3	4	5	6	7	8
Part	W	x	y	z	Wx	Wy	Wz
a	$5w$	-2.5	0	8	$-12.5w$	$0w$	$40w$
b	6	0.0	3	8	0.0	18	48
c	8	0.0	6	4	0.0	48	32
d	10	5.0	6	0	50.0	60	00
e	10	10.0	3	4	100.0	30	40
f	4	10.0	-2	8	40.0	-8	32
	$43w$				$177.5w$	$148w$	$192w$

The desired coordinates are calculated from the sums in the lowest line of the table:

$$\bar{x} = \frac{177.5w}{43w} = 4.13 \qquad \bar{y} = \frac{148w}{43w} = 3.44 \qquad \bar{z} = \frac{192w}{43w} = 4.47 \text{ in.}$$

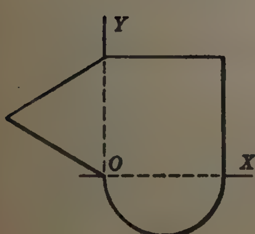


FIG. 144.

As you may have anticipated, w does not appear in these final results, and, obviously, the position of the center of gravity of any homogeneous body does not depend on the specific gravity of the body.

EXAMPLE 3. Figure 144 represents a flat sheet of tin consisting of three parts, namely, a 6-by-6-in. square, a semicircle, and an equilateral triangle. Determine the position of the center of gravity of the sheet.

Solution. The weight of the tin sheet per square inch is denoted by w in the table adjoining. For location of the centers of gravity of the triangular and semicircular parts, see

Ex. 1 and Ex. 4, Art. 75, respectively, and so arrive at the values −1.73 and −1.27 of the adjoining table. The derivation of the other values should, in view of preceding solutions, be obvious, as should also the final calculations below the table.

1	2	3	4	5	6
Part	W	x	y	Wx	Wy
Square	$36w$	3.0	3.0	$108w$	$108w$
Semicircle	14.1	3.0	−1.27	42.3	−17.9
Triangle	15.6	−1.73	3.0	−27.0	46.8
	$65.7w$			$123.3w$	$136.9w$

$$\bar{x} = \frac{123.3}{65.7} = 1.88 \qquad \bar{y} = \frac{136.9}{65.7} = 2.08 \text{ in.}$$

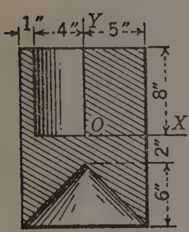


FIG. 145.

EXAMPLE 4. Figure 145 represents, in section, a cylinder of cast iron with a cylindrical hole filled with lead in one end and an (empty) conical hole in the other. Determine the position of the center of gravity of the composite body.

Solution. The weights of the parts, in pounds, are listed in column 2 of the adjoining table. The coordinates, except $-6\frac{1}{2}$, of the centers of gravity in columns 3, 4, and 5 are obvious from the figure. The distance $6\frac{1}{2}$ in. is equal to 2 in. (see Fig. 145) plus the distance from the apex to the center of gravity of the cone, or $4\frac{1}{2}$ in. (see Ex. 2, Art. 75). The derivation of the other figures (products and sums) should be obvious; also the final calculations below the table.

1	2	3	4	5	6	7	8
Part	W	x	y	z	Wx	Wy	Wz
Large cylinder	+327.5	0	0	0	0	0	0
Small cylinder removed	−26.2	−2	4	0	+52.4	−104.8	0
Cone removed	−41.0	0	$-6\frac{1}{2}$	0	0	+266.5	0
Lead cylinder poured	+41.3	−2	4	0	−82.6	+165.2	0
Actual body	301.6				−30.2	+326.9	0

$$\bar{x} = \frac{-30.2}{301.6} = -0.10 \qquad \bar{y} = \frac{326.9}{301.6} = 1.08 \qquad \bar{z} = \frac{0}{301.6} = 0 \text{ in.}$$

The calculations for $\bar{z}$ were not necessary, since it is obvious from the symmetry of the body that its center of gravity is in the xy plane, and hence $\bar{z} = 0$.

74. Centroid of a Line, Surface, or Solid. Since these geometric conceptions have no weight, they have no center of gravity in the sense of the term as defined in Art. 71. However, it is quite customary to speak of their center of gravity, meaning the center of gravity of the line, surface, or volume materialized, that is, conceived as a homogeneous slender wire, thin plate, or

body, respectively. The term **centroid** has been proposed as a substitute for center of gravity, as being more appropriate in application to lines, surfaces, and solids; it is given preference in this book.*

It is unnecessary to carry the notion of materialization into a *calculation* for coordinates of a centroid. Lengths, areas, or volumes, as the case may be, are used instead of imagined weights. For example, a calculation for the coordinates of the centroid of the heavy zigzag line in Fig. 143 would be just like that in the table of Ex. 2, Art. 73, but the headings of columns 2, 6, 7, and 8, respectively, would be L, Lx, Ly, and Lz; and there would be no w's.

Moment of a line, area, or volume. Products Lx, Ay, Vz, etc., are called moments because they resemble certain products which are moments of forces with respect to a line or point. They are called moments of L, A, or V with respect to the coordinate plane or axis from which x, y, or z is measured. If the line or area is wholly in the xy plane, say, then Lx and Ay are moments with respect to the y axis and the x axis respectively; see Ex. 1. Sometimes they are called **statical moments** to distinguish them from moments of forces. Though statical moments have nothing to do with rotation, they have signs which are fixed by the signs of the factors in the products (see Exs. 3 and 4).

There is a **principle of moments** for lines, areas, and volumes, namely: *the moment of any line, area, or volume with respect to a coordinate plane is equal to the algebraic sum of the moments of all its component parts with respect to that plane.*

EXAMPLE 1. Prove that the distance from the centroid C of a trapezoid (area) to its longer base (see Fig. 146*a*) is

$$\bar{y} = \frac{B + 2b}{3(B + b)} a$$

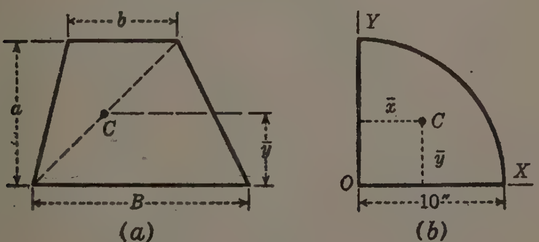


FIG. 146.

Solution. We regard the trapezoid as consisting of two triangles as shown. The areas of the smaller and larger triangles respectively are $\frac{1}{2}ba$ and $\frac{1}{2}Ba$. The distance between the centroid and the base of either triangle is $\frac{1}{3}a$. The area of the trapezoid is $\frac{1}{2}(b + B)a$, and its

* The word area is sometimes used as a synonym of geometrical surface. Thus one speaks of areas of definite shape (square, semicircular, etc.), and also of the centroids of such areas. The word volume is used analogously but less commonly as a synonym of geometrical solid.

moment with respect to the longer base is $\frac{1}{2}(b + B)a\bar{y}$. Since this moment is equal to the sum of the moments of the triangles,

$$\tfrac{1}{2}(b + B)a\bar{y} = \tfrac{1}{2}Ba \times \tfrac{1}{3}a + \tfrac{1}{2}ba \times \tfrac{2}{3}a$$

This equation yields the stated value of $\bar{y}$ above.

EXAMPLE 2. Locate the centroid of the heavy line $OXYO$ of Fig. 146b, XY being a circular arc.

Solution. Obviously the two coordinates of the centroid of the (whole) line are equal; we calculate $\bar{x}$. The lengths of the three parts of the line and their sum are recorded in the second column of the table below; and the x coordinates of the centroids of the parts are in the

1	2	3	4
Part	L	x	Lx
Horizontal	10	5	50
Vertical	10	0	0
Arc	15.7	6.37	100
Whole	35.7		150

third column (see Ex. 3 of Art. 75 for the coordinate 6.37). The moments of these parts with respect to the y axis and their sum are in the last column. (Since the lengths L and the coordinates x are in inches, the moments Lx are in *inches squared.*) The moment of the whole line is $35.7\bar{x}$; since it is equal to the sum of the moments of the parts,

$$35.7\bar{x} = 150 \qquad \text{or} \qquad \bar{x} = 4.20 \text{ in.}$$

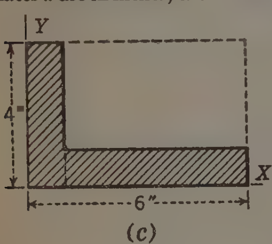


FIG. 146.

EXAMPLE 3. Locate the centroid of the shaded area in Fig. 146c. The figure represents the cross section of an "angle," a piece of structural steel; the size is described as 6 by 4 by 1.

Solutions. 1. We regard the section as consisting of two rectangles (vertical and horizontal). Their areas, coordinates of centroids, and moments with respect to the coordinate planes are recorded in the table. Hence

Parts	A	x	y	Ax	Ay
4 × 1	4	0.5	2	2.0	8.0
5 × 1	5	3.5	0.5	17.5	2.5
	9			19.5	10.5

$$\bar{x} = \frac{19.5}{9} = 2.17 \qquad \text{and} \qquad \bar{y} = \frac{10.5}{9} = 1.17 \text{ in.}$$

2. We regard the angle section as consisting of a large rectangle 6 by 4 minus a smaller rectangle 5 by 3. The table below shows the figures for this plan. The sums in the last line

Parts	A	x	y	Ax	Ay
6 × 4	24	3	2	72	48
5 × 3	−15	3.5	2.5	−52.5	−37.5
	9			19.5	10.5

agree with those of the preceding table and yield the same results.

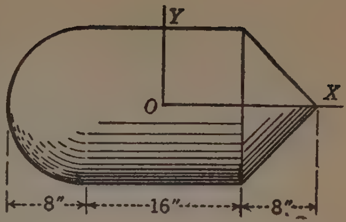


FIG. 147.

EXAMPLE 4. Locate the centroid of a cylinder, a hemisphere, and a cone, as represented in Fig. 147.

Solution. Obviously the centroid is on the axis of the cylinder, chosen as x axis; therefore $\bar{y} = 0$. We assume that you know how each number of the table below was calculated, except perhaps 10 and -11. Since the centroid of the cone is 2 in. from its base (see Ex. 2, Art. 75),

Part	V	x	Vx
Cylinder	3217.0	0	0
Cone	536.2	10	5,362
Hemisphere	1072.3	−11	−11,795
	4825.5		−6,433

x for the cone is $2 + 8$ or 10; and, since the centroid of the hemisphere is 3 in. from its base (see Ex. 5, Art. 75), x for the hemisphere is $-(3 + 8)$ or -11. Finally

$$\bar{x} = -6433 \div 4825.5 = -1.33 \text{ in.}$$

(Since the volumes are in cubic inches and the coordinates are in inches, the moments are in *biquadratic inches.*)

75. Centroid Determined by Integration. The locations of the centroids of common geometrical forms or figures are given in many books, especially in various engineering handbooks. The way in which these locations are determined is explained in textbooks for beginners in calculus. We show here that the method used may be based on the principle of moments, Art. 19; but in problems requiring integration, it is necessary to regard the line, area, or volume as consisting of elementary parts, *infinite* in number. To illustrate briefly, let it be required to find the x coordinate of the centroid of the curve AB in Fig. 148. The length of the line is $\int ds$, and the moment of the line with respect to the y axis is $\bar{x}\int ds$. The moment of any ds, as the one at P, is $(ds)x$, and the algebraic sum of the moments of all elements is $\int x\,ds$. The principle of moments gives

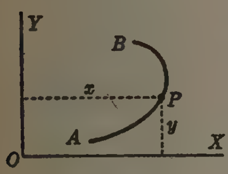


FIG. 148.

$$\bar{x}\int ds = \int x\,ds$$

If in a given problem the integrations can be performed, the equation above furnishes the required coordinate.

EXAMPLE 1. Prove that the distance from the centroid of a triangle to any side is $\frac{1}{3}a$, where a is the altitude measured to that side.

Solution. Let OAB (Fig. 149*a*) be the triangle and C its centroid. For simplicity the x axis is taken perpendicular to a side, AB. The moment of the triangle with respect to the y axis is $(\frac{1}{2}ab)\bar{x}$. Now conceive the triangle to consist of strips as indicated. The area of the strip is $(b/a)x\,dx$ (from similar triangles), and the moment of the strip with respect to the y axis is $[(b/a)x\,dx]x$; the algebraic sum of the moments of all strips comprising the triangle is $\int_0^a (b/a)x^2\,dx$. The principle of moments gives

$$\tfrac{1}{2}ab\bar{x} = \frac{b}{a}\int_0^a x^2\,dx$$

Integrating and solving gives $\bar{x} = \frac{2}{3}a$; hence, $a - \bar{x} = \frac{1}{3}a$.

(Since the centroids of all strips lie on the median to AB, the centroid of the triangle is on that median; and, since the centroid is on each median, it is at their intersection.)

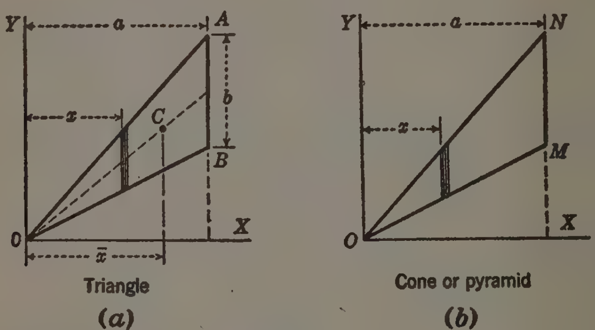


FIG. 149.

EXAMPLE 2. Prove that the distance from the centroid of any solid cone or pyramid to its base is $\frac{1}{4}a$, where a denotes altitudes of the cone or pyramid.

Solution. Let O (Fig. 149*b*) be the apex and $OMNO$ a section of the solid (cone or pyramid) perpendicular to the base; MN is the projection of the base on this section. Let A denote the area of the base. Then the volume of the solid is $\frac{1}{3}Aa$, and the moment of the solid with respect to the yz plane is $\frac{1}{3}Aa\bar{x}$, where $\bar{x}$ is the x coordinate of the centroid of the solid. Now conceive the solid as consisting of elementary laminas parallel to the base. The area of either base of the lamina indicated is Ax^2/a^2; and the volume of the lamina is $(Ax^2/a^2)\,dx$. The moment of the lamina with respect to the yz plane is $(Ax^2/a^2)\,dx$ times x, and the algebraic sum of the moments of all laminas is

$$\frac{A}{a^2}\int_0^a x^3\,dx$$

The principle of moments gives

$$\tfrac{1}{3}Aa\bar{x} = \frac{A}{a^2}\int_0^a x^3\,dx$$

Integrating and solving gives $\bar{x} = \frac{3}{4}a$; hence, etc.

(Since the centroids of all laminas is on the line joining the apex of the solid with the centroid of the base, the centroid of the solid is on that line.)

EXAMPLE 3. Prove that the distance of the centroid of the circular arc AB (Fig. 150*a*) from the center of the circle is

$$\bar{x} = \frac{\sin \alpha}{\alpha} r$$

See the figure for r and α. When the formula is applied to a numerical example, the α in the denominator must be expressed in radians.

Solution. The length of the arc AB is $2r\alpha$, and the moment of the arc with respect to the y axis is $2r\alpha \cdot \bar{x}$. Conceive the arc as consisting of elementary lengths ds. The moment of the element at any point as P is $ds \cdot x$, and the sum of the moments of all elements is $\int x\,ds$. And so

$$2r\alpha\bar{x} = \int x\,ds$$

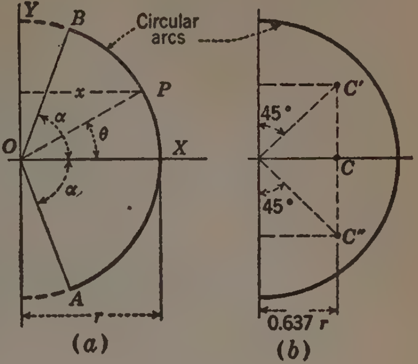


FIG. 150.

Changing to polar coordinates furnishes a simple integration; thus

$$\int x\,ds = \int_{-\alpha}^{+\alpha} r \cos\theta \cdot r\,d\theta = r^2 \int_{-\alpha}^{+\alpha} \cos\theta\,d\theta = 2r^2 \sin\alpha$$

Hence, $\quad 2r\alpha\bar{x} = 2r^2 \sin\alpha$, etc.

(For a semicircular arc, $\alpha = 90° = \frac{1}{2}\pi$ radians. Hence $\bar{x} = 2r/\pi = 0.637r$. C in Fig. 150*b* is the centroid of this arc; C' and C'' are the centroids of the quadrantal arcs.)

EXAMPLE 4. Prove that the distance of the centroid of the circular sector (Fig. 151*a*) $OABO$ from the center of the circle is

$$\bar{x} = \frac{2}{3} \frac{\sin\alpha}{\alpha} r$$

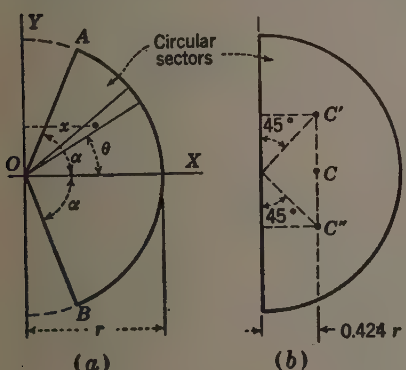


FIG. 151.

See the figure for r and α. When the formula is applied to a numerical example the α in the denominator must be expressed in radians.

Solution. The area of the sector is $r^2\alpha$, and its moment with respect to the y axis is $r^2\alpha\bar{x}$. Now conceive the sector as consisting of elementary sectors; the area of the one indicated is $\frac{1}{2}r\,ds$ or $\frac{1}{2}r^2\,d\theta$. The x coordinate of the centroid of the sector is $\frac{2}{3}r\cos\theta$; the moment of the sector with respect to the y axis is $\frac{1}{2}r^2\,d\theta \times \frac{2}{3}r\cos\theta$; and the algebraic sum of the moments of all such sectors is

$$\frac{1}{3}r^3 \int_{-\alpha}^{+\alpha} \cos\theta\,d\theta = \frac{2}{3}r^3 \sin\alpha$$

Hence, $r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin\alpha$; etc.

(For a semicircular sector, $\alpha = 90° = \frac{1}{2}\pi$ radians. Hence $\bar{x} = 4r/3\pi = 0.424r$. C in Fig. 151*b* is the centroid of this sector; C' and C'' are the centroids of the quadrantal sectors.)

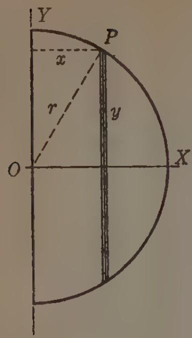


FIG. 152.

EXAMPLE 5. It is required to show that the distance from the centroid of a hemisphere to its base is $\frac{3}{8}r$, where r is the radius of the sphere.

Solution. O, Fig. 152, is the center of the sphere and the origin of a set of axes as indicated. The volume of the hemisphere is $\frac{2}{3}\pi r^3$, and its moment with respect to the yz plane is $\frac{2}{3}\pi r^3 \cdot \bar{x}$. Conceive the hemisphere as consisting of laminas parallel to the base; the volume of the lamina indicated is $\pi(r^2 - x^2)\,dx$. The principle of moments gives

$$\frac{2}{3}\pi r^3 \bar{x} = \int_0^r \pi(r^2 - x^2)x\,dx$$

Integrating and solving for $\bar{x}$ gives the required result.

CHAPTER VII

SUSPENDED CABLES

76. Preliminary. When a uniform flexible cable is suspended from two points and allowed to hang freely under its own weight or under a load uniformly distributed along its *length*, it assumes the form of a curve called a **catenary.** When a cable is thus suspended and subjected to a load that is uniformly distributed along the *horizontal*, it assumes the form of a **parabolic arc.** A transmission line is an example of a catenary, whereas suspension bridge cables are usually assumed to be parabolic. We now discuss both catenary and parabolic cables, deriving for each the equation of the curve and formulas by means of which the tension in the cable and the relations between its length, span, and sag can be calculated. For each type of loading we limit the discussion to the symmetrical case, in which the supports are at the same level, and we consider first the parabolic cable because it presents the simpler problem.

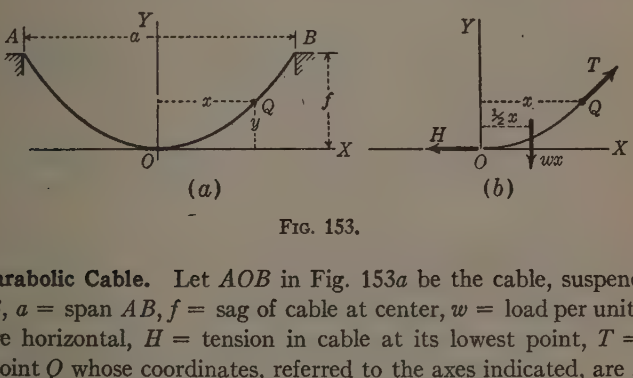


FIG. 153.

77. Parabolic Cable. Let AOB in Fig. 153a be the cable, suspended from A and B, a = span AB, f = sag of cable at center, w = load per unit distance along the horizontal, H = tension in cable at its lowest point, T = tension at any point Q whose coordinates, referred to the axes indicated, are x and y.

To derive the equation of the curve we isolate a portion of the cable OQ and draw the fbd (Fig. 153b). The forces acting are H, T, and the distributed load wx, which acts at midlength of x. These forces are in equilibrium; therefore $\Sigma M_Q = -Hy + wx(x/2) = 0$; whence

$$x^2 = \frac{2H}{w}y \qquad \text{or} \qquad y = \frac{w}{2H}x^2 \tag{1}$$

This is the standard form of the equation of a parabola; the axis of the parabola coincides with the y axis, and the vertex is at O.

To derive a formula for H we substitute in Eq. 1 for x and y their values for the point A, where $x = \frac{1}{2}a$ and $y = f$. This gives

$$H = \frac{wa^2}{8f} \tag{2}$$

Substituting this value of H, we can write Eq. 1

$$x^2 = \frac{a^2}{4f}y \qquad \text{or} \qquad y = \frac{4f}{a^2}x^2 \tag{1a}$$

To derive a formula for T we let ϕ = slope of the curve at Q and note that $T \sin \phi = wx$ and $T \cos \phi = H$. Squaring each of these equations and adding gives

$$T^2 = w^2x^2 + H^2 = w^2x^2 + \frac{w^2a^4}{64f^2} \tag{3}$$

At the points of suspension A and B, $x = \pm\frac{1}{2}a$, and substitution of either of these values for x in Eq. 3 shows that at these points, where the tension is maximum,

$$T_{\max} = \frac{1}{2}wa\left(1 + \frac{a^2}{16f^2}\right)^{\frac{1}{2}} \tag{4}$$

It is convenient to express $T_{\max}$ as a function of the total load wa and the sag ratio f/a, which we denote by n. The following table gives values of $T_{\max}/wa$ for various values of n:

$n =$	1	0.50	0.25	0.125	0.10	0.05	0.01
$\frac{T_{\max}}{wa} =$	0.515	0.559	0.707	1.118	1.346	2.550	12.81

It should be noted that the horizontal component of T is the same at all points of the cable and is equal to H.

LENGTH OF CABLE. To derive a formula for the length L of the cable from A to B we let ds = length of an elementary portion and make use of the relation $ds^2 = dx^2 + dy^2$, which holds for all plane curves. Since $ds = (dx^2 + dy^2)^{\frac{1}{2}}$ and, from Eq. 1a, $dy/dx = (8f/a^2)x = 8nx/a$, $ds = [1 + (8nx/a)^2]^{\frac{1}{2}}\, dx$, and so

$$\begin{aligned} L &= 2\int_0^{\frac{1}{2}L} ds = 2\int_0^{\frac{1}{2}a}\left[1 + \left(\frac{8nx}{a}\right)^2\right]^{\frac{1}{2}} dx \\ &= a\left\{\frac{1}{2}(1 + 16n^2)^{\frac{1}{2}} + \frac{1}{8n}\log_e\,[4n + (1 + 16n^2)^{\frac{1}{2}}]\right\} \end{aligned} \tag{5}$$

An *approximate formula for L*, much more convenient to use than Eq. 5 and sufficiently accurate for small sag ratios, may be derived as follows: Expanding

the coefficient of dx above by the binomial theorem gives

$$ds = 1 + \frac{1}{2}\left(\frac{8nx}{a}\right)^2 - \frac{1}{8}\left(\frac{8nx}{a}\right)^4 + \frac{1}{16}\left(\frac{8nx}{a}\right)^6 - \cdots$$

and integrating between limits as before gives

$$L = a\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4 + \frac{256}{7}n^6 - \cdots\right) \tag{6}$$

The following table gives values of L/a by the exact formula 5 and the approximate formula 6 for several values of n:

$n =$	1.0	0.50	0.25	0.125	0.10	0.05	0.01
L/a (exact) =	2.3234	1.4789	1.1478	1.0402	1.0260	1.0066	1.0003
L/a (approx.) =		1.8381	1.1505	1.0402	1.0260	1.0066	1.0003

Even when the sag ratio is as much as 0.25 the error in the approximate formula is only about 0.25 per cent, and for sag ratios less than 0.25 the formula yields very accurate results even if only the first three terms are used.

78. Catenary Cable. Let AOB in Fig. 153a represent the cable, suspended from A and B, a = span AB, f = sag, w = weight of cable or load per unit length of cable. Also let x and y be the coordinates of any point Q of the cable with respect to the axes indicated, ϕ = slope of the cable at Q, s = length of cable from O to Q, H = tension at O, and T = tension at Q.

To derive the equation of the curve we consider the portion of the cable OQ; the fbd is exactly like Fig. 153b except that the force wx is replaced by a force ws acting through the center of gravity of the cable segment OQ. The forces H, T, and ws are in equilibrium, and

$$\Sigma F_x = 0 \quad \text{gives } H = T\cos\phi = T\frac{dx}{ds} \quad \text{or} \quad T = H\frac{ds}{dx} \tag{1}$$

$$\Sigma F_y = 0 \quad \text{gives } ws = T\sin\phi = T\frac{dy}{ds} \quad \text{or} \quad T = ws\frac{ds}{dy} \tag{2}$$

Therefore $ws/H = \tan\phi = dy/dx$, or $H\,dy/dx = ws$; and differentiating gives $H\,d^2y/dx^2 = w\,ds/dx$. Substituting $(dx^2 + dy^2)^{\frac{1}{2}}$ for ds, and denoting dy/dx by u, this equation can be written

$$H\frac{du}{dx} = w(1 + u^2)^{\frac{1}{2}} \quad \text{or} \quad \frac{du}{(1+u^2)^{\frac{1}{2}}} = \frac{w}{H}dx$$

Integration gives

$$\log_e[u + (u^2+1)^{\frac{1}{2}}] = \log_e\left\{\frac{dy}{dx} + \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^{\frac{1}{2}}\right\} = \frac{w}{H}x + C$$

Since $dy/dx = 0$ when $x = 0, C = 0$. Denoting H/w by c, where c is the length of a cable segment whose weight $wc = H$, we write this equation

$$\frac{dy}{dx} + \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^{\frac{1}{2}} = e^{x/c}$$

Solving for dy/dx gives

$$\frac{dy}{dx} = \tfrac{1}{2}(e^{x/c} - e^{-x/c})$$

whence by integration

$$y = \tfrac{1}{2}(e^{x/c} + e^{-x/c})c + C = \tfrac{1}{2}(e^{x/c} + e^{-x/c})c - c$$

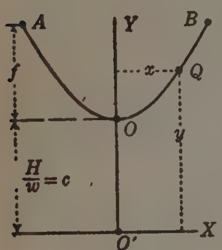


FIG. 154.

(The constant of integration is found to equal $-c$ from the condition that $y = 0$ when $x = 0$.)

With respect to a new origin O', distance c below O (Fig. 154), this equation can be written

$$y = \tfrac{1}{2}(e^{x/c} + e^{-x/c})c \tag{3}$$

or

$$y = c \cosh \frac{x}{c} \tag{4}$$

Equations 3 and 4 are the two common forms of the equation of the symmetrical catenary.

Equations 3 and 4 are not definitive, because both include the unknown quantity c. It is, in fact, not possible to derive explicit equations from which H, T, and L can be solved for directly, as was done for the parabolic cable. But we can develop relationships between f, a, L, c, and T from which, by trial solutions, any of the unknowns can be determined. This we now do.

The tension at any point Q is, by Eq. 1, $T = H(ds/dx) = H[1 + (dy/dx)^2]^{\frac{1}{2}}$, and, since, by Eq. 4, $dy/dx = \sinh x/c$,

$$T = H\left[1 + \left(\sinh \frac{x}{c}\right)^2\right]^{\frac{1}{2}} = H \cosh \frac{x}{c} = wy \tag{5}$$

The maximum value of T occurs at the supports, where $y = f + c$, and is $T_{\max} = w(f + c)$.

From Eq. 1, $wy = H\,(ds/dx) = wc\,(ds/dx)$, whence $(\cosh x/c)\,dx = ds$. Integrating and noting that the constant of integration $= 0$ because $s = 0$ when $x = 0$, we get $c \sinh x/c = s$, and, since $s = \tfrac{1}{2}L$ when $x = \tfrac{1}{2}a$, $c \sinh a/2c = \tfrac{1}{2}L$, or

$$\tfrac{1}{2}a = c \sinh^{-1} \frac{L}{2c} \tag{6}$$

By substituting the coordinates of $B(x = \frac{1}{2}a, y = f + c)$ in Eq. 5, we get

$$f + c = c \cosh \frac{a}{2c} \tag{7}$$

Substituting $[(\cosh a/2c)^2 - 1]^{\frac{1}{2}}$ for $\sinh a/2c$ in the equation $c \sinh a/2c = \frac{1}{2}L$ (just preceding Eq. 6) and then replacing $\cosh a/2c$ by $(f + c)/c$ from Eq. 7, we get

$$(f + c)^2 = c^2 + \tfrac{1}{4}L^2 \tag{8}$$

For convenience, the more important of the above equations are grouped below, and some of them are given in alternative form.

$$y = c \cosh \frac{x}{c} \tag{4}$$

$$T = wy \qquad T_{\max} = w(f + c) \tag{5}$$

$$\tfrac{1}{2}a = c \sinh^{-1} \frac{L}{2c} \qquad \text{or} \qquad \frac{a}{2c} = \sinh^{-1} \frac{L}{2c} \tag{6}$$

$$f + c = c \cosh \frac{a}{2c} \qquad \text{or} \qquad 1 + \frac{f}{c} = \cosh \frac{a}{2c} \tag{7}$$

$$(f + c)^2 = c^2 + \tfrac{1}{4}L^2 \qquad \text{or} \qquad \frac{c}{f} = \frac{1}{8}\left(\frac{L}{f}\right)^2 - \frac{1}{2} \tag{8}$$

The way in which these equations are to be used depends upon which dimensions are given and which are to be found. Thus, if L and f are given, c can be found directly from Eq. 8, then y from 4, T from 5, and a from 6. If a and f are given, c can be found from 7 by trial, and then y, T, and L can be found directly. If a and L are given, c can be found from 6 by trial and then y, T, and f can be found directly.

Trial solutions of Eqs. 7 and 8 are likely to be tedious, and the solution of some catenary problems can be greatly facilitated by means of diagrams plotted to show how T and L vary with the sag ratio. Various diagrams of this kind have been prepared for use in designing transmission lines.

DYNAMICS

CHAPTER VIII

RECTILINEAR MOTION

79. Preliminary. A **particle** is a body, or portion of a body, so small that no distinction need be made between the positions or motions of different points within it. Indeed, a particle may be conceived of as a materialized point, having substance but no dimensions, and it is sometimes convenient to speak of a particle as a point. By *elementary particle* is meant one of the infinite number of infinitesimal parts of which a body may be conceived as made up. For many purposes an entire body, even though not small, may be regarded as a particle if its dimensions and manner of motion are such that all parts of it move alike or nearly alike. In the discussions of this chapter any actual body used for purposes of illustration, such as a ball or car or projectile, may be thought of as a particle unless the contrary is definitely implied.

The line along which a moving particle travels is called the path of the particle or path of the motion. If the path is a straight line the motion is rectilinear; if it is a curved line the motion is curvilinear.

80. Position; Displacement. The position, at any instant, of a particle that has rectilinear motion is conveniently specified by its distance from a chosen fixed origin in the path. We call this distance the **s-coordinate**, and customarily denote it by s. To indicate which side of the origin the moving particle is on, s is given a sign. Either direction along the path may be taken as positive and the other as negative. Unless the contrary is stated, we assume for convenience that rectilinear motions discussed or referred to are horizontal, and we take direction to the right as positive.

The **displacement** of a particle for any given interval of time is Δs, the increment* in its s-coordinate; it is equal to the distance between the initial and final positions of the particle, and is positive when the final position is to the right of the initial position.†

If the direction of the motion does not change during a given interval of time, the displacement of the particle for that interval is equal to the distance it

* By increment in any quantity that varies with time is meant the change that quantity undergoes in some given interval of time; it is computed by subtracting the initial value of the quantity from the final value. Thus if at one instant t_1 there are 12 gallons of water in a vessel and at a later instant t_2 there are 10 gallons, the increment in the content of the vessel for the interval t_1 to t_2 is $10 - 12 = -2$ gallons.

† In some books the word " displacement " is defined differently, being used as we here use " s-coordinate."

travels. But if the motion is reversed during the interval the distance traveled is greater than the displacement. Thus, if a particle starts at $s = 3$, moves out to $s = 10$, and then back to $s = -6$, the displacement for the entire motion is $\Delta s = (-6) - (+3) = -9$, whereas the distance traveled is 7 (right) + 16 (left) = 23.

If a particle moves so that it undergoes equal displacements in all equal intervals of time it has *uniform motion;* if it moves in any other way it has *nonuniform motion.*

THE s–t EQUATION AND THE s–t GRAPH. If a particle moves so that s varies with time in accordance with some definite law the motion can be described by an s–t equation. Thus suppose that a particle starts from a point 10 ft to the right of the origin and moves to the right in such a way that at any later instant its distance in feet from the starting point is equal to twice the square of the elapsed time in seconds. Then the s–t equation for the motion is $s = 10 + 2t^2$, where t denotes elapsed time in seconds and s is in feet.

A graph showing the relation between s and t is called an s–t graph for the motion. It is customary to plot t horizontally and s vertically. Figure 155 is the s–t graph for the first 4 sec of the motion described above; it was plotted from the s–t equation of the motion. If the s–t equation of a motion is unknown but the motion can be observed for several pairs of corresponding or simultaneous values of s and t, then these values plotted afford an approximate s–t graph for the motion.

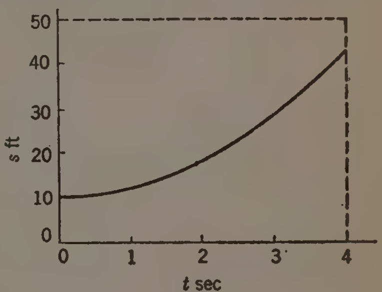


FIG. 155.

81. Velocity. The velocity of a moving particle is the time rate at which it changes position. This rate has magnitude, according to how fast the particle moves, and also direction, according to whether the particle moves to the right or to the left. The magnitude of the velocity is expressed in units of distance per unit of time, as miles per hour (mi/hr), feet per second (ft/sec), etc. The direction of the velocity is conveniently indicated by sign, and, in accordance with the rule for sign given in Art. 80, the velocity is considered positive when to the right and negative when to the left. We denote velocity by v.

In any uniform motion the velocity is constant and may be computed by dividing the displacement for any interval of time by the interval. That is,

$$v = \frac{\Delta s}{\Delta t} \tag{1}$$

where Δs denotes the displacement that occurs in the interval Δt.

In any nonuniform motion the velocity is not constant but changes continuously, and Eq. 1 does not, in general, give the velocity at any particular instant; it gives only the *average* velocity for the interval Δt. That is,

$$v_a = \frac{\Delta s}{\Delta t} \qquad (2)$$

where v_a denotes average velocity. The velocity at a particular instant is the limit of the average velocity for an interval that includes the instant in question as the interval is taken smaller and smaller. This limit is ds/dt, hence

$$v = \frac{ds}{dt} \qquad (3)$$

The above equations indicate that v_a and v are positive or negative according to the sign of Δs or ds. This is consistent with the rule for the sign of velocity already given; thus if Δs is positive the particle has moved to the right, and if ds/dt is positive s is increasing and the particle is moving to the right.

Speed. It is sometimes desirable to speak of the rate of motion without regard to direction, that is of the magnitude of velocity without regard to sign. We use the word "speed," in this sense, to express how fast a particle moves. If two cars are running at 60 mi/hr, one eastward and the other westward, their speeds are the same but their velocities differ in sign. We can express speed by $|ds/dt|$.

The average speed for an interval of time is the distance traveled during that interval divided by that interval. Speed and average speed as here defined have exactly the same meanings as in everyday speech.

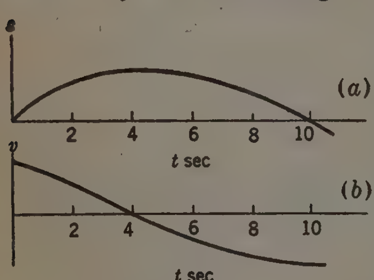


FIG. 156.

Query. Under what circumstances is the average velocity of a particle for a given interval equal to its average speed for that interval?

THE v–t EQUATION AND THE v–t GRAPH. Just as the positions of a particle at successive instants can be described by an s–t equation or an s–t graph, its velocity can be described by a v–t equation or by a v–t graph, which is a curve plotted with t as abscissa and v as ordinate.

The s–t and v–t graphs for a motion are related in an obvious way, for it follows from the equation $v = ds/dt$ that the velocity is represented by the slope of the s–t graph. From this relationship it is easy to tell, in a general way, how the velocity varies and what the v–t curve is like simply from inspection of the s–t graph. For example, suppose that Fig. 156a is the s–t graph for a certain motion in a straight path (the path is not shown). It is apparent that for the first 4 sec of the motion the velocity is positive, because the graph

slopes upward, and that the speed decreases because the graph becomes less and less steep. At $t = 4$ the velocity becomes zero, because the s–t graph becomes horizontal. From then on the velocity is negative, because the graph slopes downward, but the speed increases because the graph becomes steeper and steeper. We could describe the motion thus: When $t = 0$ the particle is at the origin and is moving to the right; it moves to the right, going slower and slower, for 4 sec, and then moves to the left, going faster and faster, until after a total elapsed time of 10 sec it returns to the origin.

We can also, by inspection of the s–t graph in Fig. 156a, determine the approximate shape of the v–t graph. We know that it must slope downward somewhat as shown in Fig. 156b, and that it must pass through zero at $t = 4$. But it may be straight, or it may curve otherwise than as shown. The way in which the slope of the s–t graph is measured and interpreted to determine the actual value of the velocity is explained in the next article.

82. Calculation of Velocity. If the s–t equation for a motion is known the v–t equation can be found by differentiation, and the value of v at any instant can be calculated therefrom.

EXAMPLE 1. In the example of Art. 80 a particle moves according to the equation $s = 10 + 2t^2$, the units being the foot and second. It is required to derive the v–t equation for this motion, and to determine the velocity when $t = 4$ and also when the particle has traveled 40 ft. It is further required to determine the average velocity for the interval $t = 0$ to $t = 4$.

Solution. Since $v = ds/dt$, $v = 4t$, and this is the general equation for the velocity. When $t = 4$, $v = 16$ ft/sec.

Since the v–t equation shows that v is always positive, the particle always moves to the right, and, since $s = 10$ initially (when $t = 0$), $s = 50$ after the particle has traveled 40 ft. Substituting 50 for s in the s–t equation and solving for t gives $t = \sqrt{20}$, and substituting $\sqrt{20}$ for t in the v–t equation gives $v = 17.9$ ft/sec.

When $t = 0$, $s = 10$, and when $t = 4$, $s = 10 + 2(4^2) = 42$; therefore Δs for the interval $t = 0$ to $t = 4$ is $42 - 10 = +32$, and for this interval $v_a = \Delta s/\Delta t = +32/4 = +8$ ft/sec.

EXAMPLE 2. Figure 157 represents a rigid rod of length L; it is made to move so that its lower end B slides along the horizontal floor OX while its upper end C slides along the vertical wall OY. Suppose that B starts at O and moves to the right with a constant velocity v_B so that its distance from O at any instant is $x = v_B t$. It is required to derive a formula for the velocity of C and to determine this velocity for the numerical data $L = 20$ ft, $v_B = 10$ ft/sec, $\theta = 40°$.

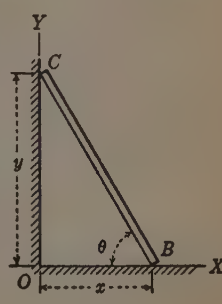


FIG. 157.

Solution. The s-coordinate of C, measured from O, is here denoted by y, and the upward direction is considered positive. Then

$$y = (L^2 - x^2)^{\frac{1}{2}} = (L^2 - v_B^2 \cdot t^2)^{\frac{1}{2}}$$

Therefore the velocity of C is

$$v_C = \frac{dy}{dt} = \frac{1}{2}(L^2 - v_B^2t^2)^{-\frac{1}{2}}(-2v_B^2t)$$

This is the v–t formula for the motion of C. You should show that it is readily simplified to

$$v_C = -v_B \cot \theta$$

When $\theta = 40°$ and $v_B = 10$ ft/sec, substitution in the formula gives $v_C = -11.92$ ft/sec.

The simplified expression shows that the velocity of C is wholly determined by the slope of the rod and the velocity of the lower end; it also shows that v_C becomes infinite when $\theta = 0$, and that therefore an actual rod could not be made to move all the way to the horizontal position in the manner described.

In the above derivation it is assumed that B is at O when $t = 0$ and that v_B is to the right. You should show that the formula applies no matter where B starts and no matter whether B moves to the right or to the left, provided that v_B corresponds in sign to the direction of the motion.

Graphical solution. If the relation between s and t cannot be expressed by an equation, then the formula $v = ds/dt$ cannot be used directly to find the velocity. But if the s–t graph can be drawn its slope can be evaluated directly in terms of v.

EXAMPLE 3. In a certain launching the ship moved through the distances given after s in the times given after t in the schedule below:

$t = 0$	2	4	6	8	10	12	14	16 sec
$s = 0$	3.4	9.3	17.3	27.4	39.6	53.4	69.4	88.0 ft

It is required to determine the velocity of the ship at $t = 6$ sec.

Solution. The s–t graph for the motion is constructed to some convenient scale (Fig. 158), and a tangent is drawn* to this curve at the point corresponding to $t = 6$. The slope of the curve at this point is the slope of the tangent, which can be found by taking any arbitrary distance AB along the tangent and dividing its vertical projection CB by its horizontal projection AC. If CB is measured in terms of feet and AC in terms of seconds, then this ratio gives the velocity directly in terms of feet per second. As drawn in Fig. 158, $CB = 45.5$ ft and $AC = 10$ sec, and so $v = 45.5 \div 10 = 4.55$ ft/sec. (It is convenient to take AC arbitrarily as some convenient whole number of time units, as was done in this example.)

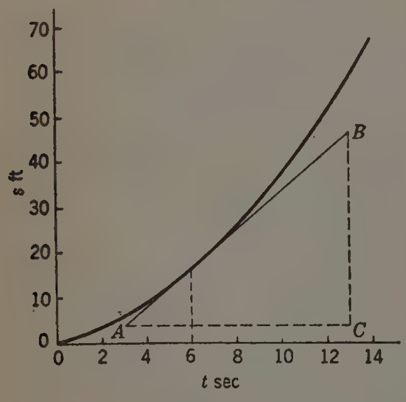


FIG. 158

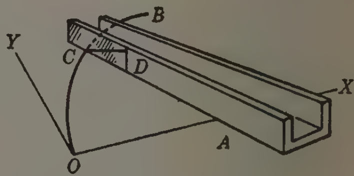


FIG 159.

* A tangent to a curve can be thus drawn by eye or by means of any of several instruments that have been devised for the purpose. One such instrument is represented in Fig. 159. It consists of a metal straight-edge A with a portion of one side polished to a mirror. OB represents a curve on a piece of paper across which the straight-edge is laid at random but so that a portion of the curve is reflected from the mirror. The image CD and the curve CO are not smoothly continuous; there is a cusp at C. If the instrument is turned about C until the cusp disappears, the curve merging smoothly into its image, then the straight-edge A is normal to the curve OB at C. The normal at C having been located, the tangent is easily drawn. The

VELOCITY AS THE LIMIT OF THE AVERAGE VELOCITY. From data such as are given in the schedule of the above example, it is possible to determine the approximate velocity at any instant by making use of the relation (pointed out in Art. 81) that the velocity at an instant is the limit of the average velocity for an interval that includes that instant as the interval is taken shorter and shorter. As an illustration of the method Ex. 3 will be solved in this way. In the table are given the average velocities for intervals terminating at the end of the sixth second ($t = 6$). It is apparent that v_a increases as the interval

INTERVAL	t	Δs	$v_a = \Delta s/\Delta t$
0 to 6	6	$17.3 - 0 = 17.3$	2.88
2 to 6	4	$17.3 - 3.4 = 13.9$	3.48
4 to 6	2	$17.3 - 9.3 = 8.0$	4.00

shortens, and, if the reasonable assumption is made that v_a continues to increase in about the same way as the interval approaches zero, it appears that the limiting value must be about 4.5 ft/sec; this is an approximate value of v at $t = 6$. Obviously we could "close in" on the instant in question in any one of three ways, namely, by using intervals terminating at that instant, or intervals beginning at that instant, or intervals "centered" at that instant. Note that v_a for any interval is represented graphically by the slope of the corresponding secant line on the s–t graph, and that the slope of the secant approaches the slope of the tangent as the interval approaches zero.

83. Acceleration. It is pointed out in Art. 81 that if a particle has non-uniform motion its velocity is continuously changing; in each successive time interval the particle acquires or takes on some increment of velocity. The time rate at which the velocity changes is the **acceleration** of the particle. This rate has magnitude, according to how much velocity is being taken on per unit of time, and direction, according to the direction of the velocity that is being taken on. The magnitude of the acceleration is expressed in units of velocity per unit of time, as miles per hour per minute (mi/hr/min) or feet per second per second (ft/sec/sec or ft/sec²). The direction of the acceleration is conveniently indicated by sign, plus when to the right, minus when to the left.

If the velocity changes uniformly (equal velocity-increments in all equal intervals of time), then the acceleration is constant and may be computed by dividing the velocity-increment for any interval of time by the interval. That is,

$$a = \frac{\Delta v}{\Delta t} \tag{1}$$

where Δv denotes the velocity-increment for the interval Δt.

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principle of this instrument is the basis of Wagener's derivator (see Gramberg's *Technische Messungen*), by means of which the slope of a curve at any point can be read directly, without drawing the tangent or normal. An autographic form of (mirror) derivator has been devised by A. Elmendorf (see *Sci. Am. Suppl.* for Feb. 12, 1916).

If the velocity does not change uniformly, then the acceleration is not constant but changes continuously, and Eq. 1 does not, in general, give the acceleration at any particular instant but gives only the *average* acceleration for the interval Δt. That is,

$$a_a = \frac{\Delta v}{\Delta t} \tag{2}$$

where a_a denotes average acceleration. The acceleration at a particular instant is the limit of the average acceleration for an interval that includes the instant in question as the interval is taken smaller and smaller. This limit is dv/dt; that is,

$$a = \frac{dv}{dt} \tag{3}$$

If we substitute for v its value ds/dt, Eq. 3 becomes $a = d^2s/dt^2$.

The above equations indicate that a_a and a are positive or negative according to the sign of Δv or dv, and this is consistent with the rule for the sign of acceleration given above. It should be particularly noted that the sign of the acceleration does not depend merely on whether the speed is increasing or decreasing.* If a particle is moving to the right and going faster and faster it has positive acceleration, but it also has positive acceleration when moving to the left and going slower and slower. In both cases positive velocity is being taken on and the direction of the acceleration is to the right. The magnitude of the acceleration, without regard to sign, represents the rate of change of speed.

THE a–t EQUATION AND THE a–t GRAPH. The acceleration of a particle can be described by an equation or by a graph just as the position and the velocity can. The a–t graph, constructed with t as abscissa and a as ordinate, has the same relation to the v–t graph that the v–t graph has to the s–t graph, for it follows from Eq. 3 that the acceleration is represented by the slope of the v–t curve.

Suppose that Fig. 160a is the v–t graph for a certain motion along a straight path. It is apparent from this graph that for the first 5 sec the acceleration is negative, because the graph slopes downward, and that the magnitude of the acceleration decreases because the graph becomes less and less steep. At $t = 5$ the acceleration is zero because the slope is zero. During the last 10 sec the acceleration is positive, because the graph slopes upward, and the magnitude of the acceleration increases because the graph becomes steeper and steeper. We could describe the motion somewhat as follows: The particle is initially

* It is customary in everyday speech to use the word "acceleration" to connote rate of increase of speed or "pick up," and the word "deceleration" to connote rate of decrease of speed or retardation. This convenient usage is also common among engineers, but it is not appropriate in a general discussion of motion.

moving to the left, and it moves to the left faster and faster for 5 sec, then slower and slower for the next 5 sec. It then reverses the direction of its motion and moves to the right faster and faster for the last 5 sec. From $t = 0$ to $t = 5$ the speed increases, less and less rapidly; from $t = 5$ to $t = 10$ the speed decreases, more and more rapidly; from $t = 10$ to $t = 15$ the speed increases, more and more rapidly.

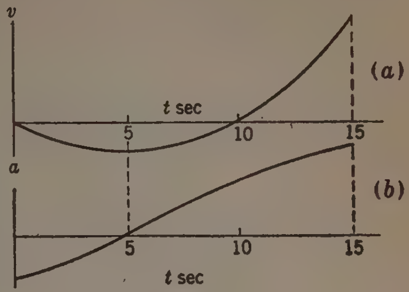

FIG. 160.

One can, by inspection of the v–t graph, determine the approximate form of the a–t graph (Fig. 160b); evidently it starts with some negative value and rises throughout the 15-sec interval, passing through zero at $t = 5$. But it may be made up of straight lines, or may curve otherwise than as shown. The way in which the slope of the v–t graph is measured and interpreted to determine the actual value of the acceleration is explained in the next article.

84. Calculation of Acceleration. If the v–t equation for a motion is known, the a–t equation can be found by differentiation, and the value of a at any instant can be calculated therefrom.

EXAMPLE 1. A particle moves in the positive direction along a straight line so that the velocity in miles per hour is always equal to one-tenth the square of the time in seconds after starting. It is required to develop the formula for the acceleration of the particle and to determine the acceleration 3 sec after motion starts.

Solution. From the description of the motion it is evident that $v = 0.1t^2$. Therefore $a = dv/dt = 0.2t$. This is the a–t equation for the motion. When $t = 3$, $a = 0.6$ mi/hr/sec.

EXAMPLE 2. With reference to Ex. 2, Art. 82, it is required to derive the a–t equation for the motion of C, and to determine the acceleration of C for the special case $L = 10$ ft, $v_B = 5$ ft/sec, when $\theta = 90°$, 45°, and 5°.

Solution. The v–t equation for the motion of C was found to be

$$v_C = \tfrac{1}{2}(L^2 - v_B{}^2t^2)^{-\frac{1}{2}}(-2v_B{}^2t)$$

Therefore

$$a_C = \frac{dv_C}{dt} = -\frac{(L^2 - v_B{}^2t^2)^{\frac{1}{2}}v_B{}^2 - v_B{}^2t[\frac{1}{2}(L^2 - v_B{}^2t^2)^{-\frac{1}{2}}(-2v_B{}^2t)]}{L^2 - v_B{}^2t^2}$$

This is the a–t equation for the motion of C. It is readily simplified to the form $a_C = -v_B{}^2/(L \sin^3 \theta)$ (you should perform this reduction).

If $L = 10$ and $v_B = 5$, substitution of these values and of the value of $\sin\theta$ gives $a_C = -2.5$ ft/sec² when $\theta = 90°$, $a_C = -7.07$ ft/sec² when $\theta = 45°$, and $a_C = -3770$ ft/sec² when $\theta = 5°$. The minus sign means that a_C is downward, that having been taken as the negative direction in the original example. Since v_C is also negative, the results mean that the speed of C is increasing at each instant, and that it increases faster and faster as the bar becomes more and more nearly horizontal.

(We could have obtained the formula for a_C by differentiating the simplified formula

$v_C = -v_B \cot\theta$, but this would involve evaluating $d\theta/dt$, which you are not yet prepared to do. The meaning and evaluation of $d\theta/dt$ are explained in Arts. 97 and 110.)

Graphical solution. If the relation between v and t cannot be expressed by an equation, then the formula $a = dv/dt$ cannot be used directly to find the acceleration. But if the v–t graph can be drawn its slope can be evaluated in terms of a.

EXAMPLE 3. Corresponding values of velocity and time after the starting of an electric railway car were found to be as given in the schedule below:

$t = 0$	1	2	3	4	5	6	7	8	9	10	sec
$v = 0$	2.8	5.3	7.7	9.9	11.9	13.7	15.2	16.4	17.3	18.0	mi/hr

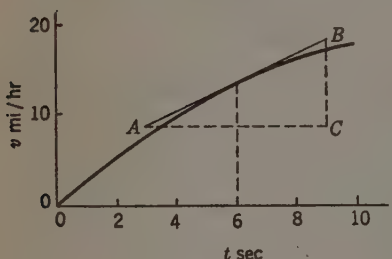

FIG. 161.

It is required to determine the approximate acceleration of the car at $t = 6$ sec.

Solution. The v–t graph for the motion is constructed to some convenient scale (Fig. 161), and a tangent is drawn to this curve at the point corresponding to $t = 6$. The slope of this tangent is measured in terms of acceleration by dividing AC (arbitrarily taken as 6 sec) into CB (which, according to the v scale, represents 10 mi/hr). The acceleration is therefore $a = 10 \div 6 = 1.67$ mi/hr/sec.

ACCELERATION AS THE LIMIT OF THE AVERAGE ACCELERATION. From data such as are given in the schedule of the above example, it is possible to determine the approximate value of the acceleration at any instant by estimating the limit of the average acceleration, following a procedure similar to that illustrated in Art. 82, where the velocity is found in this way from average velocities. To solve Ex. 3 in this way the average accelerations are calculated for shorter and shorter periods, all starting at $t = 6$. The necessary computations are tabulated.

INTERVAL	Δt	Δv	$a_a = \Delta v/\Delta t$
6 to 10	4	$18.0 - 13.7 = 4.3$	1.075
6 to 9	3	$17.3 - 13.7 = 3.6$	1.20
6 to 8	2	$16.4 - 13.7 = 2.7$	1.35
6 to 7	1	$15.2 - 13.7 = 1.5$	1.50

The average acceleration increases as the interval shortens, and it is apparent that its limiting value at $t = 6$ is about 1.65 mi/hr/sec; this is an approximate value of a at $t = 6$.

85. Study of a Motion by Integration. In the preceding article we established the relations $v = ds/dt$ and $a = dv/dt$, and showed how, by differentiation, the v–t law can be determined from the s–t law, and the a–t law from the v–t law. By the reverse process of integration one can determine the v–t law

from the a–t law and the s–t law from the v–t law, using the relations

$$ds = v\,dt \quad \text{or} \quad s = \int v\,dt \qquad \text{and} \qquad dv = a\,dt \quad \text{or} \quad v = \int a\,dt$$

Integration of these equations leads to formulas for s and v, provided that the constants of integration can be found from the known circumstances of the motion. In any event, the increment in abscissa and the increment in velocity for any time interval t_1 to t_2 can be found by integrating between limits; thus

$$\Delta s = \int_{t_1}^{t_2} v\,dt \qquad \text{and} \qquad \Delta v = \int_{t_1}^{t_2} a\,dt$$

EXAMPLE 1. A passenger elevator, going up, starts from rest and attains its maximum velocity in 3.6 sec and in three stages. For the first 0.3 sec it moves according to the equation $a = 20t$; for the next 3 sec the acceleration is constant; for the last 0.3 sec the acceleration decreases linearly to zero. (Units are the foot and second.) It is required to derive the a–t, v–t, and s–t equations for the motion, dating time from the instant of starting, and to determine v and s at $t = 0.3$, 3.3, and 3.6 sec.

Solution. For the interval $t = 0$ to $t = 0.3$, $a = 20t$. Therefore

$$v = \int 20t\,dt = 10t^2 + C_1 = 10t^2 \quad (C_1 = 0 \text{ since } v = 0 \text{ when } t = 0)$$

and

$$s = \int 10t^2 = \tfrac{10}{3}t^3 + C_2 = \tfrac{10}{3}t^3 \quad (C_2 = 0 \text{ since } s = 0 \text{ when } t = 0)$$

We now have the a–t, v–t, and s–t equations for the initial period, and for $t = 0.3$ they give: $a = 6$ ft/sec²; $v = 0.9$ ft/sec; $s = 0.09$ ft.

For the interval $t = 0.3$ to $t = 3.3$, $a = 6$. Therefore

$$v = \int 6dt = 6t + C_3$$

It is known that $v = 0.9$ when $t = 0.3$, and on substituting these simultaneous values of v and t and solving for C_3 it is found that $C_3 = -0.9$. Hence the v–t equation is

$$v = 6t - 0.9$$

Therefore

$$s = \int (6t - 0.9)dt = 3t^2 - 0.9t + C_4$$

It is known that $s = 0.09$ when $t = 0.3$, and on substituting these simultaneous values of s and t and solving for C_4 it is found that $C_4 = 0.09$. Hence the s–t equation is

$$s = 3t^2 - 0.9t + 0.09$$

You should now be able to complete the solution by deriving the motion equations for the interval $t = 3.3$ to $t = 3.6$. Do this, and also sketch the a–t, v–t, and s–t graphs for the entire 3.6-sec period, plotting the values found and making use of the slope relationships discussed in Arts. 81 and 83 to determine the approximate shapes of the curves.

EXAMPLE 2. A particle moves with an acceleration given by $a = 2t^2 - 10$. It is required to determine the increment in the velocity for the interval $t = 1$ to $t = 6$.

Solution. $\Delta v = \int_1^6 (2t^2 - 10)dt = \tfrac{2}{3}t^3 - 10t\Big]_1^6 = 93.3$

Note that it is not necessary to know the actual velocity at either instant in order to determine how much it changes during the interval.

Graphical solution. If the a–t and v–t laws cannot be expressed by equations, then solution by integration, as illustrated in the preceding examples, is not possible. But if adequate data are available the a–t and v–t graphs can be constructed and integrated graphically, as in Ex. 3 below. Even if the given a–t or v–t graph is linear, so that its equation can readily be written, it may be easier to employ graphical integration, as in Ex. 4 below.

EXAMPLE 3. Figure 162a represents a portion of the a–t graph for a projectile fired vertically upward with an initial velocity of 792 m/sec; the acceleration, being downward, is regarded as negative. It is required to determine the velocity of the projectile and its distance above the starting point 5 sec after motion commences.

Solution. The areas of the strips between the successive vertical lines drawn on the graph represent increments in velocity for 1-sec intervals. These strips are nearly trapezoidal, and hence each Δv can be found with sufficient accuracy by multiplying the mid-ordinate of the corresponding strip (measured in meters per second per second) by the width of the strip (measured in seconds).* These products are recorded on the figure. The total increment

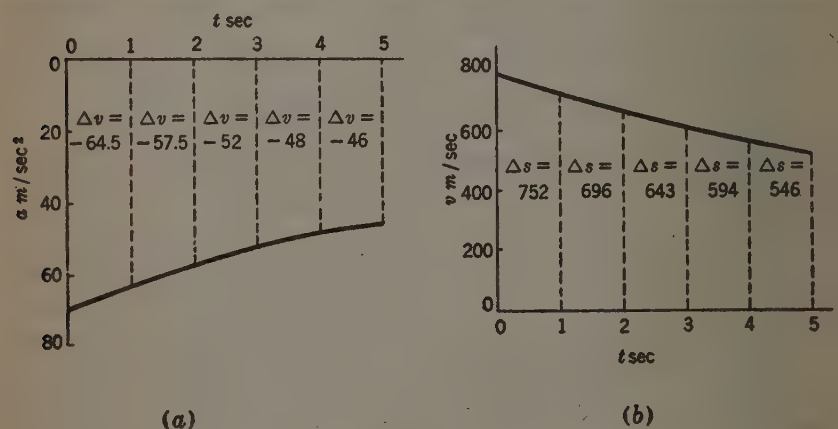

FIG. 162.

* To determine the area under a given curve, various methods may be employed. Thus if the area to be found is divided into a large number of narrow vertical strips, each strip may be regarded as a trapezoid, and its area found by multiplying the ordinate at its middle by its width. Or the arithmetical mean of all such mid-ordinates may be taken as the average ordinate to the part of the curve in question. The accuracy of this method is greater the larger the number of strips taken.

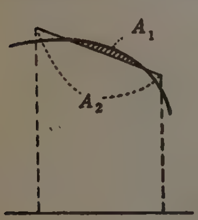

FIG. 163.

Another method is to divide the area into vertical strips as before and then to replace the bounding curve by a straight line so drawn (by eye) as to make the area between this line and the curve the same above as below the curve. Thus in Fig. 163 the straight line is drawn so as to make the shaded area A_1 equal to the total unshaded area A_2. Then the area of the trapezoid is equal to the area under the curve, and the mean of the mid-ordinates of all such trapezoids represents the average ordinate to the curve.

in velocity for the 5-sec period is the sum of these quantities, or −268 m/sec. Since the initial velocity is 792 m/sec, the final velocity is 792 − 268 = 524 m/sec.

To find the displacement of the projectile we construct the v–t graph and integrate it graphically. The necessary calculations for getting values of v are tabulated thus:

INTERVAL	Δv (from a–t graph)	CUMULATIVE Δv (from $t = 0$ to end of interval)	v AT END OF INTERVAL (792 + cumulative Δv)
0 to 1	−64.5	−64.5	727.5
1 to 2	−57.5	−122.0	670.0
2 to 3	−52.0	−174.0	618.0
3 to 4	−48.0	−222.0	570.0
4 to 5	−46.0	−268.0	524.0

From these calculated values of v, the v–t graph (Fig. 162*b*) is constructed, and the displacement for each successive 1-sec interval is then computed by multiplying the mid-ordinate of each vertical strip (measured in meters per second) by its width (measured in seconds). These products are recorded on the figure. The total displacement for the 5-sec interval is the sum of these quantities, or 3231 m. Therefore the distance of the projectile above its starting point 5 sec after motion commences is 3231 m.

EXAMPLE 4. A particle moves for 13 sec in accordance with the a–t graph shown in Fig. 164*a*; when $t = 0$, $v = -12$ and $s = -40$. It is required to construct the v–t and s–t graphs directly from the a–t graph, without making use of equations.

Solution. Areas under the a–t graph for the time intervals 0–5, 5–7, and 7–13 represent increments of velocity for these intervals; calculation of these areas gives the values of Δv indicated on Fig. 164*a*. With these the v–t graph is built up; starting with $v = -12$, the Δv's are added successively, yielding the curve shown in Fig. 164*b*. Note that the slope of this v–t graph is everywhere proportional to a.

Areas under this v–t graph for the time intervals 0–2, 2–5, 5–7, and 7–13 represent increments of s for these intervals, and calculation of these areas gives the values of Δs indicated on Fig. 164*b*. With these the s–t graph is built up; starting with $s = -40$, the Δs's are added successively. Values of s are now known for $t = 0$, 2, 5, 7, and 13. Through the five points thus located the s–t curve is drawn, use being made of the fact that its slope is everywhere proportional to the velocity. Thus the s–t curve (Fig. 164*c*) initially slopes downward, becomes less steep as v approaches zero, becomes horizontal at $t = 2$ when $v = 0$, then slopes upward more and more steeply as long as v increases ($t = 2$ to 5), then is a straight line as long as v is constant ($t = 5$ to 7), then has a diminishing slope as v decreases, and again becomes horizontal at $t = 13$ when again $v = 0$.

It is, of course, easy to write the a–t equations for the three stages of the motion and to derive the v–t and s–t equations from them as is done in Ex. 1. Thus:

($t = 0$ to 5) $a = 6$, $v = 6t + C$, where $C = -12$, since $v = -12$ when $t = 0$
$s = 3t^2 - 12t + C$, where $C = -40$, since $s = -40$ when $t = 0$

($t = 5$ to 7) $a = 0$, $v = 0 + C$, where $C = 18$, since $v = 18$ when $t = 5$
$s = 18t + C$, where $C = -115$, since $s = -25$ when $t = 5$

($t = 7$ to 13) $a = -3$, $v = -3t + C$, where $C = 39$, since $v = 18$ when $t = 7$
$s = -1.5t^2 + 39t + C$, where $C = -188.5$, since $s = 11$ when $t = 7$

You should check these equations by testing them for $t = 2$, 5, 7, and 13.

Areas may also be found, of course, by direct measurement with a planimeter. When this is done, the scale of the area is the product of the scales of the abscissas and ordinates. Thus, if on the a–t graph 1 in. = m ft/sec² and 1 in. = n sec, 1 sq in. = mn ft/sec.

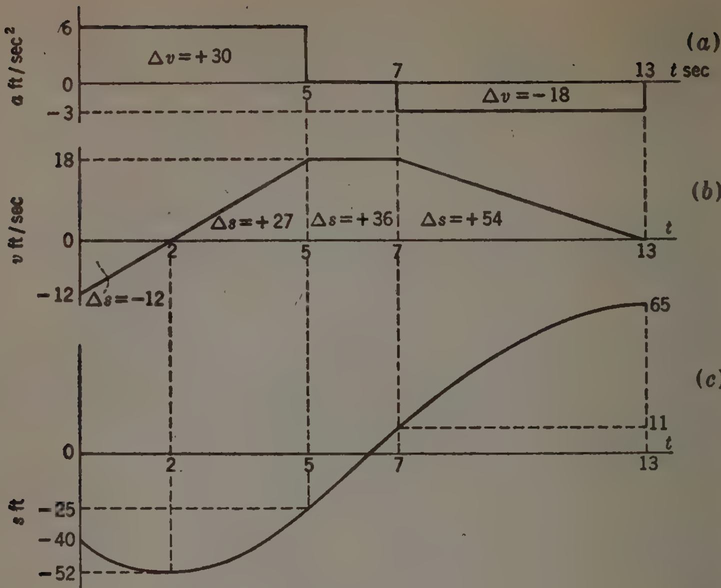

FIG. 164.

USE OF THE a–s RELATIONSHIP. If a particle moves so that a bears a known relationship to s, then the v–s relationship can be determined, because $a = dv/dt = (dv/ds)(ds/dt) = (dv/ds)v$, and so $a\,ds = v\,dv$. If the a–s equation is known, it is therefore possible to derive the v–s equation by integrating. Thus $\int a\,ds = \int v\,dv$, or, with corresponding limits,

$$\int_{s_1}^{s_2} a\,ds = \int_{v_1}^{v_2} v\,dv \qquad \text{hence} \qquad \int_{s_1}^{s_2} a\,ds = \tfrac{1}{2}v_2^2 - \tfrac{1}{2}v_1^2$$

If the a–s law cannot be expressed by an equation, the value of $\int a\,ds$, which represents half the increment in the square of the velocity, can be determined by graphical integration if sufficient data are available to establish the a–s graph.

EXAMPLE 5. A particle moves in accordance with the equation $a = 10 - 4s$, where a is in feet per second per second and s is in feet. When $s = 0$, $v = 20$. It is required to determine v when $s = 10$, and to determine the value of s when v reverses direction.

Solution. Since $a\,ds = v\,dv$, we have, on substituting for a the given equation and for the limits the appropriate values,

$$\int_0^s (10 - 4s)\,ds = \int_{20}^{v} v\,dv$$

whence $10s - 2s^2 = \frac{1}{2}(v^2 - 400)$, or $v = (400 + 20s - 4s^2)^{\frac{1}{2}}$. This is the v–s equation. On substituting 10 for s, we find $v = 14.14$ ft/sec. Setting the expression for v equal to zero and solving for s, we find $s = 12.81$ and -7.81. Since v reverses direction for each of these

values of s, the particle simply moves back and forth continuously between these limiting positions.

EXAMPLE 6. Figure 165 is an a–s graph (from computed data) for the motion of a projectile along the bore of a gun, from starting point to muzzle. It is required to determine the approximate muzzle velocity.

Solution. The area under the curve is divided into vertical strips as indicated in the figure, and the area of each such strip is determined as described in the footnote on p. 124. These areas are recorded on the figure. The total area is the sum of these, equal to 1,280,000 (ft/sec)². This represents half the increment in the square of the velocity, and since the initial velocity is zero the square of the final velocity is 2 × 1,280,000, and so the muzzle velocity is 1600 ft/sec.

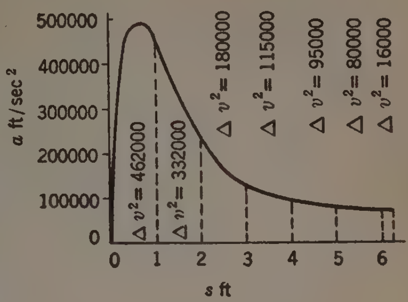

FIG. 165.

86. Components of Displacement, Velocity, and Acceleration. Since in rectilinear motion there are, for displacement, velocity, and acceleration, only two possible directions, and since these two directions can be distinguished by sign, displacement, velocity, and acceleration have thus far been treated as algebraic quantities varying only in magnitude and sign. But actually they are vector quantities, and each can be represented by a vector of appropriate length and direction.* These vectors can be resolved into components which

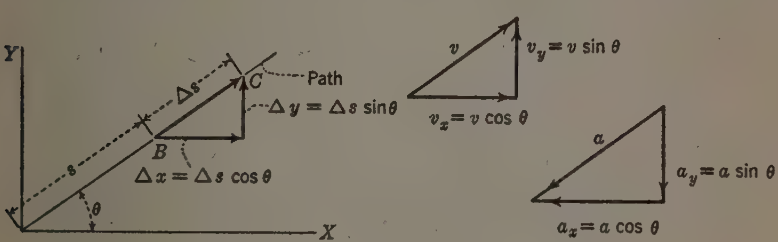

FIG. 166.

represent components of the displacement, velocity, or acceleration. The components most often useful are those parallel to rectangular axes.

Consider a particle that moves up and to the right with decreasing speed along the straight path indicated in Fig. 166. During a given interval the particle moves from B to C, and at some particular instant it has a velocity v and an acceleration a. Assume rectangular axes x and y. In the figure are

* Displacement, velocity, and acceleration are each completely described by a corresponding vector drawn anywhere, since these quantities are not regarded as having position. In this respect they differ from a force, which is a localized vector quantity, completely described only when its line of action, as well as its magnitude and direction, is given.

shown vectors that represent Δs and its components Δx and Δy, the velocity v and its components v_x and v_y, and the acceleration a and its components a_x and a_y. Obviously Δx and Δy represent increments in the coordinates x and y of the particle. Now v_x and v_y, as defined, are the time rates at which x and y respectively are changing, for

$$v_x = v \cos \theta = \frac{ds}{dt} \cos \theta = \frac{ds \cos \theta}{dt} = \frac{dx}{dt}$$

and similarly $v_y = dy/dt$. Also a_x and a_y, as defined, are the time rates at which v_x and v_y respectively are changing, for

$$a_x = a \cos \theta = \frac{dv}{dt} \cos \theta = \frac{dv \cos \theta}{dt} = \frac{dv_x}{dt}$$

and similarly $a_y = dv_y/dt$.

87. Mass. In the foregoing discussion nothing is said concerning the cause of motion; it is simply assumed that a particle moves in some particular way. In the next article it is explained that the motion of a particle is governed by the forces that act upon it and by its own mass. The nature and occurrence of forces are discussed in preceding chapters; in this article we discuss the meaning of mass.

By **mass** of a body is meant the amount of material or substance in that body. Since weight is proportional to mass, the mass of a body is most easily determined by comparing its weight with the weight of another body arbitrarily chosen as a unit or standard of mass. "The primary standard of mass for this country is United States Prototype Kilogram 20, which is a platinum-iridium standard kept at the National Bureau of Standards."* This is an accurate copy of the International Prototype Kilogram, kept at the International Bureau of Weights and Measures. All other United States standards of mass are multiples or submultiples of Kilogram 20; thus the avoirdupois pound† = 0.4535924277 kg, the gram = 0.001 kg, etc. But there exist many more or less accurate physical pounds, grams, etc., for use in weighing.

The terms kilogram, pound, etc., are also used to designate units of force, the unit of force in each instance being the weight of the corresponding unit of mass. But the mass of a body is invariable, as long as no material is added to or taken from it, whereas the weight of a body varies slightly according to geographical location (Art. 3). Hence, in order to afford an accurate measurement of mass, weighing must consist in a direct comparison between the weight of the body under consideration and the weight, at the same place, of a body of known mass.

It should be noted that a body may be weighed for either of two purposes,

* *Letter Circular LC449*, National Bureau of Standards.

† The British Imperial Pound mass is a platinum standard kept in the Standards Department of the Board of Trade, London.

namely, (i) to find out how much there is of it, or (ii) to find out how heavy it is. The weighing can be done either by a beam scale or by a spring scale; both are, in principle, devices for finding how heavy a body is, and both are used to measure not only earth-pulls but other forces also. For example, the force required to break a sample of steel is measured by a beam or spring scale in some types of testing machines, and the pulling strength of draft horses is measured by spring scales. A good beam scale indicates the same weight for a given body no matter where the weighing is done; it correctly tells "how much" but incorrectly "how heavy." An accurate spring scale indicates different weights for a given body at different places; it correctly tells "how heavy" and incorrectly "how much." But in either case the error noted is small and for most purposes negligible.

MASS-CENTER. In Art. 71 the center of gravity of a body is defined as that point through which the resultant of the weights of all the constituent particles of the body always acts. For some purposes the mass of a body may be considered as concentrated at its center of gravity, and in discussions not involving weight it is convenient and appropriate to call this point the center of mass or **mass-center.**

In Art. 73 it is also shown that $W_1x_1 + W_2x_2 + W_3x_3 \cdots = W\bar{x}$, where W_1, W_2, etc., are the weights, x_1, x_2, etc., are the x coordinates of the constituent particles of the body, W is the weight of the body, and $\bar{x}$ is the x coordinate of its center of gravity. Since mass is proportional to weight we may write $dm_1 \cdot x_1 + dm_2 \cdot x_2 + dm_3 \cdot x_3 \cdots = m\bar{x}$, where dm_1, dm_2, etc., are the masses of the constituent particles, m is the mass of the body, and $\bar{x}$ is the x coordinate of the center of gravity or mass-center.

88. Laws of Motion of a Particle. The relationship between the motion of a particle and the forces that act on it can be stated thus: (i) A particle acted on by no forces, or by a system whose resultant is nil, is either at rest or in uniform rectilinear motion. (ii) A particle acted on by a single force, or by a system whose resultant is a single force, is accelerated; the acceleration is in the direction of the force and is directly proportional to the force and inversely proportional to the mass of the particle.*

* The fundamental facts were discovered by Galileo (1564–1642), who made the first scientific experiments in dynamics of which we know. (See *Dialogues Concerning Two New Sciences*, translated by H. Crew and Alfonso De Salvio, The Macmillan Company, New York.) The formal presentation of the principles was made by Newton (1642–1727), who stated them in the form of these three axioms known as Newton's laws of motion:

Law I. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

Law II. Change of motion is proportional to force applied, and takes place in the direction of the straight line in which the force acts.

Law III. To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed. (Thomson and Tait, *Treatise on Natural Philosophy*.)

With respect to a particle, therefore, rest and uniform rectilinear motion may be thought of as *natural* states which persist if not interfered with, and accelerated motion may be thought of as an unnatural or *forced* state. The property by virtue of which a particle tends to remain in either of the natural states, and to resist being accelerated, is called **inertia.**

The first of the above laws was partially stated in statics (Art. 5), where it was explained that the force system acting on a body at rest is said to be in equilibrium. The force system acting on a particle that has uniform rectilinear motion has a zero resultant, according to the first law of motion, and so such a system also is said to be in equilibrium or to be balanced.

The most familiar example of acceleration is afforded by a falling body. For a small, heavy body which has just started to fall, air resistance is negligible, and so such a body may be regarded as acted upon by a single force — its own weight. It is, therefore, accelerated, and, since weight and mass are proportional, this acceleration is, at any given place, the same for all bodies. But, since the mass of a given body is constant and its weight varies slightly with location, the acceleration caused by its weight, that is, the acceleration with which it falls, likewise varies according to location. This acceleration, denoted by g, has been very accurately determined at various places on the earth. The "standard value of gravity," corresponding closely to the value of g at latitude 45° and sea level, is 980.665 cm/sec² or 32.1739 ft/sec².* There are few engineering calculations in which it is necessary either to take into account the slight variation in g or to express its value more exactly than to three significant figures, and so we shall from now on use 32.2 as the value of g in feet per second per second.

EQUATION OF MOTION FOR A PARTICLE. Both the laws of motion stated above are expressed by the equation

$$a = K\frac{F}{m} \qquad \text{or} \qquad F = \frac{1}{K}ma \tag{1}$$

where F denotes the force or resultant acting on the particle, m the mass of the particle, a the acceleration of the particle, and K a numerical coefficient whose value depends upon the units in which F, m, and a are expressed.

Units. In American engineering practice the use of the foot, the second, and the pound force as units is well established. It is therefore convenient to retain these units, and it is also convenient to adopt as unit mass a mass so large that a force of 1 lb will give it an acceleration of just 1 ft/sec². This unit mass is called a **slug**; it is 32.2 times as large as a pound mass, because the acceleration given it by a pound force is only 1/32.2 as great as the acceleration given a pound mass. With these units the term K in the equation of motion is 1, and we have simply

$$a = \frac{F}{m} \qquad \text{or} \qquad F = ma \tag{2}$$

* *Letter Circular LC449,* National Bureau of Standards.

where F is in pounds, m is in slugs, and a is in feet per second per second. This system of units will be used consistently in this book.

It is perfectly feasible, of course, to use other systems of units. For example, in the cgs system the centimeter, gram, and second are taken as the units of length, mass, and time, and the unit force, called *dyne*, is defined as that force which will give a mass of 1 gram an acceleration of 1 cm/sec². Although commonly used in physics, the dyne is too small (about 0.00000225 lb) to be suitable in most engineering computations, and a larger unit is preferable. One recently proposed is the *newton*, defined as that force which will give a mass of 1 kg an acceleration of 1 m/sec². When the kilogram (force) is taken as the unit of force it is convenient to adopt as the unit of mass the *metric slug*, which is such a mass as will be given an acceleration of 1 m/sec² by a force of 1 kg and is about 9.8 times as large as the kilogram mass, since the acceleration of gravity is 9.8 m/sec² approximately. (See Appendix A, Art. 1, for an extensive list of units used in mechanics.)

If applied to a freely falling body, Eq. 2 takes the form $W = mg$, where W is the weight of the body and g is the acceleration of gravity. Therefore $m = W/g$, or the mass of a body in slugs is equal to its weight in pounds of force divided by the acceleration of gravity in feet per second per second. But if a body weighs W pounds it has W pounds of mass and $W/32.2$ slugs of mass. Therefore, when substituted for m in Eq. 2, $W/32.2$ may be regarded either as weight divided by acceleration of gravity, or as mass in pounds divided by the pure number 32.2.

89. Acceleration along Any Axis. Figure 167 represents a particle P moving with acceleration a under the action of forces whose resultant is a single force F. Let x be any axis, making with the path of the particle the angle θ. Obviously, since F and a have the same direction, $F \cos\theta = ma \cos\theta$. But $F \cos\theta$ is the component along the x axis of F, or of the force system of which F is the resultant, and $a \cos\theta$ is the x component of the acceleration, a_x. Hence the component along any axis of the system of forces acting on a particle is equal to the product of the mass of the particle and the component of its acceleration along that axis, or

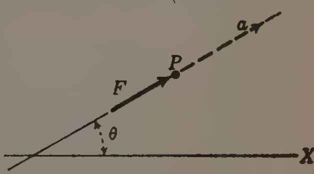

FIG. 167.

$$\Sigma F_x = ma_x$$

where ΣF_x represents the component of the system, and a_x the component of the acceleration, along the axis x. This equation, written for each of three chosen rectangular axes, provides three independent equations which can be used to determine the acceleration caused by given forces, or to determine the forces needed to cause a given acceleration.

EXAMPLE. Figure 168 is a view from above of a particle (small body), weighing 2 lb, which lies on a smooth horizontal floor and has applied to it the horizontal forces shown. It is required to determine the resulting acceleration of the particle.

Solution. We choose horizontal x and y axes as shown; the z axis is vertical. Then

$$\Sigma F_x = 5\ (\cos 30°) + 3\ (\cos 45°) - 4(\cos 60°) = \frac{2}{32.2} a_x \quad \text{whence}$$

$$a_x = +71.7 \text{ ft/sec}^2$$

$$\Sigma F_y = -5\ (\sin 30°) + 3\ (\sin 45°) + 4(\sin 60°) = \frac{2}{32.2} a_y \quad \text{whence}$$

$$a_y = +49.7 \text{ ft/sec}^2$$

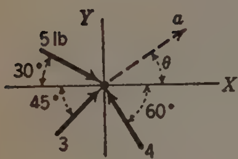

FIG. 168.

From the circumstances of the problem, it is obvious that there is no vertical or z acceleration; therefore the upward reaction of the floor N can be found from the equation

$$\Sigma F_z = -2 + N = \frac{2}{32.2} 0$$

whence $N = 2$ lb. The acceleration of the particle is now found from its x and y components. Its magnitude is $a = (71.7^2 + 49.7^2)^{\frac{1}{2}} = 87.3$ ft/sec², and it is directed as shown, making with the x axis an angle $\theta = \tan^{-1}(49.7/71.7) = 34.7°$.

Instead of determining a_x and a_y separately as is done above, the resultant F of the forces could be found by the methods of statics (Art. 27) and the acceleration then found from the relation $F = ma$, but the resultant F cannot be found until all the forces on the particle, including N, are known, and to find N the equation $\Sigma F_z = ma_z$ must be used.

We could also solve this example by determining the acceleration that each one of the forces, if acting alone, would give the particle, and then adding these accelerations vectorially to get the resultant acceleration. This method would be correct because the acceleration imparted to a particle by a given force is independent of any other forces that may act on the particle and is also independent of the motion of the particle.

90. Rectilinear Translation of a Rigid Body. A rectilinear translation is such a motion of a rigid body that every straight line of the body remains fixed in direction and every point in the body has rectilinear motion (curvilinear translation is defined in Art. 105). It is obvious that all points of the body move alike, and so by displacement, velocity, or acceleration of a body in rectilinear translation is meant the displacement, velocity, or acceleration respectively of any point of the body. We can write, for every elementary particle of such a body, the equation of motion $\Sigma F_x = dm \cdot a_x$ where dm is the mass of the elementary particle. The sum of the left-hand terms of all such equations represents ΣF_x for *all* forces, internal and external, that act on the body; since the internal forces occur in pairs of equal and opposite actions and reactions, they mutually cancel and the sum of the left-hand terms represents ΣF_x for the *external* forces that act on the body. The sum of the right-hand terms of all the motion equations is $\Sigma dm \cdot a_x$; since a_x is the same for all particles, $\Sigma dm \cdot a_x = a_x \Sigma dm = ma_x$, where m is the mass of the body. There-

fore the equation of motion for a particle can be applied to a body of any size that has rectilinear motion of translation.

In the examples and problems of the next article it is assumed that the bodies dealt with have motion of translation, that is, that they do not tip over or otherwise turn. In Art. 92 we discuss the conditions that the external forces must satisfy in order for this to be true.

91. Typical Problems; Examples. The equation of motion for a particle can be used to solve simple problems in which it is required to determine how a particle (or body regarded as a particle) will move when acted on by specified forces, or to determine the forces that will make a particle move in a specified way. Since only rectilinear motion has so far been considered, the only problems that can as yet be discussed are problems in which the particle is known to move in a straight path.

The examples below illustrate the recommended procedure, which may be outlined thus: *Draw a fbd for the particle or body, and represent thereon, as completely as possible, all external forces and the acceleration. It is advisable to represent the acceleration by a broken-line arrow to distinguish it from the forces. Next write down the equation $\Sigma F_x = ma_x$ for as many axes as necessary, choosing such axes as are most convenient. Then solve the equation or equations for the desired unknowns.*

Example 1. A block weighing 50 lb rests on a horizontal floor; the coefficient of kinetic friction between the block and the floor is 0.3. A constant horizontal force of 30 lb is applied to the block. It is required to determine the resulting acceleration.

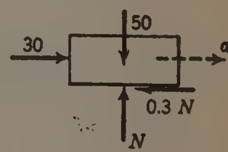

Fig. 169.

Solution. The fbd for the block is shown in Fig. 169. The forces acting are the weight of the block, the applied force of 30 lb, and the reaction of the floor, consisting of the normal component N and the friction component, the last being equal to $0.3N$ since slipping occurs. The acceleration is horizontal and in the direction of the 30-lb force as shown. Choosing vertical and horizontal axes, we have

$$\Sigma F_y = -50 + N = 0 \quad \text{whence } N = 50 \text{ lb and } 0.3N = 15 \text{ lb}$$
$$\Sigma F_x = 30 - 15 = (50 \div 32.2)a \quad \text{whence } a = 9.66 \text{ ft/sec}^2$$

Example 2. A particle of mass 10 lb rests on a smooth horizontal floor. It is required to determine (*a*) the least constant force that will give the particle a velocity of 100 ft/sec to the right in 5 sec, and (*b*) the least constant force that will bring the particle to rest from that velocity within a distance of 200 ft.

Solution. (*a*) We first find the necessary acceleration. Since the accelerating force is constant, the acceleration is constant; it is $a = \Delta v/\Delta t = 100/5 = 20$ ft/sec² to the right. The least force required to give the particle this acceleration would be a force in the direction of the acceleration, and this force would be $F = (10 \div 32.2)\,20 = 6.23$ lb to the right.

(*b*) Again we first find the necessary acceleration. Since the stopping force is constant, the stopping acceleration is constant; the velocity therefore varies linearly with time and its average value is the mean of its initial and final values, or $v_a = (100 + 0) \div 2 = 50$ ft/sec. The time required for stopping is $\Delta t = \Delta s/v_a = 200 \div 50 = 4$ sec, and the constant acceleration is equal to the average acceleration, or $a = a_a = \Delta v/\Delta t = (-100)/4 = -25$ ft/sec².

The negative sign signifies that the acceleration is to the left, it having been originally assumed that the direction of the velocity (to the right) was the positive direction. The required force is $F = (10 \div 32.2)(-25) = -7.77$ lb; the negative sign signifies that this force acts to the left.

EXAMPLE 3. A block A (Fig. 170a) is placed on the horizontal top of a second block B, which in turn rests on a smooth horizontal floor. The coefficient of friction between A and B is 0.12. By means of a horizontal force P applied to B, slipping is made to occur as between A and B. It is required to determine the acceleration of both blocks at the instant slipping impends.

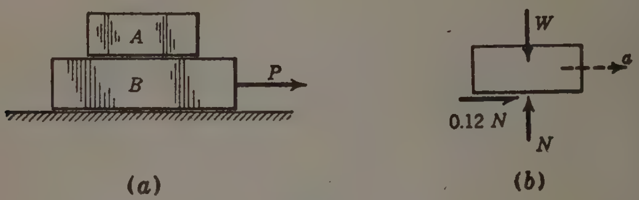

FIG. 170.

Solution. Block A is considered, and the fbd is drawn (Fig. 170b). The forces are the weight of A, denoted by W, the normal component of the reaction from B, denoted by N, and the friction, equal to $0.12N$ when slipping impends. Choosing vertical and horizontal axes, we have

$$\Sigma F_y = -W + N = 0 \quad \text{whence } N = W \text{ and friction} = 0.12W$$

$$\Sigma F_x = 0.12W = \left(\frac{W}{g}\right)a \quad \text{whence } a = 0.12g = 3.86 \text{ ft/sec}^2$$

EXAMPLE 4. Figure 171a represents a block A on an inclined plane and a second block B on the horizontal top of A. The plane is smooth, but the top of A is rough. A weighs 100 lb; B weighs 50 lb. A horizontal force of 200 lb is applied to A as shown. It is required to determine how large the coefficient of friction between A and B must be to prevent B from slipping on A.

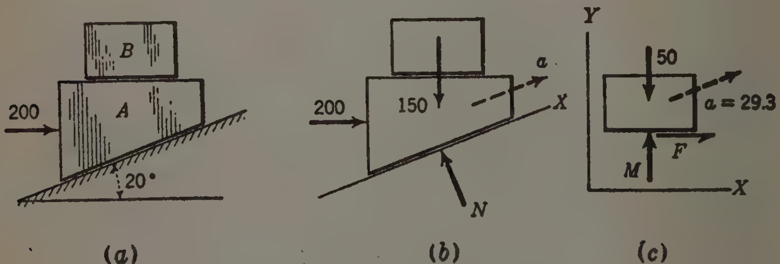

FIG. 171.

Solution. As long as there is no slipping between A and B, the two blocks move alike and can be regarded as a single body. The fbd for the two blocks so regarded is shown in Fig. 171b. The external forces are the weight, the applied force of 200 lb, and the normal reaction N of the smooth plane. The acceleration is assumed to be up the plane (if this assumption is

wrong a negative result will be obtained for a, indicating that its direction is opposite to that assumed). Writing the equation of motion with respect to an axis parallel to the plane, we have

$$\Sigma F_x = 200 \cos 20° - 150 \sin 20° = (150 \div 32.2)a \quad \text{whence } a = 29.3 \text{ ft/sec}^2$$

(In writing an equation such as this the choice of sign is purely arbitrary; we could just as well have taken direction up the plane as negative. Usually it is convenient to take the direction of the acceleration as the positive direction, as was done here, but the essential thing is to *have the signs of the forces and the sign of the acceleration consistent.*)

We now consider block B. The fbd is shown in Fig. 171c; the external forces are the weight, and the reaction from A, represented by its normal component M and its friction component F. The acceleration is, of course, the same as the acceleration of the blocks regarded jointly, and is so indicated in the figure. We now choose horizontal and vertical axes x, y, and write and solve the equations of motion thus:

$$\Sigma F_x = F = (50 \div 32.2)(29.3 \cos 20°) \quad \text{whence } F = 42.7 \text{ lb}$$
$$\Sigma F_y = -50 + M = (50 \div 32.2)(29.3 \sin 20°) \quad \text{whence } M = 65.5 \text{ lb}$$

If slipping impends, as assumed, F is the limiting friction and so the coefficient of friction is equal to $F \div M = 0.652$. This is the least value of μ that will prevent slipping.

Effect of a variable force. In each of the preceding examples the applied forces have been constant, but the equation $F = ma$ is equally applicable to a variable force and can be used to find the a–t law if the F–t law is known. When the a–t equation has been thus found, velocity and displacement can be determined by the methods of Art. 85.

EXAMPLE 5. Referring to Ex. 1 above, assume that the applied force, instead of being constant, is initially zero and increases uniformly at the rate of 5 lb/sec. It is required to determine the velocity and position of the block 8 sec after the force is applied. (The coefficient of static as well as kinetic friction may be assumed to be 0.3.)

Solution. Obviously the magnitude of the applied force is given by the equation $F = 5t$. Since the limiting friction is 15 lb, the block will not move until $F = 15$, or $t = 3$. For the first 3 sec, then, $a = 0$, $v = 0$, and $s = 0$. After $t = 3$, the friction force is 15 lb, and the equation of motion is $\Sigma F = 5t - 15 = (50 \div 32.2)a$, whence $a = 3.22t - 9.66$. Therefore $v = \int a\,dt = 1.61t^2 - 9.66t + C$, where C is found to be 14.5 by setting $v = 0$ when $t = 3$. When $t = 8$, this equation gives $v = 40.2$ ft/sec. The equation for s is $s = \int v\,dt = 0.537t^3 - 4.83t^2 + 14.5t + C$, where C is found to be -14.5 by setting $s = 0$ when $t = 3$. When $t = 8$, this equation gives $s = 67.5$ ft.

92. Force Criterion for Rectilinear Translation. It is shown in Art. 90 that a body in rectilinear translation can be treated as a particle as far as the use of the equation $\Sigma F_x = ma_x$ is concerned. But a body will not have motion of translation unless the external force system that acts upon it is such as not to cause any rotation or turning of the body. It is perhaps apparent from experience and intuition that any couple will produce rotation, and that a force will produce rotation unless it acts through the mass-center of the body. This conclusion, which is presently shown to be correct, may be stated thus: *The resultant of all external forces acting on a body that has rectilinear motion of trans-*

lation is a single force whose line of action passes through the mass-center of the body.

Proof. Let a be the acceleration of the body, and m_1, m_2, m_3, etc., the masses of the constituent particles. Then the resultant forces on these particles equal respectively m_1a, m_2a, m_3a, etc., all directed like a (Fig. 172a). Therefore the resultant forces on the particles make up a system of parallel forces which are proportional to the masses of the particles. This system is thus similar to the system made up of the weights of the particles (Fig. 172b) and, like these weights, has a resultant that passes through the mass-center G. This resultant is the resultant of all forces, internal and external, that act on the body. But the internal forces occur in pairs of equal, opposite, and collinear actions and reactions; therefore they contribute nothing to the resultant, which may be regarded as the resultant of the *external* forces that act on the body.

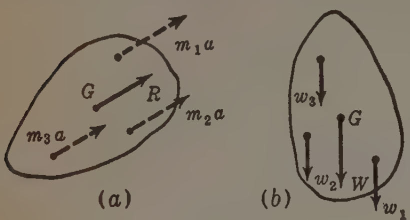

FIG. 172.

Since the resultant of the external forces passes through the mass-center *the moment of the external forces about any axis through the mass-center equals zero,* or $\Sigma M_G = 0$, where the subscript G denotes any axis through the mass-center. This fact yields three independent moment equations, since the summation of moments may be set equal to zero for each of three noncoplanar axes through the mass-center.

It should be noted that, although a body will not have motion of translation unless the resultant of the external forces passes through the mass-center, it will not have motion of translation even when this condition is satisfied unless initially without rotation, for, if already turning, it will continue to turn. Thus a pitched baseball meant to curve spins as it travels through the air because a rotary motion is given it when it is thrown.

In the examples and problems of Art. 91 the lines of actions of the forces and the dimensions of the bodies are not specified; it is simply *assumed* that the bodies in question have motion of translation. In the examples and problems below, it is required to determine forces, dimensions, etc., that will *insure* motion of translation, and use must be made of $\Sigma M_G = 0$ as well as of $\Sigma F_x = ma_x$.

EXAMPLE 1. A homogeneous rectangular block 6 ft tall and 3 ft square in section stands on end on a horizontal floor. The coefficient of kinetic friction between the block and the floor is 0.1. It is required to determine the maximum acceleration the block can be given by a horizontal force applied at the top without causing it to overturn.

Solution. We assume that the block can be made to slip without tipping by a force P. The fbd for the block is shown in Fig. 173. The forces are the weight W, the applied force P, taken as acting to the right, and the reaction of the floor, represented by its normal com-

ponent N and its friction component, which is $0.1N$ since there is slipping. Since the block is on the point of tipping over to the right, this floor reaction will act at the right edge of the bottom surface as shown. The acceleration a is to the right since P was assumed to act in that direction. Three independent equations are available for determining the three unknowns N, P, and a; they are

$$\Sigma F_y = -W + N = 0$$
$$\Sigma M_G = -P(3) - 0.1N(3) + N(1\tfrac{1}{2}) = 0$$
$$\Sigma F_x = P - 0.1N = (W \div 32.2)a$$

The first equation gives $N = W$; the second equation then gives $P = 0.4W$, and the third equation then gives $a = 9.66$ ft/sec².

Had a negative value been obtained for a, it would have meant that the block would tip before slipping. You should show that the least value of μ which would cause this condition to obtain is 0.25.

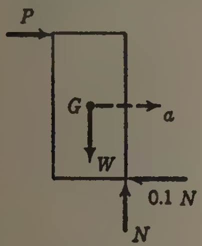

FIG. 173.

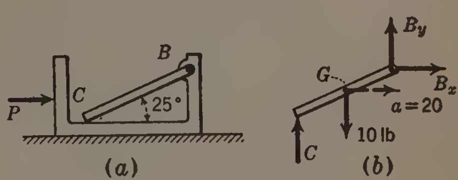

FIG. 174.

EXAMPLE 2. Figure 174a represents a slender rod BC which is pinned at B to one end of a box that rests on a horizontal floor; the other end of the rod rests on the smooth horizontal bottom of the box. Box and rod are given a horizontal acceleration to the right of 20 ft/sec² by a push P applied to the box. The rod weighs 10 lb and is 6 ft long. It is required to determine all forces on the rod.

Solution. The fbd for the rod is shown in Fig. 174b; the forces are the weight, the upward reaction C of the smooth floor, and the horizontal and vertical components B_x, B_y of the pin reaction. To solve for the three unknowns C, B_x, and B_y there are available three independent equations, namely:

$$\Sigma F_x = B_x = (10 \div 32.2)20$$
$$\Sigma F_y = B_y + C - 10 = 0$$
$$\Sigma M_G = B_y(3\cos 25°) - B_x(3\sin 25°) - C\,(3\cos 25°) = 0$$

Solution of these equations gives $B_x = 6.21$, $B_y = 6.45$, $C = 3.55$ lb. Had a negative value been obtained for C, it would have meant that the lower end of the rod would rise unless held down.

93. Inertia; Some Simple Experiments. In Art. 88 inertia was defined as that property by virtue of which a particle (or body) resists being accelerated. In this sense, inertia is an absolute quality possessed in equal degree by all bodies, because all bodies are completely inert. But since it is harder — that is, takes a greater force — to give a certain acceleration to a heavy body than to a light one, it may be said that the heavy body has more inertia than the

light one. Thus, regarded quantitatively, the inertia of a body is seen to be proportional to the amount of substance in it. The mass of a body therefore could be determined by comparing its inertia with the inertia of a unit mass, the ratio of these inertias being equal to the ratio of the forces required to give the body and the unit mass respectively the same acceleration.

Simple manual experiments such as those now to be described can be of great aid to an understanding of inertia and of the significance of the equation of motion.

Tie a cord to a small but reasonably heavy body, such as a brick or paper weight or large nut, so that the body can be lifted by means of the cord. If the body weighs, say, 2 lb, you would ordinarily think of the upward pull required to lift it as being 2 lb also. But note that if you lift quickly, as by jerking, the force exerted is much greater than when you lift slowly, so much so that you will perhaps break a cord capable of sustaining several times as much as 2 lb. In thus jerking you give the object a large upward acceleration, and the upward pull of the cord must exceed the downward pull of gravity by an amount equal to the mass of the object times this acceleration. Or, to put it differently, the upward pull must overcome not only the weight of the object but also its resistance to being accelerated. This resistance is not a force exerted *on* the body; it is a force exerted *by* the body on the cord.

Now hold the body suspended by the cord and lower it, not at a uniform speed but faster and faster. You will note that the upward pull is now less than that required to hold it still; the body has a downward acceleration, and the weight exceeds the upward pull by an amount equal to the mass times this acceleration. If you lower your hand very quickly, the cord will go slack; there is now no upward force at all and the object is simply falling with an acceleration of 32.2 ft/sec^2 under the pull of gravity. If, while the body is moving downward, you check the motion suddenly, the upward force required is again much greater than the weight, because in thus stopping it you give the body an upward acceleration. If the body is dropped on the floor it is stopped very quickly, the upward acceleration is very great during the stopping period, and the upward force the floor exerts on the body is correspondingly great. This is proved by the fact that the floor, unless very hard, is indented by the equal and opposite downward force the body exerts upon it. The harder the floor, the quicker the body is stopped, the greater the corresponding acceleration, and the greater the upward force. It is for this reason that a bottle or other brittle object will break if dropped on a concrete floor and will not break if dropped on soft ground; the concrete floor stops it quicker and hence exerts upon it a much greater force.

As another experiment, lay a book flat on a table and place upon it some small object like a penknife. Push the book gently so as to slide it along the table. The penknife will remain in place on the book without slipping, because the book, being rough, can give it an acceleration equal to that you have given the

book. But now push more violently; the acceleration thus given the book is greater than that which the book can give the penknife, and so the knife slips on the book. It actually moves in the same direction as the book, but with less acceleration, and so lags behind.

Again, lay an eraser in the palm of the open hand, and then turn the hand quickly so that the surface of the palm is vertical, at the same time swinging the hand horizontally toward the eraser. You will find, after a few trials, that it is easy to move the hand several feet without having the eraser slip down; in accelerating the eraser the hand exerts a horizontal pressure upon it, and, if this pressure times the coefficient of friction exceeds the weight of the eraser, it will not slip.

The resistance which a body, by virtue of its inertia, offers to being given a certain acceleration is a force, which can, of course, be measured. It is equal to ma, the mass of the body times the impressed acceleration, but is directed *opposite* to the acceleration and is exerted not *on* the body but *by* it on some other body. In the last of the experiments described above, the pressure exerted by the eraser on the hand is such a force, and it is equal and opposite to the accelerating force exerted by the hand on the eraser.

CHAPTER IX

CURVILINEAR MOTION

94. Preliminary. In rectilinear motion, as pointed out in Arts. 81 and 83, velocity and acceleration may be regarded as algebraic quantities; they need be treated as vectors only when their components are to be found. In curvilinear motion, however, velocity and acceleration can have any direction and so must always be treated as vector quantities. In the articles that follow they will be defined and discussed on this basis.

95. Position; Speed; Velocity. The position of a particle on a curved path can be specified by its rectangular coordinates, by its polar coordinates, or by its distance from some fixed origin in the path, this distance being measured along the path and being considered positive when the particle is to one side of the origin and negative when it is to the other side. When this last method is used we call the distance of the particle from the origin the s-coordinate and denote it by s, as in rectilinear motion.

In curvilinear motion, as in rectilinear motion, we use the word speed to signify how fast a particle moves, without regard to direction. Hence speed = $|ds/dt|$. It is often helpful to think of speed as being indicated by a speedometer on the moving particle.

The velocity of a particle in curvilinear motion is a vector quantity whose magnitude is the speed of the particle and whose direction is along the tangent to the path. We denote velocity by V and magnitude of velocity by v.* It will be convenient to drop the symbol $||$ in the expression for v and write $v = ds/dt$; then this formula applied to a numerical problem will give not only the magnitude of the velocity but also, by the sign of the computed ds/dt, its *sense* along the tangent to the path.

Since the equation $v = ds/dt$ holds for both curvilinear and rectilinear motion, all the other relations — algebraic and graphical — between s and v that hold for rectilinear motion hold also for curvilinear motion. Thus in curvilinear motion $s = \int v\,dt$, or $\Delta s = \int_{t_1}^{t_2} v\,dt$, and v can be found from the slope of the s–t graph, or Δs from the area under the v–t graph.

* This scheme of notation, in which a vector quantity is denoted by a capital letter and its magnitude by the same letter in lower case, will be followed generally in this discussion of curvilinear motion. Also, since the component along a fixed axis of a vector quantity can be regarded as an algebraic quantity (like velocity in rectilinear motion), we shall denote any such component of a vector quantity by the appropriate lower-case letter and subscript.

EXAMPLE 1. A particle P moves in the circle shown (Fig. 175) according to the law $s = 24 - 2t^2$, s being measured in feet from Q (positive direction counterclockwise) and t being time in seconds. It is required to determine the velocity of the point when $t = 3$.

Solution. The magnitude of the velocity is found by applying the foregoing formula for v. Since $s = 24 - 2t^2$, $v = ds/dt = -4t$. Hence, when $t = 3$, $v = -12$ ft/sec. The negative sign means that the sense of the velocity along the tangent to the path is negative, that is, clockwise.

The direction of the velocity is found by determining the position of the point in the path and ascertaining the direction of the tangent there. When $t = 3$, $s = 24 - 2 \times 3^2 = +6$ ft; therefore the point is 6 ft from Q (measured counterclockwise), and the velocity is directed along the tangent to the path at P as shown. The angle V makes with the vertical is $\theta = 6/10$ rad $= 34° 20'$.

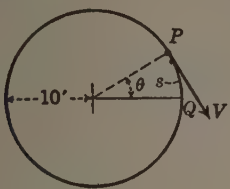

FIG. 175.

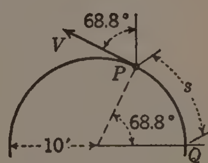

FIG. 176.

EXAMPLE 2. A particle moves in the circular path (Fig. 176) in accordance with the equation $v = t^2 - 10$, the units being the foot and second and the positive direction being counterclockwise. When $t = 0$ the particle is at Q. It is required to determine the velocity when $t = 6$.

Solution. When $t = 6$, $v = 36 - 10 = 26$ ft/sec. This gives the magnitude and sense (counterclockwise) of the velocity. To determine its direction it is necessary to determine the position of the particle, which is done by computing s, measured from Q, the most convenient origin. Since $s = \int v\,dt$, $s = \frac{1}{3}t^3 - 10t + C$. The constant C is found from the condition that $s = 0$ when $t = 0$, whence $C = 0$. Therefore, when $t = 6$, $s = 12$ ft. Therefore the particle is 12 ft from Q in the counterclockwise direction, and V, being directed along the tangent to the path there, makes an angle of $(12 \div 10)$ rad $= 68.8°$ to the vertical as shown.

96. Axial Components of Velocity. In curvilinear motion, as in rectilinear motion, velocity can be resolved into components, and when these components are parallel to rectangular axes x, y, and z their magnitudes and senses are given by

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt}$$

If, therefore, the circumstances of a motion are such that x, y, and z can be expressed by equations in terms of t, one can by differentiation derive equations for the component velocities. The velocity can then be found by computing the resultant of these components. And, if the equations for the component velocities v_x, v_y, v_z are given, then by integration the equations for x, y, and z can be derived.

If one of the component velocities is at all times zero, the path is a plane curve

in the plane of the other two components, and the motion is called *uniplanar.* If the velocity has a component along each of the three axes the path is a tortuous curve in space, and the motion is called *three-dimensional.* Uniplanar motion is the more common type of motion encountered in engineering problems, and it is the kind considered in most of the problems and examples of this book.

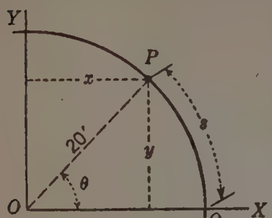

FIG. 177.

EXAMPLE 1. A particle P starts at Q and moves counterclockwise in the circle of Fig. 177 according to the law $s = 2t^3$, where s is measured in feet from Q (positive counterclockwise) and t = time in seconds after starting. It is required to develop the general formulas for the x and y components of the velocity, and to determine these components when $t = 2$.

Solution. It is plain from the figure that with origin at O

$$x = 20 \cos \theta \qquad \text{and} \qquad y = 20 \sin \theta$$

Now $\theta = s/20 = 2t^3/20 = 0.1t^3$, hence

$$x = 20 \cos (0.1t^3) \quad \text{and} \quad y = 20 \sin (0.1t^3)$$

therefore

$$v_x = \frac{dx}{dt} = -20 \sin (0.1t^3)0.3t^2 = -6t^2 \sin (0.1t^3)$$

and

$$v_y = \frac{dy}{dt} = 20 \cos (0.1t^3)0.3t^2 = +6t^2 \cos (0.1t^3)$$

The above are the general formulas for the x and y components of the velocity. When $t = 2$

$$v_x = -6 \times 4 \sin (0.8 \text{ rad}) = -24 \sin 45° \, 48' = -17.2 \text{ ft/sec}$$

and

$$v_y = +6 \times 4 \cos (0.8 \text{ rad}) = +24 \cos 45° \, 48' = +16.7 \text{ ft/sec}$$

The signs indicate that the x component of the velocity is directed toward the left and that the y component is directed upward.

(You should check the results obtained in the above solution by ascertaining, by the methods of Art. 95, the magnitude and direction of V when $t = 2$, and then resolving V into its components v_x and v_y.)

EXAMPLE 2. A point starts from the origin and moves along the curve $y = \frac{1}{8}x^2$ in such a way that the horizontal component of the velocity is always equal to 6 ft/sec. It is required to determine the velocity of the point 2 sec after the start.

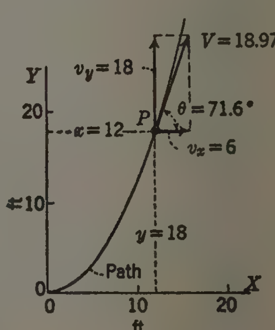

FIG. 178.

Solution. Since $v_x = 6$, $x = \int v_x \, dt = 6t + C$. Since $x = 0$ when $t = 0$, $C = 0$. Therefore $x = 6t$. And, since $y = \frac{1}{8}x^2 = 4.5t^2$, $v_y = dy/dt = 9t$. When $t = 2$, $v_x = 6$ and is directed toward the right, and $v_y = 18$ and is directed upward. The velocity V is therefore directed toward the right and upward. Its magnitude is

$$v = (6^2 + 18^2)^{\frac{1}{2}} = 18.97 \text{ ft/sec}$$

The angle that V makes with the horizontal is $\theta = \tan^{-1} 18 \div 6 = 71° 36'$. The results of the solution are represented in Fig. 178. (You should check the value obtained for θ in the above solution by determining the slope of the path, dy/dx, at the place where the point is when $t = 2$.)

97. Radial and Transverse Components of Velocity. When polar coordinates are used to describe the path of a particle or to specify its position, the velocity can often be most easily studied by means of its components parallel and normal to the radius vector; these are called *radial* and *transverse* components, respectively, and are denoted by V_ρ and V_θ, their magnitudes and senses being denoted by v_ρ and v_θ with appropriate sign. We shall now derive formulas for v_ρ and v_θ for a particle having uniplanar motion.

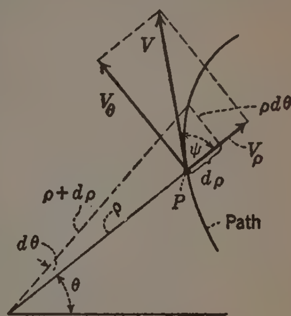

FIG. 179.

Figure 179 represents the path of a particle P whose position is defined by the polar coordinates ρ and θ. The velocity of P, represented by the vector V, is directed along the tangent to the path and makes with the radius vector the angle ψ. The component of V parallel to the radius vector is obviously equal to $v \cos \psi$, and the component normal to the radius vector is equal to $v \sin \psi$. Now from the figure it can be seen that $\cos \psi = d\rho/ds$; therefore

$$v_\rho = v \cos \psi = \frac{ds}{dt}\frac{d\rho}{ds} = \frac{d\rho}{dt} \tag{1}$$

Hence the radial component of the velocity is the rate at which the length of the radius vector is changing. If the length is increasing, $d\rho/dt$ is positive and V_ρ is directed away from the origin; if the length is decreasing, $d\rho/dt$ is negative and V_ρ is directed towards the origin.

From the figure it can also be seen that $\sin \psi = \rho\, d\theta/ds$; therefore

$$v_\theta = v \sin \psi = \frac{ds}{dt}\frac{\rho\, d\theta}{ds} = \rho\frac{d\theta}{dt} \tag{2}$$

Now $d\theta/dt$ is the time rate of change of θ. This is the rate at which the radius vector is turning; it is called the *angular velocity* of the radius vector and is denoted by ω. The transverse component of the velocity is therefore $\rho\omega$, the product of the length of the radius vector ρ and its angular velocity ω expressed in radians per unit time. The sign of v_θ is determined by the sign of ω; if ω is positive it means that θ is increasing, and if ω is negative it means that θ is decreasing. The sense of v_θ is readily determined by inspection as soon as it is known which way the radius vector is turning.

EXAMPLE. The rod AB (Fig. 180) is made to turn counterclockwise about an axis through A in such a manner that $\theta = \frac{1}{2}t$, where θ, measured as shown, is in radians and t is in seconds. A small sphere P slides along the rod, starting at A when $\theta = 0$ and moving outward so that ρ, its distance from A in feet, is given by $\rho = \frac{1}{3}t^2$. It is required to determine the position and velocity of P at the end of 3 sec.

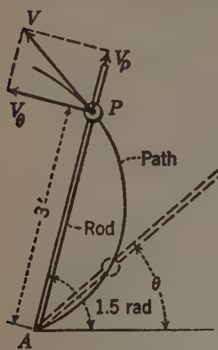

FIG. 180.

Solution. When $t = 3$, $\theta = 1\frac{1}{2}$ rad and $\rho = 3$ ft. The position of P is therefore as shown in the figure. (Positions of P at several instants between $t = 0$ and $t = 1$ are thus readily determined, and the path of P is constructed by drawing a smooth curve through these successive positions.) The equation for the radial component of the velocity is

$$v_\rho = \frac{d\rho}{dt} = \tfrac{2}{3}t \tag{2}$$

which, when $t = 3$, gives $v_\rho = 2$ ft/sec. The plus sign means that this velocity component is directed outward along the radius vector (or rod) as shown.

The equation for the angular velocity of the radius vector is $\omega = d\theta/dt = \frac{1}{2}$; that is, the angular velocity is constant and is $\frac{1}{2}$ rad/sec counterclockwise. The equation for the transverse component of the velocity of P is therefore

$$v_\theta = \rho\omega = (\tfrac{1}{3}t^2)\tfrac{1}{2} = \tfrac{1}{6}t^2$$

which, when $t = 3$, gives $v_\theta = 1.5$ ft/sec. This velocity component, since ω is counterclockwise, is directed as shown. The velocity of P is the resultant of V_ρ and V_θ; hence its magnitude is $v = (2^2 + 1.5^2)^{\frac{1}{2}} = 2.5$ ft/sec., and its direction is such that it makes with the rod an angle $\psi = \tan^{-1}(1.5 \div 2) = 36.9°$. This velocity V is, of course, tangent to the path of P.

98. Acceleration. The acceleration of a particle is the time rate of change of its velocity; it is a vector quantity whose magnitude and direction can be defined as follows:

Let Fig. 181*a* represent the path of a moving particle P. Imagine that, as P moves along its path, its changing velocity is represented at all times in

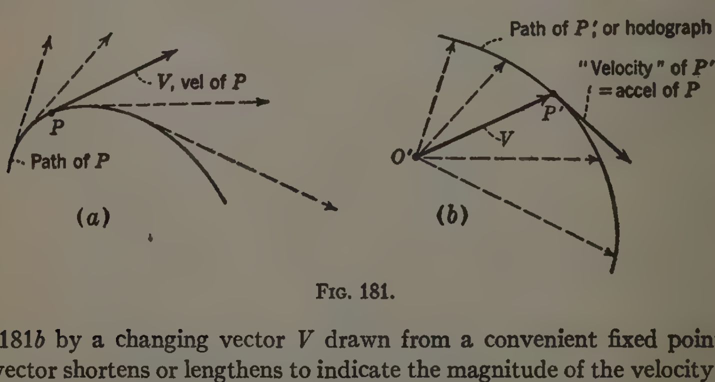

FIG. 181.

Fig. 181*b* by a changing vector V drawn from a convenient fixed point O'; this vector shortens or lengthens to indicate the magnitude of the velocity, and swings around to have the same direction as the velocity. (The variation is suggested by the broken-line vectors, which represent V at successive instants.)

Obviously any change in the velocity is accompanied by a movement of P', the tip of the velocity vector, which traces out a path, called a **hodograph**, in the vector diagram while the particle P follows its path on the space diagram. Thus at any instant the way in which the velocity of P is changing is completely indicated by the way in which the vector tip P' is moving: P' moves slowly if V is changing slowly; it moves rapidly if V is changing rapidly; and the direction in which P' moves is the direction in which V is changing. That is to say, the acceleration of P is represented by the " velocity " of P', this " velocity "* being measured in terms of distance along the hodograph per unit time (feet per second per second) and being directed along the tangent to the hodograph. If the motion of P is uniplanar, the hodograph is a plane curve; if the motion of P is three-dimensional, the hodograph is in general (not always) a tortuous curve.

99. Direct Calculation of Acceleration. The relationship described in Art. 98 is helpful to an understanding of acceleration and to an understanding of the variation of any vector quantity, but it does not provide a generally convenient means of actually computing acceleration. It is indeed possible, and for some motions easy, to determine acceleration directly by finding the " velocity " of the tip of the velocity vector, but it is nearly always far easier to determine an acceleration from its *components*, as described in the next two articles. We shall, however, give two examples of the use of the direct method.

First, consider a particle P that moves with uniform speed v in a circular path of radius r (Fig. 182a). The velocity vector V is constant in length and changes only in direction, swinging around through 360° while P travels once around its path. The hodograph is therefore a circle of radius v (Fig. 182b). Now P' goes once around the hodograph while P goes once around its path; hence the " velocity " of P' is to the velocity of P as the radius of the hodograph is to the radius of the path, or

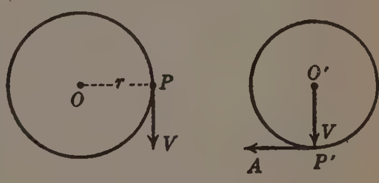

FIG. 182.

$$\frac{\text{" Velocity " of } P' \ (=a)}{\text{Velocity of } P \ (=v)} = \frac{\text{Radius of hodograph } (=v)}{\text{Radius of path } (=r)}$$

hence $a = v^2/r$.

* We use the quotation marks to emphasize the fact that the " velocity " of P' is not a true velocity, since it is not rate of change of position in space. Assuming the vector V to vary as described, its tip P' would move in the vector diagram; the actual velocity of P' would equal the number of inches per second it moved in this diagram, and, if the vector diagram were drawn to a scale of 1 in. = 10 ft/sec, and P' moved in this diagram with a velocity of 5 in./sec, this would represent an acceleration of P of $10 \times 5 = 50$ ft/sec². This concept of the rate of change of any vector quantity as the " velocity " of the tip of the vector representing that quantity is very useful and is employed later in discussing momentum.

The direction of the acceleration of P is the direction of the "velocity" of P', that is, along the tangent to the hodograph, and hence perpendicular to V and to the path. Therefore if a particle moves with uniform speed in a circular path its acceleration is equal to v^2/r and is always directed toward the center of the path.

Again, consider an airplane descending in a spiral glide. Assume that the speed v is constant and that the path is a helix, lying in a cylindrical surface of radius r and inclined to the horizontal at the constant angle θ. The velocity vector is of constant length v, and, as it swings about to correspond in direction with the direction of the airplane's velocity, it is always inclined at the angle θ to the horizontal and its tip P' travels in a horizontal plane along a circular hodograph of radius $v \cos \theta$. Now P' goes once around the hodograph while the plane makes one circuit of the helical path. The length of the hodograph is $2\pi v \cos \theta$; the length of one circuit of the helical path is $2\pi r/\cos \theta$. Therefore

$$\frac{\text{"Velocity" of } P' \ (=a)}{\text{Velocity of plane } (=v)} = \frac{2\pi v \cos \theta}{2\pi r/\cos \theta},$$

whence $a = (v \cos \theta)^2/r$.

The direction of the acceleration of the airplane is the direction of the "velocity" of P', that is perpendicular to the horizontal component of V, or straight in towards the vertical axis of the helical path.

100. Axial Components of Acceleration. Acceleration, like velocity, can be resolved into components. We first discuss acceleration components that are parallel to arbitrarily selected rectangular axes x, y, z.

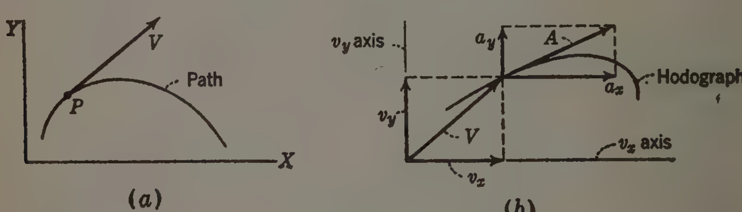

FIG. 183.

Figure 183a represents the path and Fig. 183b the hodograph for a motion, assumed uniplanar for convenience in representation. The acceleration is represented by the vector A in (b), and the axes x and y are arbitrarily taken horizontal and vertical. Now, since A is the "velocity" of P', the x component of A is the x component of the "velocity" of P'; this is the rate at which the horizontal coordinate of P' is changing and, as is evident from the figure, the rate at which v_x is changing. Similarly the y component of A is the rate at which v_y is changing; and, if the motion were three-dimensional, the z component of A would be the rate of change of v_z. Therefore the magni-

tudes and senses of the acceleration components are given by

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt}$$

It is apparent, therefore, that with respect to acceleration as well as to velocity a curvilinear motion can be analyzed as three (or two) independent rectilinear motions. If the x–t, y–t, and z–t laws are given, then by differentiation the v_x, v_y, and v_z equations and the a_x, a_y, and a_z equations can be derived. From its rectangular components, A can be found in the usual way. If the a_x, a_y, and a_z equations are given, then the v_x, v_y, and v_z equations can be derived by integration, and the x, y, and z equations by a second integration.

EXAMPLE 1. A particle P starts at O (Fig. 184) and moves in the xy plane in accordance with the equations $x = 2t^3$, $v_y = 10t$. Units are the foot and the second. It is required to determine the position, velocity, and acceleration of the particle when $t = 3$.

Solution. For the x component of the motion:

$$x = 2t^3 \qquad v_x = \frac{dx}{dt} = 6t^2 \qquad a_x = \frac{dv_x}{dt} = 12t$$

When $t = 3$, $x = 54$, $v_x = 54$, $a_x = 36$.

For the y component of the motion:

$$v_y = 10t \qquad y = \int v_y\,dt = 5t^2 + (C = 0) \qquad a_y = \frac{dv_y}{dt} = 10$$

When $t = 3$, $y = 45$, $v_y = 30$, $a_y = 10$.

Therefore when $t = 3$ the particle is at $x = 54$, $y = 45$; its velocity is $V = (54^2 + 30^2)^{\frac{1}{2}} = 61.8$ ft/sec, directed up and to the right at $\tan^{-1}(30 \div 54) = 29.1°$ to the horizontal; its acceleration is $A = (36^2 + 10^2)^{\frac{1}{2}} = 37.4$ ft/sec² directed up and to the right at $\tan^{-1}(10 \div 36) = 15.5°$ to the horizontal. These results are indicated in Fig. 184, as is also the approximate form of the path, determined by drawing a smooth curve through positions of the particle computed and plotted for $t = 0, 1, 2, 3$, and 4. (You should derive the equation of this curve in x and y.)

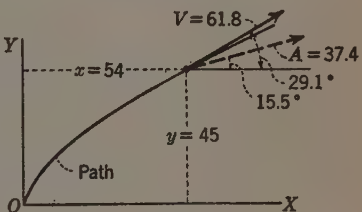

FIG 184.

EXAMPLE 2. A particle P starts at Q and moves counterclockwise in the circle (Fig. 185) according to the law $s = 2t^3$ (this is the motion described in Ex. 1 of Art. 96). It is required to develop the general formulas for the x and y components of the acceleration, and to determine these components and the acceleration when $t = 2$.

Solution. It is shown in the example referred to that

$$v_x = -6t^2 \sin(0.1t^3) \qquad \text{and} \qquad v_y = 6t^2 \cos(0.1t^3)$$

Therefore

$$a_x = \frac{dv_x}{dt} = -12t \sin(0.1t^3) - 1.8t^4 \cos(0.1t^3)$$

and

$$a_y = \frac{dv_y}{dt} = 12t \cos(0.1t^3) - 1.8t^4 \sin(0.1t^3)$$

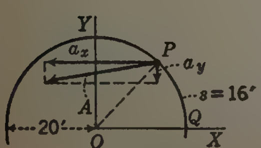

FIG. 185.

These are the general formulas for the x and y components of the acceleration. When $t = 2$, $a_x = -37.3$ ft/sec² and $a_y = -3.95$ ft/sec². The negative signs mean that the x component is directed to the left and the y component downward; A is therefore directed to the left and downward. Its magnitude is $a = (37.3^2 + 3.95^2)^{\frac{1}{2}} = 37.5$ ft/sec², and the angle it makes with the horizontal is $\tan^{-1}(3.95 \div 37.3) = 6°$. For the sake of clearness the acceleration and its components are indicated in Fig. 185 by vectors drawn from the point where the particle is at the instant in question, this position having been found from the equation $s = 2t^3$.

EXAMPLE 3. A particle starts from rest at the origin and moves along the parabola $y^2 = 36x$ (x and y in feet) in such a way that the y component of the acceleration is constant and equal to 6 ft/sec². It is required to determine the position, velocity, and acceleration of the particle 2 sec after starting.

Solution. With respect to the y component of the motion, the following equations apply:

$$a_y = 6 \qquad v_y = \int a_y\,dt = 6t \qquad y = \int v_y\,dt = 3t^2$$

(The initial conditions show that the constants of integration are zero.) With respect to the x component of the motion, the following equations apply:

$$x = \frac{1}{36}y^2 = \frac{1}{4}t^4 \qquad v_x = \frac{dx}{dt} = t^3 \qquad a_x = \frac{dv_x}{dt} = 3t^2$$

Substitution of $t = 2$ in the above equations gives

$$\begin{array}{lll} x = 4 \text{ ft} & v_x = 8 \text{ ft/sec} & a_x = 12 \text{ ft/sec}^2 \\ y = 12 \text{ ft} & v_y = 12 \text{ ft/sec} & a_y = 6 \text{ ft/sec}^2 \end{array}$$

Therefore when $t = 2$ the point is 4 ft to the right of, and 12 ft above, the origin. The magnitude of the velocity is $v = (8^2 + 12^2)^{\frac{1}{2}} = 14.43$ ft/sec; it is directed to the right and up; the angle it makes with the horizontal is $\tan^{-1}(12 \div 8) = 56.3°$. The magnitude of the acceleration is $a = (12^2 + 6^2)^{\frac{1}{2}} = 13.43$ ft/sec²; it is directed to the right and up; the angle it makes with the horizontal is $\tan^{-1}(6 \div 12) = 26.6°$.

101. Tangential and Normal Components of Acceleration. In many motions the acceleration can be determined most readily from its components A_t and A_n along the tangent and normal to the path of the particle. It will now be shown that

$$a_t = \frac{dv}{dt} \qquad \text{and} \qquad a_n = \frac{v^2}{r}$$

That is, the tangential acceleration at any instant is equal to the time rate at which the speed is changing then, and the normal acceleration is equal to the square of the speed divided by the radius of curvature of the path at the then position of the particle.

Consider the path and hodograph of the motion to be represented by Figs. 186*a* and *b*. O is the center of curvature of the path at the moving particle P, O' is a chosen origin of the velocity vectors, and OQ and $O'Q'$ are convenient lines of reference from which the angles θ and ϕ are measured; v and ϕ are polar coordinates of P'.

Since the radius vector $O'P'$ is parallel to the tangent at P, A_t and A_n are the components of A along and normal to $O'P'$; they are indicated in Fig. 186*b*. And, since A is represented by the "velocity" of P', A_t and A_n respectively

are represented by the components of that "velocity" along and normal to $O'P'$. Now these component "velocities" correspond to the radial and transverse velocities of Art. 97; that is (in Fig. 186*b*), A_t is the radial "velocity" of P', equal to dv/dt, and A_n is the transverse "velocity" of P', equal to $v\omega'$, where ω' is the rate at which $O'P'$ turns. Now, referring to both Figs. 186*a*

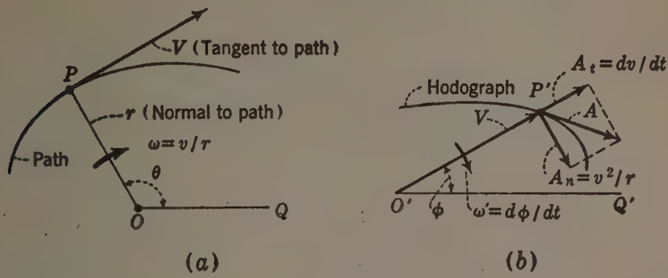

FIG. 186.

and 186*b*, it is apparent that $O'P'$ is at all times perpendicular to OP, drawn from the center of curvature O of the path to the particle P. Therefore OP turns at the same rate as $O'P'$, or $d\theta/dt = d\phi/dt = \omega'$; and, since $d\theta = ds/r$, where r = radius of curvature of the path,

$$\omega' = \frac{ds}{r\,dt} = \frac{v}{r} \qquad \text{and} \qquad a_n = v\omega' = \frac{v^2}{r}$$

The sense of A_t along the tangent and the sense of A_n along the normal are apparent from the figure. A_t has the same direction as V if the speed is increasing (vector $O'P'$ lengthening), and the opposite direction if the speed is decreasing ($O'P'$ shortening). The turning of $O'P'$ always causes P' to move transversely in a direction toward the inside of the path, and so A_n *is always directed towards the center of curvature O.*

EXAMPLE 1. A particle starts at Q and moves counterclockwise in the circle (Fig. 187) according to the law $s = 2t^3$ (this is the motion described in Ex. 1 of Art. 96 and Ex. 2 of Art. 100). It is required to develop general formulas for the tangential and normal components of the acceleration, and to determine, by means of these formulas, the acceleration when $t = 2$.

Solution. Since $s = 2t^3$, $v = 6t^2$; hence

$$a_t = \frac{dv}{dt} = 12t \qquad \text{and} \qquad a_n = \frac{v^2}{r} = 1.8t^4$$

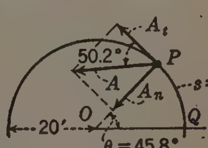

FIG. 187.

These are the general formulas for the magnitudes of A_t and A_n, and the positive sign of a_t shows that A_t is directed counterclockwise.

When $t = 2$, the above formulas give $a_t = 24$ ft/sec² and $a_n = 28.8$ ft/sec². Therefore the magnitude of the acceleration is $a = (24^2 + 28.8^2)^{\frac{1}{2}} = 37.5$ ft/sec². The direction is found as follows: Since $s = 2t^3 = 16$ ft, $\theta = \frac{16}{20}$ rad $= 45.8°$. Therefore A_t makes with the hori-

zontal an angle of 44.2°. The angle between A and A_t is $\tan^{-1}(a_n/a_t) = \tan^{-1}(28.8 \div 24) = 50.2°$, as shown in Fig. 187. Therefore A is directed down and to the left at 6° to the horizontal.

(You should compare the above solution of this problem with the solution made by using axial components in Ex. 2 of Art. 100.)

EXAMPLE 2. A car is rounding a curve of 400-ft radius at a speed of 40 mi/hr. It is required to determine the maximum rate at which the driver could slow down without having the acceleration of the car momentarily exceed 10 ft/sec².

Solution. At a speed of 40 mi/hr or 58.7 ft/sec the normal acceleration of the car is $a_n = 58.7^2 \div 400 = 8.61$ ft/sec². The total acceleration, at the moment the brakes are applied, is $a = 10 = (8.61^2 + a_t^2)^{\frac{1}{2}}$; hence $a_t = 5.08$ ft/sec². This is the maximum rate at which the speed could be decreased at first; as the speed diminished, a_n would diminish and a_t could be progressively increased.

102. Average Speed; Average Velocity; Average Acceleration. In most engineering problems involving motion the position, speed, velocity, and acceleration of a particle at a particular instant are of paramount importance, and so we defined and discussed the *instantaneous* value of each of these quantities in the preceding articles. But occasionally there is reason to speak of the *average* speed, velocity, or acceleration for a particular interval of time, and we now define these averages. Although the definitions are given with reference to curvilinear motion, they apply as well to rectilinear motion, which may be regarded as a special case of curvilinear motion in which the radius of curvature of the path is infinite.

The *average speed* of a particle for a given interval of time is the distance traveled during that interval divided by the interval; it is a scalar quantity having magnitude only. According to the definition

$$\text{Average speed} = \frac{\text{Distance traveled}}{\Delta t}$$

The *displacement* of a particle for a given interval of time is the vector joining the positions of the point at the beginning and end of the interval, the sense of the vector being from the initial to the final position. The *average velocity* for the interval is the displacement divided by the interval; it is a vector quantity directed like the displacement. It may be thought of as such a velocity as, constantly maintained for the given interval of time, would result in the same displacement as is produced by the actual varying velocity. Denoting average velocity by V_a, we have, according to the definition,

$$V_a = \frac{\text{Displacement}}{\Delta t}$$

The *increment of velocity* for a given interval of time is the change in velocity that occurs during that interval; it is the vector difference between the initial and final velocities, or the velocity that, added vectorially to the initial velocity, would give the final velocity. The *average acceleration* for the interval is the

increment of velocity divided by the interval; it is a vector quantity directed like the increment of velocity. It may be thought of as such an acceleration as, constantly maintained throughout the given interval of time, would result in the same change in velocity as is produced by the actual varying acceleration. Denoting the average acceleration by A_a and the increment of velocity by ΔV, we have, according to the definition,

$$A_a = \frac{\Delta V}{\Delta t}$$

In Fig. 188 the quantities that have been defined are represented and their relationships restated. At the beginning of an interval Δt, particle P is at M and has velocity V_1; at the end of the interval P is at N and has velocity V_2. Displacement, average velocity V_a, increment of velocity ΔV, and average

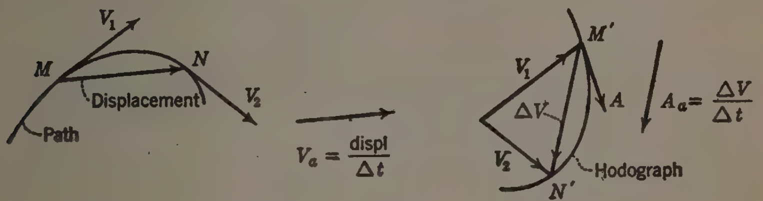

FIG. 188.

acceleration A_a for the interval are as represented. The distance traveled is arc MN if motion is not reversed during the interval; it may be any distance greater than this if motion is reversed.

As the time interval for which they are computed approaches zero, being taken shorter and shorter but always so as to include a particular instant, the average speed, velocity, and acceleration approach, as limits, the speed, velocity, and acceleration at that particular instant. Let t_1 be some particular instant during the interval Δt, say at the beginning of the interval, when P is at M, and let the interval be taken shorter and shorter but always so as to include the instant t_1. Then, as Δt approaches zero:

Limit of the average speed = lim (distance traveled/Δt) = ds/dt = speed at instant t_1.

Limit of magnitude of V_a = lim (chord $MN/\Delta t$) = lim ($\Delta s/\Delta t$) = ds/dt = magnitude of V at instant t_1, and limiting direction of V_a is limiting direction of chord MN, which is tangent to path at M, and is direction of V.

Limit of A_a = lim ($\Delta V/\Delta t$) = lim average "velocity" of P' = "velocity" of P' at instant t_1 = acceleration of P at instant t_1.

103. Simple Harmonic Motion. It is apparent from Arts. 96 and 100 that a curvilinear motion can sometimes be most easily analyzed by resolving it into

two rectilinear motions — the projections of the curvilinear motion on x and y axes. Conversely, there is one very important rectilinear motion that is most easily defined and analyzed as the projection of a curvilinear motion on a straight line. It is called **simple harmonic motion** (shm for brevity) and is of great importance in the theory of sound, whence the name. It also plays important roles in other branches of physics and in engineering, especially in the field of mechanical vibrations (see Appendix C). It is defined in various ways; we choose the following: a point which moves along the diameter of a circle and so that it is always the projection, on that diameter, of a point which is moving uniformly along the circumference has a simple harmonic motion. Thus if in Fig. 189 P is a point moving uniformly along the circle, and Q is the projection of P on the vertical diameter, Q has shm.

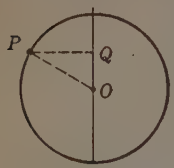

FIG. 189.

Certain facts about this motion are obvious from the definition: (i) the motion is to and fro, or vibratory; (ii) the motion is symmetrical about the midpoint O of its range, that is, at equal distances from O the velocities of Q are equal, and the accelerations also; (iii) the velocity of Q is maximum at O and zero at the ends of the range.

DEFINITIONS AND NOTATION: By **displacement** at a particular instant of a point in shm is meant the distance of the point from the middle of its range, sign (positive or negative) being used to indicate on which side of the middle the point then is; the symbol is y.* By **amplitude** is meant one-half the range of the motion, that is, the radius of the defining circle; the symbol is y_0. By **period** is meant the time of one cycle, or complete to-and-fro motion of Q corresponding to one trip of P around the circle; the symbol is T. By **frequency** is meant the number of cycles per unit time; the symbol is n. By **circular frequency** is meant the number of radians swept out per unit time by the radius OP; the symbol is ω. (This frequency is the angular velocity of the radius; see Art 97.) Evidently

$$\omega = 2\pi n = \frac{2\pi}{T} \qquad \text{and} \qquad T = \frac{1}{n}$$

VELOCITY AND ACCELERATION OF SHM. Referring to Fig. 190, let y, v, and a respectively denote the displacement, velocity, and acceleration of Q at any time t after Q was in its highest position, θ the angle YOP, V_0 the velocity of P, and A_0 the acceleration of P. V_0 is directed along the tangent and A_0 along the normal to the circle, as shown. From Art. 97, $\theta = \omega t$ and $v_0 = \omega y_0$; and from Art. 101, $a_0 = v_0^2/y_0 = \omega^2 y_0$.

* The term " displacement " is used here with a different meaning from that given it in Art. 80. As there defined, it refers to change of position during an interval of time; as here defined it is identical with the s-coordinate measured from the central point as origin. This latter usage is customary and convenient in discussions of vibrations.

Now y, v, and a are respectively the projections on OY of OP, V_0, and A_0, and so (using plus and minus to signify upward and downward, respectively)

$$y = y_0 \cos \theta = y_0 \cos \omega t \quad (1)$$

$$v = -v_0 \sin \theta = -\omega y_0 \sin \omega t \quad (2)$$

$$a = -a_0 \cos \theta = -\omega^2 y_0 \cos \omega t \quad (3)$$

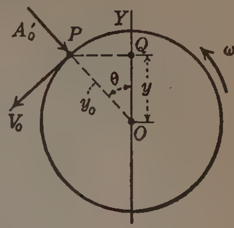

FIG. 190.

It follows from Eqs. 1 and 3 that

$$a = -y\omega^2 \quad (4)$$

Equation 4 shows that the accelerations of Q in various positions are proportional to the displacements of Q in those positions, and that, since acceleration and displacement for any position have opposite signs, the acceleration is always directed from Q toward O.

It has been pointed out that y, v, and a are the projections on the y axis of the vectors OP (or y_0), V_0, and A_0, all of which change direction or rotate together. If these vectors are drawn from a common point as in Fig. 191, then it becomes apparent that, as they rotate, vectors V_0 and A_0 respectively lead vector y_0 by 90° and 180°. And so v and a are said to lead y by these angles, and also by the corresponding time intervals $\frac{1}{4}T$ and $\frac{1}{2}T$. It is said also that v and y respectively follow or lag a by 90° and 180° respectively.

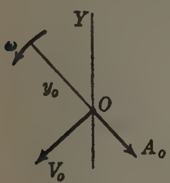

FIG. 191.

Since a particle that has shm has acceleration proportional and opposite to its displacement, the resultant force acting on such a particle is also proportional and opposite to the displacement. A deflected spring or other elastic body exerts a force that is approximately proportional to the deflection and so tends to produce a vibration which is, or closely resembles, a shm. (See Appendix C.)

EXAMPLE 1. The equation of a certain simple harmonic motion is $y = 5$ (in.) sin [3(rad/sec) $\times$ t(sec)]. What are the amplitude, maximum velocity, maximum acceleration, frequency, and period?

Solution. The amplitude is 5 in.; the maximum velocity v_0 is $5 \times 3 = 15$ in./sec; the maximum acceleration a_0 is $5 \times 3^2 = 45$ in./sec²; the frequency is $n = 3 \div 2\pi = 0.478$ cyc/sec; the period T is $1 \div 0.478 = 2.09$ sec.

EXAMPLE 2. It is required to write the equations of motion for the shm of Q (Fig. 192). The amplitude is y_0, the circular frequency is ω or $2\pi n$, and time is reckoned from the instant when P is at P_0.

Solution. The angle P_0OP, no matter how large, is ωt. Therefore at all times

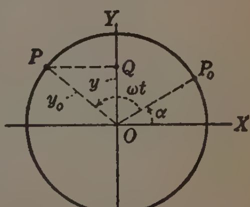

FIG. 192.

$$y = y_0 \sin (\omega t + \alpha) \qquad v = y_0\omega \cos (\omega t + \alpha) \qquad a = -y_0\omega^2 \sin (\omega t + \alpha)$$

If P_0 were at the upper end of the vertical diameter so that $\alpha = 90°$, then

$$y = y_0 \cos \omega t \qquad v = -y_0\omega \sin \omega t \qquad a = -y_0\omega^2 \cos \omega t$$

(Compare these results with Eqs. 1, 2, and 3, respectively.)

104. Force-Acceleration Relationship for a Particle in Curvilinear Motion. The fundamental equation of motion for a particle, $F = ma$ or $\Sigma F_x = ma_x$ (Art. 89), applies to a particle having any kind of motion. In problems involving curvilinear motion, the object and method of solution are essentially the same as in problems involving rectilinear motion (Art. 91). In the example and problems of this present article we shall, as in Art. 91, deal with actual bodies which may be regarded as particles because their size and manner of moving are such that all constituent particles move alike or practically alike.

FIG. 193.

EXAMPLE 1. A small lead ball weighing 0.5 lb is suspended by a cord 10 ft long and made to swing in a vertical plane like a pendulum. When it reaches the lowest point of its swing its speed is 15 ft/sec. It is required to determine the tension in the cord at that instant.

Solution. The fbd for the ball when at the lowest point of its swing is shown in Fig. 193. The forces acting are its own weight and the vertical pull T of the cord (air resistance is considered negligible). There being no force with a component along the tangent to the path, $a_t = 0$. The acceleration of the ball therefore consists of the normal acceleration $a_n = v^2/r = 15^2 \div 10 = 22.5$ ft/sec² directed upwards as indicated. Taking the direction of the acceleration as the positive direction,

$$\Sigma F_n = T - 0.5 = (0.5 \div 32.2)22.5$$

whence $T = 0.85$ lb.

CENTRIFUGAL FORCE. The term "centrifugal force" is familiar to almost everyone, but it is often used loosely and with little understanding of its real meaning. We now explain just what it does mean, using the above example for purposes of illustration.

When a particle has curvilinear motion a force must be exerted on it to give it its normal acceleration. This force, equal to mv^2/r, and directed *inward* toward the center of curvature of the path, is called a *centripetal* (meaning toward the center) force; it is exerted *on* the particle by some other body — the track or cord or whatever it is that constrains or guides it in its path. The particle exerts an equal and opposite *outward* force on the constraining or guiding body; this force is called a *centrifugal* (meaning away from the center) force. It represents the resistance the particle offers, by virtue of its inertia, to being made to change the direction of its motion. In the above example 0.5 lb of the 0.85-lb force T is needed to balance the weight of the ball; the remainder, 0.35 lb, is needed to give the ball its normal acceleration. This part of T is the centripetal force. The ball exerts an outward (downward) force on the string equal to 0.85 lb; of this, 0.5 lb is exerted by virtue of its weight and the remaining 0.35 lb is the centrifugal force the ball exerts by virtue of its inertia.

Another illustration of centripetal and centrifugal forces is afforded by Ex. 2 below, which deals with the forces acting on a box in a car that is rounding a curve. The inward or normal component R_n of the friction force on the box is a centripetal force; the equal outward force exerted by the box on the car is a centrifugal force.

In a method of analysis explained in Chapter XII the term "centrifugal force" is commonly and correctly used with a meaning somewhat different from that attributed to it here; this other usage is explained in Art. 135 but is not relevant to the present discussion. You should avoid the use of the term except when it is appropriate, and, since the examples and problems of this chapter deal with the forces exerted *on* a moving body, they will rarely involve centrifugal forces.

EXAMPLE 2. A box weighing 60 lb rests on the horizontal floor of a truck which rounds a curve of 300-ft radius. At a certain instant the speed of the truck is 50 mi/hr and it is slowing down at the rate of 8 mi/hr/sec. It is required to determine the forces exerted on the box at that instant by the bottom of the truck, and to determine the least coefficient of friction that would prevent slipping of the box on the floor of the truck.

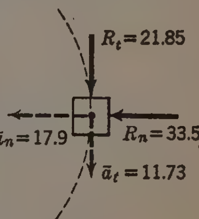

FIG. 194.

Solution. The acceleration of the box consists of a tangential component $a_t = 8$ mi/hr/sec $= 11.73$ ft/sec² and (since $v = 50$ mi/hr or 73.3 ft/sec) a normal component $a_n = 73.3^2 \div 300 = 17.9$ ft/sec². These component accelerations are represented in the (incomplete) fbd (Fig. 194), which shows the box as viewed from above. The forces acting are the weight and the normal reaction N of the truck (not shown), and the friction force represented by its components R_t and R_n, along the tangent and normal to the path respectively. Then in the vertical or y direction

$$\Sigma F_y = N - 60 = 0 \qquad \text{whence } N = 60 \text{ lb}$$

In the tangential direction

$$\Sigma F_t = R_t = (60 \div 32.2)11.73 \qquad \text{whence } R_t = 21.85 \text{ lb}$$

In the normal direction

$$\Sigma F_n = R_n = (60 \div 32.2)17.9 \qquad \text{whence } R_n = 33.5 \text{ lb}$$

The total friction force $R = (21.85^2 + 33.5^2)^{\frac{1}{2}} = 40$ lb, and the minimum coefficient of friction that would prevent slip is $40 \div 60 = 0.667$.

EXAMPLE 3. A small, heavy ball is projected from ground level upward and forward at an angle θ to the horizontal with a moderate initial velocity equal to v_0. It is required to determine the equation of the path or trajectory of the ball, the maximum height attained, the time required for the ball to return to the ground, and the distance from the starting point to the point where it strikes the ground. Air resistance may be disregarded.

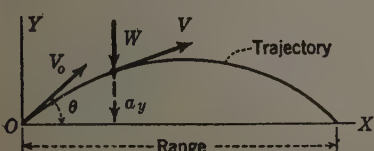

FIG. 195.

Solution. In Fig. 195 are shown the trajectory referred to horizontal and vertical axes x and y, the initial velocity V_0, and the fbd for the ball at some instant during its flight. The only force acting on the ball is its own

weight W. The horizontal and vertical components of the motion are analyzed as follows:

$$\Sigma F_y = -W = \left(\frac{W}{g}\right)a_y \qquad \text{and} \qquad \Sigma F_x = 0 = \left(\frac{W}{g}\right)a_x$$

$$a_y = -g \qquad \text{and} \qquad a_x = 0$$

Integrating:

$$v_y = -gt + C_1 \qquad \text{and} \qquad v_x = 0 + C_2$$

The constants of integration are found from the facts that, when $t = 0$, $v_y = v_0 \sin\theta$ and $v_x = v_0 \cos\theta$; hence $C_1 = v_0 \sin\theta$ and $C_2 = v_0 \cos\theta$, and so

$$v_y = -gt + v_0 \sin\theta \qquad \text{and} \qquad v_x = v_0 \cos\theta$$

Integrating again, and noting that both new constants of integration are zero because $x = 0$ and $y = 0$ when $t = 0$, we have

$$y = -\tfrac{1}{2}gt^2 + v_0 t \sin\theta \qquad \text{and} \qquad x = (v_0 \cos\theta)t$$

Substituting in the equation for y the value of t in terms of x, we have as the equation of the path

$$y = x \tan\theta - x^2 \frac{g}{2v_0^2 \cos^2\theta}$$

This equation shows that the trajectory is a parabola.

To find the maximum height attained, we note that $v_y = 0$ when the ball is at the highest point of its trajectory. Thus, when $v_y = 0$, $t = (v_0 \sin\theta)/g$ and so

$$\max y = \frac{(v_0 \sin\theta)^2}{2g}$$

To find the time of flight we set $y = 0$ and solve for t, getting

$$t = \frac{2(v_0 \sin\theta)}{g}$$

(Note that this is twice the time required for the ball to reach the top of its trajectory; hence the projectile takes just as long to traverse the " ascending branch " of the trajectory as to traverse the " descending branch." You should show that these two parts of the path are alike, and that the trajectory is symmetrical about the mid-ordinate.)

To find the range we substitute this value of t in the equation for x, getting

$$\max x = \frac{v_0^2 \sin 2\theta}{g}$$

Or we could use the equation of the trajectory, set $y = 0$, and solve for x, which gives the same result. The expression for maximum x shows that the maximum possible range is secured when $\theta = 45°$, and that this range varies as the square of the initial velocity.

(Air resistance is disregarded in the above solution. Actually, air resistance may greatly affect the motion of a body; its influence depends upon the size, mass, and form of the body, upon the density of the air, which varies markedly with altitude, and especially upon the speed of the motion. It cannot be ignored if even approximate results are to be obtained in calculations concerning the trajectories of projectiles, the range of which is greatly affected by air resistance. For example, the 95-lb shell of a certain 155-mm cannon has a muzzle velocity of 1950 ft/sec; without air resistance it would have a maximum range equal to $1950^2 \div 32.2 =$ 118,200 ft when fired at an angle of elevation of 45°. Actually its maximum range is 45,000 ft, attained at an angle of elevation of 36.3°. The 170-grain [0.02457-lb] bullet of the .30 caliber Springfield rifle has a muzzle velocity of 2700 ft/sec and according to the formula

would have a maximum range of 226,500 ft, whereas its actual maximum range is only 16,500 ft and is attained at an angle of elevation of 34.8°. In general, as these examples indicate, air resistance has more influence on a light body moving with high velocity than on a heavy body moving more slowly, but the density, cross-sectional area, and form of the body are also factors of importance.)

105. Curvilinear Translation. Rectilinear translation is defined in Art. 90. Curvilinear translation is such a motion of a rigid body that every straight line of the body remains fixed in direction and every point of the body has curvilinear motion. The side rod of a locomotive, which connects the driving wheels, has curvilinear translation when the locomotive is running on a straight track, and it would continue to have curvilinear translation if the track itself were moved up and down or sideways, as long as it remained parallel to its original position.

The motions of all points of a body in curvilinear translation are alike. For by definition a line AB joining any two points A and B of the body remains fixed in direction; therefore, if for any time interval AB and $A'B'$ represent the initial and final positions of the line, the displacement of A from A to A' is equal and parallel to the displacement of B from B to B', and, since the displacements of A and B, and hence of all points of the body, are equal and parallel for any interval, the velocities and accelerations at any instant of all points of the body are identical. By displacement, velocity, and acceleration of a body having curvilinear translation are meant the displacement, velocity, and acceleration respectively of any point of that body.

It is shown in Art. 92 that the resultant of all external forces acting on a body that has rectilinear translation is a single force whose line of action passes through the mass-center of the body. There is nothing in the proof that limits it to rectilinear translation; it is based on the fact that all particles have identical accelerations and so applies equally to curvilinear translation. Therefore a body that has curvilinear translation can be treated as a particle as far as the use of the equation $\Sigma F_x = ma_x$ is concerned, and the external forces must satisfy the criterion $\Sigma M_G = 0$, where G denotes any axis through the mass-center.

EXAMPLE. A straight uniform bar 12 ft long weighing 60 lb is suspended in a horizontal position by two vertical ropes each 20 ft long. One rope is attached to the bar at 4 ft from the left end; the other is attached at the right end. The bar is raised to such a height that when released and allowed to swing freely it will have, when in its lowest position, a velocity of 16 ft/sec. It is required to determine the tension in each rope when the bar is in this position.

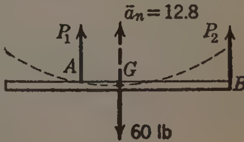

FIG. 196.

Solution. The fbd for the bar when in its lowest position is as shown in Fig. 196. The external forces are the weight of the bar, acting through the mass-center, and the vertical pulls P_1 and P_2 of the ropes at A and B respectively. The acceleration of the bar is vertical — there are no horizontal forces, therefore no horizontal acceleration — and is, simply, the normal acceleration $a_n = 16^2 \div 20 = 12.8$ ft/sec², directed

upward as shown. The following equations apply:

$$\Sigma F_y = P_1 + P_2 - 60 = \frac{60}{32.2} 12.8$$

and

$$\Sigma M_G = -(P_1 \times 2) + (P_2 \times 6) = 0$$

Solution gives $P_1 = 62.8$ lb and $P_2 = 20.9$ lb.

(It should be noted that the summation of moments is not equal to zero about either A or B, the points about which moments would naturally be taken if the bar were at rest and the values of P_1 and P_2 to be found.)

106. Motion of the Mass-Center of a Body. The equation $\Sigma F_x = ma_x$ completely defines the relation between the acceleration of a particle, or of a rigid body having motion of translation, and the forces that act on it. With this as a basis, we shall now develop equations that define the relation between the motion of the *mass-center* of any body (solid, liquid, or gaseous) having any kind of motion and the external forces that act on that body.

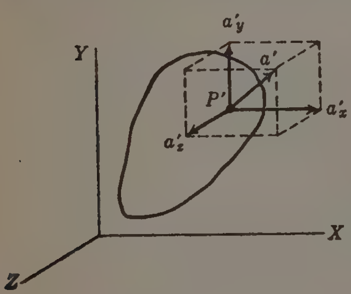

FIG. 197.

Let P' (Fig. 197) be any elementary particle of the body and OX any line making with OY and OZ a set of rectangular axes. Then for P' we can write $\Sigma F_x' = dm' \cdot a_x'$, the primes denoting that F_x, dm, and a_x pertain to the particle P'. Similarly, for any other particle P'' we can write $\Sigma F_x'' = dm'' \cdot a_x''$. Suppose such equations to be written for all particles composing the body. The sum of all the left-hand members of these equations would equal the sum of all the right-hand members, or

$$\left\{\begin{matrix}\text{The summation, parallel to any axis } x\text{, of the} \\ \text{components of all forces acting on all particles}\end{matrix}\right\} = dm' \cdot a_x' + dm'' \cdot a_x'' + \cdots$$

Now the left-hand member of this equation includes the x components of all forces, external and internal, that act on the body. But since the internal forces occur in pairs of equal, opposite, and collinear actions and reactions their components cancel, and so the left-hand member represents also the x component of the external force system.

It will now be shown that the right-hand member of the equation is equal to $m\bar{a}_x$, where m is the mass of the body and $\bar{a}_x$ is the x acceleration of its mass-center. From Art. 87

$$dm' \cdot x' + dm'' \cdot x'' + \cdots = m\bar{x}$$

where x', x'', etc., represent the x coordinates of the constituent particles at any instant, and $\bar{x}$ the x coordinate of the mass-center of the body at the same

instant. Differentiating both sides of this equation with respect to time, and noting that $dx/dt = v_x'$, etc., gives

$$dm' \cdot v_x' + dm'' \cdot v_x'' + \cdots = m\bar{v}_x$$

Differentiating again, and noting that $dv_x'/dt = a_x'$, etc., gives

$$dm' \cdot a_x' + dm'' \cdot a_x'' \cdots = m\bar{a}_x$$

We have thus shown that the component of the external system of forces along OX is equal to the product of the mass of the body and the x component of the acceleration of its mass-center. Since the axis OX represents any line, it follows that any number of equations of the form $\Sigma F_x = m\bar{a}_x$ may be written for a body, one for each direction that may be chosen. Only three such equations would be independent. They might be written for any three axes no two of which were parallel, but it is convenient to choose rectangular axes and to write

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma F_z = m\bar{a}_z$$

These are the *equations of motion of the mass-center.*

Principle of motion of the mass-center. The equations of motion of the mass-center constitute a mathematical statement of an important principle which we shall call the principle of the motion of the mass-center. It may be put into words as follows: *In any motion of a body (whether rigid or not) the algebraic sum of the components along any line of all the external forces equals the product of the mass of the body and the component of the acceleration of the mass-center along that line.* It appears, then, that the acceleration of the mass-center of a body is the same as that of a particle of equal mass acted on by forces equal to and parallel to the external forces exerted on the body.

The exact meaning and limitations of the principle of the motion of the mass-center must be understood. It completely describes the relations existing between the external forces that act on a body and the acceleration of the mass-center of the body. It shows that for forces of given magnitude and direction the mass-center of a body moves the same whether the body turns or has motion of translation; whether the body is rigid like a steel bar, or flexible like a chain, or fluid like a quantity of water. And it shows that neither the points of application nor the positions of the lines of action of the applied forces make any difference, as far as the motion of the mass-center is concerned. But the principle tells us nothing whatever concerning the motion of points of the body other than the mass-center.

107. Typical Problems; Examples. The principle of the motion of the mass-center can be used to determine something concerning the forces that make a body move in a specified way, or to determine something concerning the motion of a body that is acted on by specified forces. The procedure in solving either type of problem is much like that described in Art. 91 for the

solution of problems involving motion of a particle. The fbd for the body under consideration should show the path and the acceleration components of the mass-center as far as they are known or can be assumed, as well as all external forces that act on the body. The axes parallel to which the forces are summed up should be selected according to convenience. If the motion of the mass-center is rectilinear, it is usually convenient to choose axes parallel and normal to the path. If the motion of the mass-center is curvilinear and uniplanar, it is usually convenient to choose axes one of which is parallel to the tangent to the path, one normal to the path in the plane of the motion, and the third normal to the plane of motion. But the selection of these axes is purely arbitrary and should be made with due consideration of the conditions of the problem.

EXAMPLE 1. A slender uniform rod 3 ft long weighing 12 lb is bent at right angles 1 ft from an end and pinned at that point to the center O of a horizontal table as shown in Fig. 198. The table and rod are made to rotate uniformly about a vertical axis through O at the rate of 200 rev/min. It is required to determine the reaction of the pin on the rod.

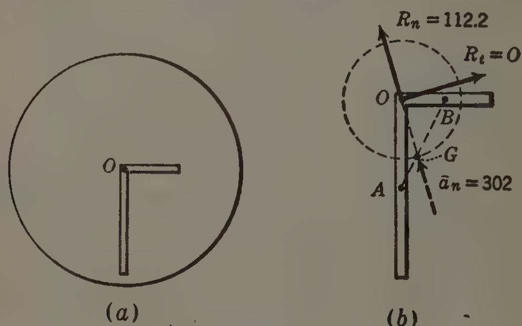

FIG. 198.

Solution. The fbd for the rod is shown in Fig. 198*b*. The coordinates with respect to O of the mass-center G of the bent rod, found in the usual way (Art. 73), are $\bar{x} = \frac{1}{6}$ ft, $\bar{y} = \frac{2}{3}$ ft. Therefore $OG = [(\frac{1}{6})^2 + (\frac{2}{3})^2]^{\frac{1}{2}} = 0.688$ ft, and this is the radius of the circular path of G. The velocity of G is equal to $2\pi(0.688)(200 \div 60) = 14.4$ ft/sec and is constant in magnitude. Since v is constant, the acceleration of G consists simply of the normal acceleration $\bar{a}_n = 14.4^2 \div 0.688 = 302$ ft/sec², directed as shown.

The forces acting on the bar are its own weight, the reaction of the table, and the reaction of the pin. The first two forces are equal and opposite, since G has no vertical acceleration. The force exerted by the pin is represented by its components R_t and R_n, parallel respectively to the tangent and the normal to the path at G. According to the principle of motion of the mass-center

$$\Sigma F_t = R_t = 0 \quad \text{and} \quad \Sigma F_n = R_n = (12 \div 32.2)302 \quad \text{whence } R_n = 112.2 \text{ lb}$$

EXAMPLE 2. A flat car is drawn past a stationary sand bin at a uniform speed of 15 ft/sec while sand is dumped on it from the bin at the uniform rate of 500 lb/sec. It is required to determine the pull P needed to keep the car moving under these circumstances. (All resistance other than that offered by the sand may be disregarded, and it may be assumed that the sand does not slip after falling upon the car.)

Solution. We take as the body under consideration all the sand. Conditions t sec after dumping has commenced are represented in Fig. 199, which is a partial fbd for the sand (shown shaded), with contiguous bodies, for clearness, indicated by dotted lines.

Since the rate of dumping is 500 lb/sec = 15.5 slugs/sec, at any instant the mass of the sand on the car is $15.5t$ slugs; and the distance moved by the car, and hence the loaded length, is $15t$ ft. Letting m denote the total mass of the sand, the mass-center G of the entire body of sand, found by taking moments about the center of the bin, is at a distance

$$\bar{x} = \frac{(15.5t)(7.5t)}{m} = \frac{116.25t^2}{m}\text{ ft}$$

from the center of the bin. The acceleration of G is

$$\bar{a}_x = \frac{d^2\bar{x}}{dt^2} = \frac{232.5}{m}\text{ ft/sec}^2$$

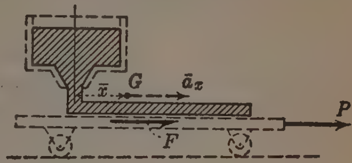

FIG. 199.

The only forces acting on the sand in the direction of the acceleration are the equal and opposite thrusts from the sides of the bin and the friction force F exerted by the car. Therefore, applying the equation of motion of the mass-center to the entire body of sand, we have

$$\Sigma F_x = F = m\frac{232.5}{m} = 232.5 \text{ lb}$$

The sand exerts an equal and opposite reaction on the car, and, since the car has no acceleration, this force is just equal to the propelling force P required to keep the car moving; hence $P = 232.5$ lb.

(It is shown in Art. 159 that this problem can be solved more easily by a different method. The above solution is given to emphasize the significance of the principle of motion of the mass-center.)

EXAMPLE 3. (Examples 1 and 2 illustrate the use of the principle of motion of the mass-center in problems where a complete solution can be obtained thereby. In some problems this principle alone is not sufficient for a complete solution but does enable one to reach certain conclusions. The two examples that follow illustrate the use of the principle in the partial solution of a problem.)

(*a*) Suppose that a uniform straight slender rod is placed on end in an almost vertical position on a smooth, horizontal floor and allowed to fall over. Does the lower end stay in place while the rod falls to the floor?

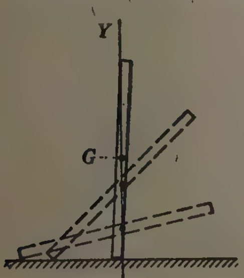

FIG. 200.

Answer. The only forces on the rod (if we disregard air resistance) are its own weight and the reaction of the smooth floor. Both these forces are vertical; therefore there is no horizontal force on the rod; therefore its mass-center has at no time any horizontal acceleration; therefore the mass-center acquires no horizontal velocity or displacement; therefore the mass-center moves vertically (along axis y) and the rod falls as indicated in Fig. 200. (We cannot, without applying principles not yet studied, find how long it will take the rod to fall, or how fast it will be moving when it strikes the floor. Indeed, the principle of the motion of the mass-center only enables us to conclude that the mass-center has no horizontal acceleration; it does not justify the conclusion that the rod falls at all, because we do not *know* that the upward reaction of the floor is less than the weight of the rod. We know from experience and common sense that the rod falls over, but to *prove* it [other

than experimentally], and to analyze the motion, we would have to use equations developed in Chapter XI, pertaining to the kind of motion the rod would have.)

(*b*) Suppose that the lower end of this rod is placed against a stop (Fig. 201) that prevents it from slipping back. Will the lower end now remain in place until the rod falls to the floor?

Answer. Let us *suppose* that the lower end does remain in place, the rod simply rotating about the point of contact, and then see if the forces present are consistent with such motion. At first, as the rod falls over, the mass-center starts to move towards the right, and so, for a time, has an acceleration towards the right. This is possible, because the horizontal reaction of the stop could account for such an acceleration. But when the rod reached the floor the mass-center would be moving vertically; it would have lost its horizontal velocity, and so, for a time, must have had an acceleration towards the left. But there is no force acting towards the left to account for such an acceleration, and so the rod could *not* move as assumed. It is clear that the maximum rightward velocity gained by the mass-center must be retained from the instant it is acquired until the fall is completed, and, since the mass-center is moving to the right at the instant the rod strikes the floor and for some unknown interval before, the lower end must move to the right instead of staying in place, and the rod behaves somewhat as indicated in Fig. 201. (Show that the same conclusion can be reached by considering the normal and tangential components of the acceleration of the mass-center of the rod.)

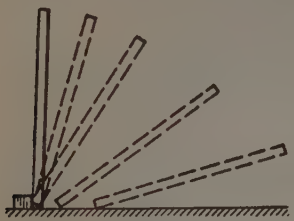

FIG. 201.

As in (*a*), we cannot analyze the motion further by the principle of motion of the mass-center.

CHAPTER X

ROTATION OF A RIGID BODY

108. Definition. A rotation is such a motion of a rigid body that one line of the body or of an extension of the body remains fixed. The fixed line is the **axis of rotation.** The motion of the flywheel of a stationary engine is rotation, and the axis of rotation is the axis of the shaft on which the wheel is mounted; the motion of an ordinary clock pendulum is a rotation, and the axis of rotation is the horizontal line through the point of support and perpendicular to the pendulum. Obviously all particles of a rotating body except those on the axis describe circles or circular arcs whose centers are in the axis and whose planes are perpendicular to the axis. The plane in which the center of gravity of the body moves is the **plane of rotation,** and the intersection of the axis of rotation and this plane is the **center of rotation.** All particles of the body on any line parallel to the axis move alike; hence the motion of the projection of the line on the plane of rotation represents the motion of all these particles. And the motion of the body itself is represented by the motion of the projection of the body on the plane of rotation.

109. Angular Displacement. By angular displacement of a rotating body during any time interval is meant the angle described during that interval by any line parallel to the plane of rotation. Obviously all such lines turn through equal angles in the same interval; it is convenient, for purposes of illustration, to select a line that cuts the axis. Let the irregular outline (Fig. 202) represent a rotating body, the plane of rotation being the plane of the paper and the center of rotation being O. Line OP is a line in the body perpendicular to the axis of rotation, and θ is the angle between OP and any fixed line of reference OX, also perpendicular to the axis of rotation. As customarily, θ is regarded as positive when measured counterclockwise from OX to OP. If θ_1 and θ_2 denote initial and final values of θ for any time interval, then the angular displacement for that interval is $\Delta\theta = \theta_2 - \theta_1$.

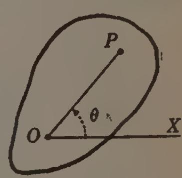

FIG. 202.

110. Angular Velocity. The angular velocity of a rotating body, commonly denoted by ω, is the time rate at which its angular displacement occurs, or the rate of change of θ.* Therefore

$$\omega = \frac{d\theta}{dt}$$

* Compare Art. 97, where the angular velocity of a line (radius vector) was thus defined.

It is evident from this equation that ω is positive or negative according to the sign of $d\theta$, and so, in conformity with the rule of sign for θ given in Art. 109, ω is considered positive when the rotation is counterclockwise and negative when the rotation is clockwise.

Angular velocity can be expressed in any units of angle and time, as revolutions per minute, degrees per second, or radians per second. In mechanics, it is usually most convenient to use radians per second, and in general ω will be so expressed in this book.

If a body rotates uniformly, describing equal angles in all equal intervals of time, then ω is constant and is equal to the angular displacement $\Delta\theta$ for any interval of time Δt divided by that interval. If the body does not rotate uniformly, then $\Delta\theta/\Delta t$ is the *average* angular velocity ω_a for the interval Δt.

EXAMPLE 1. A spoked wheel rotates so that the angle θ between a certain spoke (corresponding to the line OP in Fig. 202) and the horizontal is given by the equation $\theta = 2t^3$, where θ (measured as in Fig. 202) is in radians and t is the time in seconds after motion begins. It is required to derive the equation for the angular velocity ω of the wheel, and to calculate ω for the instant when the spoke in question first becomes vertical and also for the instant when it returns to its original horizontal position for the second time.

Solution. Since $\theta = 2t^3$, $\omega = d\theta/dt = 6t^2$. This is the equation for ω. When the spoke first becomes vertical $\theta = \pi/2$, and setting $\pi/2 = 2t^3$ and solving for t gives $t = 0.922$ sec, the time required for the spoke to reach the vertical position. Substituting this value of t in the equation for ω gives $\omega = 5.10$ rad/sec. When the spoke returns to its original position for the second time it has made two complete revolutions; therefore $\theta = 4\pi$. Setting $4\pi = 2t^3$ and solving for t gives $t = 1.84$ sec, and substituting this value of t in the equation for ω gives $\omega = 20.3$ rad/sec. Both the above values of ω are positive, indicating that θ is increasing and that the rotation is counterclockwise.

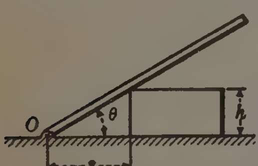

FIG. 203.

EXAMPLE 2. A straight bar pinned at one end to a horizontal floor rests upon a block as shown in Fig. 203. The block can be pushed to and fro along the floor, thus causing the bar to rotate about O. It is required to derive a general formula for the angular velocity of the bar and to determine this angular velocity when $h = 2$ ft and the block is 4 ft from O and is moving to the left at 6 ft/sec.

Solution. Let s denote the distance in feet from O to the nearer face of the block; then it is evident from the figure that $\theta = \tan^{-1} h/s$. Therefore

$$\omega = \frac{d\theta}{dt} = \left(\frac{1}{1 + (h/s)^2}\right)\left(-\frac{h}{s^2}\right)\frac{ds}{dt}$$

and, since $ds/dt = v$, the velocity of the block, the expression reduces to

$$\omega = \frac{-hv}{s^2 + h^2} \quad \text{radians per second when } v \text{ is in feet per second and } h \text{ and } s \text{ in feet}$$

This is the general formula for ω in terms of h, s, and v. When v is positive (block moving to the right), ω is negative (bar rotating clockwise); when v is negative (block moving to the left), ω is positive (bar rotating counterclockwise). When $h = 2$, $v = -6$, and $s = 4$, substitution gives $\omega = 0.6$ rad/sec.

111. Angular Acceleration. The angular acceleration of a moving body, commonly denoted by α, is the time rate of change of its angular velocity. Therefore

$$\alpha = \frac{d\omega}{dt}$$

It is evident from this equation that α is positive or negative according to the sign of $d\omega$, that is, according to whether positive or negative angular velocity is being taken on. As with angular velocity, we associate sign with direction: positive angular acceleration is counterclockwise, and negative angular acceleration is clockwise.

Angular acceleration can be expressed in any units of angular velocity and time; it is usually most convenient to use radians per second per second, and in general α will be so expressed in this book.

If the angular velocity changes uniformly, then the angular acceleration is constant and is equal to the increment in angular velocity $\Delta\omega$ for any interval of time Δt divided by that interval. If the angular velocity does not vary uniformly, then $\Delta\omega/\Delta t$ is the *average* angular acceleration ω_a for the interval Δt.

It is evident from the definitions and formulas of this article and of Arts. 81 and 83 that the relationships between θ, ω, and α are exactly like those between s, v, and a in rectilinear motion. Hence all the procedures described in the discussion of rectilinear motion — integration, graphical solutions, etc. — are equally applicable to angular motion.

EXAMPLE 1. For the wheel described in Ex. 1 of Art. 110 it is required to derive the equation for the angular acceleration α, to determine α for the instant when the selected spoke first becomes vertical, and to determine the average angular acceleration for the interval corresponding to that displacement.

Solution. In the example referred to it was found that $\omega = 6t^2$, that the spoke first reached the vertical position when $t = 0.922$ sec, and that at that instant $\omega = 5.10$ rad/sec. Since $\omega = 6t^2$, $\alpha = d\omega/dt = 12t$. This is the equation for α. Substituting $t = 0.922$ gives $\alpha = 11.06$ rad/sec². Since α is positive the angular acceleration is counterclockwise, and since ω is also counterclockwise, the wheel is turning faster and faster.

When $t = 0$, $\omega = 0$; therefore, for the interval $t = 0$ to $t = 0.922$, $\Delta\omega = 5.10$, and the average angular acceleration for the interval is $\alpha_a = \Delta\omega/\Delta t = 5.10 \div 0.922 = 5.53$ rad/sec².

EXAMPLE 2. It is required to derive a general formula for the angular acceleration of the bar described in the example of Art. 110, and to determine this angular acceleration when $h = 2$ ft, $s = 4$ ft, and the block is moving with a velocity $v = 6$ ft/sec to the left and with an acceleration $a = 3$ ft/sec² to the left.

Solution. It was found in the example referred to that $\omega = \dfrac{-hv}{s^2 + h^2}$. Differentiating with respect to t, and noting that both s and v are variable and that $ds/dt = v$ and $dv/dt = a$, it is found that

$$\alpha = \frac{2hs}{(h^2 + s^2)^2}v^2 - \frac{h}{h^2 + s^2}a$$ radians per second per second when h is in feet, v in feet per second, and a in feet per second per second

This is the general formula for the angular acceleration of the bar, and it should be noted that α depends upon all four quantities h, s, v, and a.

When $h = 2$, $s = 4$, $v = -6$, $a = -3$ (the conditions stated above), substitution gives $\alpha = 1.74$ rad/sec² counterclockwise. It was found in the example of Art. 110 that at the instant in question the bar was rotating counterclockwise; since α is also counterclockwise the bar is turning faster and faster.

EXAMPLE 3. A wheel which is rotating clockwise at 100 rev/min is given a counterclockwise angular acceleration which increases with time according to the equation $\alpha = 4t$, the units being radians and seconds. It is required to determine how many revolutions the wheel will make before its angular velocity becomes zero.

Solution. Since $\alpha = d\omega/dt$, $\omega = \int \alpha\, dt = \int 4t\, dt = 2t^2 + C$. The constant of integration C is found from the fact that, when $t = 0$, $\omega = 100$ rev/min clockwise $= -10.47$ rad/sec. Therefore $C = -10.47$ and $\omega = 2t^2 - 10.47$. Setting this equal to zero and solving for t, we find $t = 2.29$ sec. Since $\omega = d\theta/dt$

$$\theta = \int \omega\, dt = \int (2t^2 - 10.47)\, dt = \tfrac{2}{3}t^3 - 10.47t + C$$

Arbitrarily taking $\theta = 0$ when $t = 0$ gives $C = 0$; and substituting $t = 2.29$ gives $\theta = -16$ rad. The wheel therefore makes $16 \div 2\pi = 2.55$ revolutions clockwise before $\omega = 0$.

The angle turned through by the wheel could also have been found by integrating between limits, thus: $\Delta\theta = \int_0^{2.29} (2t^2 - 10.47)\, dt = -16$ rad.

112. Motion of Any Particle of a Body in Rotation. There are simple relationships between the angular velocity and angular acceleration of a rotating body and the velocity and acceleration of any particle of that body. Let P be any particle of the rotating body (Fig. 204), r the distance of P from the axis of rotation O, s the s-coordinate of P measured from any arbitrarily selected origin B on its path, and θ the angle between OB and OP, measured in radians. Then

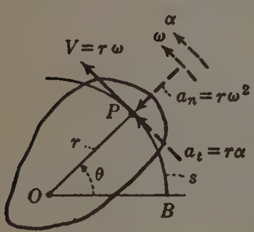

FIG. 204.

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \qquad (1)^*$$

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad (2)$$

$$a_n = \frac{v^2}{r} = r\omega^2 \qquad (3)$$

These formulas give the magnitudes of the velocity, the tangential acceleration, and the normal acceleration of P, and they show that for different particles of a rotating body these magnitudes are directly proportional to the distances of the particles from the axis of rotation. The formulas for v and a_t also indicate, by sign, the sense, along the tangent, of V and A_t. For ω and α positive as shown, the directions of V and A_t are obviously as shown. If ω were reversed, V would be reversed; and if α were reversed, A_t would be reversed. A_n is, of course, always directed towards O.

It should be noted that consistent units must be employed in the above

* Compare Art. 97, where the transverse velocity of the tip of a rotating radius vector was shown to be equal to the length of the vector times its angular velocity.

equations. Thus, if v is to be in feet per second and a_t and a_n in feet per second per second, r must be in feet, ω must be in radians per second, and α must be in radians per second per second.

EXAMPLE. Figure 205 represents a wheel and attached drum mounted upon an axle; the wheel is 4 ft and the drum 1.5 ft in diameter. By means of a rope wound about the drum the wheel is made to rotate. It is required to determine the magnitudes of the velocity, tangential acceleration, and normal acceleration of a point on the perimeter of the wheel at an instant when the rope is being pulled to the right with a velocity of 30 ft/sec and an acceleration of 12 ft/sec². (The thickness of the rope is negligible in comparison with the radius of the drum.)

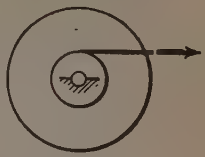

FIG. 205.

Solution. Obviously the speed of a point on the perimeter of the drum is the same as the speed of a point on the rope; therefore, from Eq. 1, the angular velocity of the drum is $\omega = 30 \div 0.75 = 40$ rad/sec. This is also the angular velocity of the wheel. Any point on the perimeter of the wheel therefore has a velocity of magnitude $v = 2 \times 40 = 80$ ft/sec.

Obviously the tangential acceleration of a point on the perimeter of the drum is just as great as the acceleration of a point on the straight part of the rope. Therefore, from Eq. 2, the angular acceleration of the drum is $\alpha = 12 \div 0.75 = 16$ rad/sec². This is also the angular acceleration of the wheel. Any point on the perimeter of the wheel therefore has a tangential acceleration $a_t = 2 \times 16 = 32$ ft/sec².

Any point on the perimeter of the wheel has a normal acceleration of magnitude $a_n = r\omega^2 = 2 \times 40^2 = 3200$ ft/sec².

113. Forces on a Body in Rotation. The principle of motion of the mass-center (Art. 106) applies to a body in rotation as to any other body. It is necessary also to establish the relationship between the external forces that act on a rotating body and the rotation of the body. It will now be shown that *the angular acceleration of a rotating body is directly proportional to the moment of the external forces about the axis of rotation and inversely proportional to the moment of inertia of the body about that axis.* That is, if ΣM_0 denotes the moment of the external forces about the axis of rotation, I_0 the moment of inertia* of the body about that axis, and α the angular acceleration, then α is proportional to $\Sigma M_0 \div I_0$, and, if consistent units are used, then

$$\alpha = \frac{\Sigma M_0}{I_0} \quad \text{or} \quad \Sigma M_0 = I_0\alpha = Mk_0^2\alpha$$

This equation we call the *equation of rotation.*

Let Fig. 206 represent a body rotating either clockwise or counterclockwise in the plane of the paper about an axis indicated by O, with an angular velocity ω and a counterclockwise angular acceleration α. Let P represent any elementary particle of the body, of mass dm and distant r from the axis of rotation.

* The moment of inertia of a body with respect to an axis is the sum of the products obtained by multiplying the mass dm of each elementary particle of the body by the square of its distance r from the axis. This sum is $\int dm \cdot r^2$. See Appendix B for further discussion of moment of inertia.

The acceleration of P is completely represented by the tangential and normal components $a_t = r\alpha$ and $a_n = r\omega^2$, directed as shown. The resultant of all forces acting on P is therefore completely represented by a tangential component $dm \cdot r\alpha$ and a normal component $dm \cdot r\omega^2$. Of these, only the tangential component has a moment about O, and so the moment of all forces acting on P about the axis of rotation is $dm \cdot r\alpha \cdot r = dm \cdot r^2\alpha$. The moment of all forces that act on all the particles of the body is therefore equal to $\int dm \cdot r^2\alpha = \alpha\int dm \cdot r^2 = \alpha I_0$. Now the system of forces acting on all the particles consists of internal and external forces. The internal forces jointly have no moment, since they consist of pairs of equal, opposite, and collinear forces. Hence the moment of the external forces about the axis of rotation equals $I_0\alpha$, as stated.

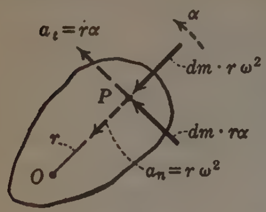

FIG. 206.

114. Typical Problems; Examples. The equation of rotation is used to determine the angular acceleration of a rotating body under given circumstances or to determine the circumstances under which a body will have a given angular acceleration. The body in question may be part of a more or less complicated system, involving several unknown quantities — velocities, accelerations, forces, dimensions, or masses. The equation of motion of the mass-center and the equation of rotation, together with the formulas that express the relations between angular and linear velocity and acceleration, will usually suffice for a solution.

EXAMPLE 1. A disk of cast iron, 4 in. thick, 3 ft in diameter, and weighing 1053 lb, is supported on a frictionless horizontal shaft 3 in. in diameter as shown in Fig. 207*a*. A cord (weight negligible) is wrapped around the disk, and a pull of 100 lb is applied thereto. It is required to determine the angular acceleration of the disk.

Solution. The external forces acting on the disk (Fig. 207*b*) are its weight, the force exerted by the cord (equivalent to a tangential force $P = 100$ lb), and the reaction R of the axle. Of these, only P has a moment about the axis of rotation. The square of the radius of gyration of the disk with respect to the axis of rotation is $\frac{1}{2}(1.5^2 + 0.125^2) = 1.133$ ft² (Appendix B); therefore its moment of inertia about that axis is $(1053 \div 32.2)\ 1.133 = 37.0$ slug-ft². Applying the equation of rotation

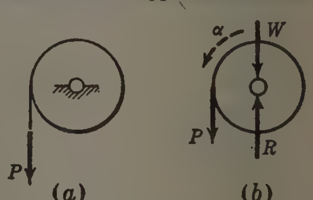

FIG. 207.

$$\Sigma M_0 = 100 \times 1.5 = 37.0\alpha \quad \text{whence } \alpha = 4.05 \text{ rad/sec}^2$$

EXAMPLE 2. Suppose that the disk of Ex. 1 is made to turn by means of a body B suspended from the cord, the weight of B being 100 lb. It is required to determine the angular acceleration of the disk and the tension in the cord under these circumstances.

Solution. Obviously, B moves with a downward acceleration; therefore the upward pull exerted on it by the cord is less than 100 lb and so α is less than 4.05 rad/sec². The tension

in the cord being unknown, the angular acceleration of the disk cannot be found directly by means of a single equation as in Ex. 1; it is necessary to consider both bodies.

The forces acting on the disk (Fig. 208*a*) are its own weight, the downward pull P of the cord (equal to the tension in the cord), and the reaction of the axle. The forces acting on the body B (Fig. 208*b*) are its own weight and the upward pull of the cord P.

For the disk, $\quad \Sigma M_0 = P \times 1.5 = 37\alpha$

For the body B, $\quad \Sigma F_y = 100 - P = \dfrac{100}{32.2}a$

and (Art. 112)

$$a = 1.5\alpha$$

Simultaneous solution of the above three equations gives $\alpha = 3.41$ rad/sec² and $P = 84.1$ lb.

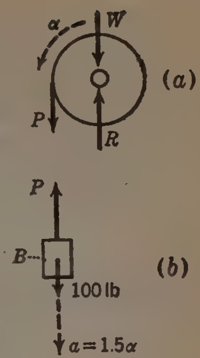

FIG. 208.

(In problems such as the foregoing it is essential that the signs of the forces and accelerations be consistent. Thus, in the equation for the body B, the acceleration, being downward, must agree in sign with the 100-lb force, not with P.)

AXLE REACTIONS. In each of the above examples the rotating body turns about an axis that passes through the mass-center, and so the mass-center has no acceleration. The external forces on the rotating body therefore satisfy the condition $\Sigma F = 0$ along any axis. Thus in Ex. 1 the constraining force R exerted by the shaft or bearings on the rotating body, called the axle reaction, is found from $\Sigma F_y = -1053 - 100 + R = 0$ to be 1153 lb, and in Ex. 2 the axle reaction is found from $\Sigma F_y = -1053 - 84.1 + R = 0$ to be 1137 lb. (Note that the equation $\Sigma F_y = 0$ is written for the forces on the rotating body alone; for the whole system — pulley, rope, and weight — ΣF_y does *not* equal zero, because the mass-center of the whole system has a downward acceleration.) In each of these examples R is constant in direction and magnitude. But, if the body rotates about an axis that does not pass through the mass-center, then the mass-center has an acceleration and the component of the axle reaction in any direction x must be found from the equation $\Sigma F_x = m\bar{a}_x$, and, since the magnitude and direction of $\bar{a}$ are in general changing, the axle reaction (or other constraining forces) also will in general change in magnitude and direction. This condition is illustrated in Exs. 3 and 4 below.

In each of the examples and problems of this chapter the rotating body under consideration has a plane of symmetry and rotates in that plane, and under these circumstances the axle reaction is always a single force in the plane of rotation. But this is not generally true if the plane of rotation is *not* a plane of symmetry of the body. (We do not consider this nonsymmetrical condition in this chapter; it is discussed in Chapter XII.) But the equation $\Sigma M_0 = I_0\alpha$ applies to the rotation of any rigid body, no matter what its form and no matter where the axis of rotation may be located.

EXAMPLE 3. Figure 209 represents a top view of a uniform homogeneous steel bar that lies on a smooth horizontal floor, to which it is pinned at O so that it can rotate about a vertical

axis at that point. The bar is 6 ft long, 4 in. wide, 1 in. thick, and weighs 77 lb. It lies flat on the floor, and O is 1 ft from one end. A clockwise couple C in the horizontal plane, of constant magnitude 40 ft-lb, is applied to the bar, which is initially in the position shown at the left. It is required to determine the resulting angular acceleration, the position of the bar when C has acted for 3 sec, and the pin reaction at that instant.

Solution. The moment of inertia of the bar about O is

$$I = \left\{\frac{1}{12}\left[6^2 + \left(\frac{1}{3}\right)^2\right] + 2^2\right\}\frac{77}{32.2} = 16.75 \text{ slug-ft}^2 \quad \text{(See Appendix B.)}$$

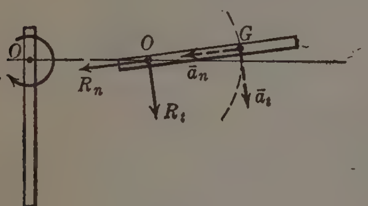

FIG. 209.

The angular acceleration of the bar is found from the equation of rotation to be $\alpha = 40 \div 16.75 = 2.39$ rad/sec². Since α is constant, $\omega = \alpha t$ and $\theta = \frac{1}{2}\alpha t^2$. When $t = 3$, $\omega = 7.17$ rad/sec and $\theta = 10.76$ rad $= 617°$ (measured from the initial position of the bar). Therefore the bar, at the end of the 3-sec interval, is in the position shown at the right.

The acceleration of the mass-center G of the bar is represented by its components $\bar{a}_t = 2\,(2.39) = 4.78$ ft/sec² and $\bar{a}_n = 2\,(7.17^2) = 102.8$ ft/sec², directed as shown. The tangential and normal components of the pin reaction, R_t and R_n, are therefore given by

$$\Sigma F_t = R_t = (77 \div 32.2)(4.78) = 11.43 \text{ lb}$$

$$\Sigma F_n = R_n = (77 \div 32.2)(102.8) = 246.0 \text{ lb}$$

Both R_t and R_n change continuously in direction as the bar rotates; R_t is constant in magnitude, since α is constant; but R_n increases, since ω increases.

EXAMPLE 4. A slender uniform rod BC (Fig. 210*a*) is pinned to a horizontal table at C and restrained at B by nails driven into the table alongside the rod. (The rod fits loosely between the nails.) The rod is 4 ft long, weighs 40 lb, and its center is 2 ft from the center of the table. The table is made to rotate about a vertical axis O through its center. It is required to determine the pressures on the rod at B and at C for the instant when the angular velocity is 30 rev/min and the angular acceleration is 40 rev/min/sec, both clockwise.

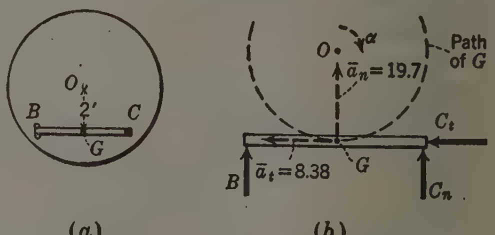

FIG. 210.

Solution. The circumstances at the instant in question are as represented in the fbd (Fig. 210*b*). The path of the mass-center of the rod is a circle of 2-ft radius. Since $\omega = 30$ rev/min $= 3.14$ rad/sec, $\bar{a}_n = 2 \times 3.14^2 = 19.7$ ft/sec². Since $\alpha = 40$ rev/min/sec $= 4.19$ rad/sec², $\bar{a}_t = 2 \times 4.19 = 8.38$ ft/sec². The moment of inertia of the rod about the axis of rotation is

$$I_0 = \frac{1}{12}\frac{40}{32.2}4^2 + \frac{40}{32.2}2^2 = 6.62 \text{ slug-ft}^2$$

The directions of the force B and the components C_n and C_t are assumed to be as shown.

The following equations apply:

$$\Sigma M_0 = B\,(2) - C_n\,(2) + C_t\,(2) = (6.62)(4.19)$$
$$\Sigma F_n = B + C_n = (40 \div 32.2)(19.7)$$
$$\Sigma F_t = C_t = (40 \div 32.2)(8.38)$$

Solution of these equations gives $C_t = 10.4$ lb; $C_n = 10.5$ lb; $B = 14.0$ lb. (Had a negative result been obtained for B, it would have meant that the nail on the other side of the rod exerted a pressure, in the direction opposite to that assumed for B.)

115. Center of Percussion. Figure 211 represents a body which has a plane of symmetry* and which hangs freely from a horizontal frictionless axle O that is perpendicular to that plane of symmetry; G is the center of gravity of the body, vertically beneath O and distant b therefrom. If a horizontal force F in the plane of symmetry is applied to the body a horizontal axle reaction R_x will usually be produced. If F is applied close to the axle, R_x will have a direction opposite to that of F; if F is applied at or near the lower end of the body, R_x will have the same direction as F. If F is applied at a certain distance d from O, then at the instant of application no reaction at all will be produced at the axle. The line of action of F then intersects the line OG at a point Q which is called the **center of percussion** of the body with respect to the axis O. We now determine the position of this point.

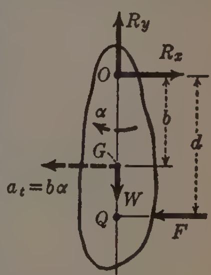

FIG. 211.

The forces on the body are F, its own weight W, and the vertical component R_y of the axle reaction. (The horizontal component R_x of the axle reaction is, by definition, zero.) The motion of the body is a rotation about O. Since F has a moment about the axis of rotation there is an angular acceleration α and a corresponding tangential acceleration of the mass-center $\bar{a}_t = b\alpha$. At the instant F is applied, $\omega = 0$, and so $\bar{a}_n = 0$. The following equations apply:

$$\Sigma F_n = R_y - W = 0 \qquad \text{whence } R_y = W$$
$$\Sigma M_0 = Fd = I_0\alpha = mk_0^2\alpha$$

where I_0 is the moment of inertia and k_0 the radius of gyration of the body with respect to the axis O, and

$$\Sigma F_t = F = m\bar{a}_t = mb\alpha$$

Eliminating F from the last two equations gives $mb\alpha d = mk_0^2\alpha$, whence

$$d = \frac{k_0^2}{b}$$

* This discussion also applies to a body that has no plane of symmetry provided that it is suspended so that the axis of rotation is a principal axis (see Appendix B, Art. B6) through the center of rotation of the body.

It should be noted that, since $k_0 > b$, $d > b$; that is, the center of percussion is always farther from the axis of suspension than the mass-center is. Furthermore, since $d = \infty$ when $b = 0$, a body has no real center of percussion with respect to an axis through the mass-center. (Prove this last statement also by the principle of motion of the mass-center.)

It is easily shown that for an axis at Q the center of percussion is at O, that is the center of percussion and center of suspension are interchangeable. You should supply proof.

A good illustration of the significance of the center of percussion is afforded by the batting of a baseball. You have probably sometimes struck a ball in such a way that the bat stung your hands, the reason being that the point of impact P was either too near or too far from the point O where you held the bat. When P is at Q, the center of percussion of the bat with respect to an axis at O, no reaction is produced at O when the impact occurs and no stinging sensation is felt by the hands. In general the bat would have plane motion rather than motion of rotation about O, but as shown in Art. 122 the position and significance of Q with respect to O are the same for plane motion as for rotation.

EXAMPLE 1. It is required to locate the center of percussion of a uniform circular disk with respect to an axis through its edge and normal to its plane.

Solution. If r denotes the radius of the disk, the radius of gyration with respect to the axis through the edge is $k_0^2 = \bar{k}^2 + r^2 = \frac{3}{2}r^2$. Since the distance from the axis to the center of gravity is r, we have $d = \frac{3}{2}r$.

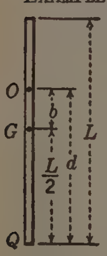

FIG. 212.

EXAMPLE 2. It is required to determine for what axis the center of percussion of a slender uniform rod is at the extreme end.

Solution. Referring to Fig. 212, it is evident that $d = b + \frac{1}{2}L$. The radius of gyration about O is given by $k_0^2 = \bar{k}^2 + b^2 = \frac{1}{12}L^2 + b^2$. Therefore $d = (\frac{1}{12}L^2 + b^2) \div b$. Equating the two expressions for d and solving for b, it is found that $b = \frac{1}{6}L$. Therefore O, the axis that was to be found, is $\frac{1}{6}L$ above the center of the rod.

116. Gravity Pendulum. By this term is meant the common pendulum, that is, a body suspended on a horizontal axis so that it can be made to oscillate freely under the influence of gravity. A real pendulum is sometimes called a compound or physical pendulum to distinguish it from an imaginary one, consisting of a particle suspended by a massless cord, called a simple or mathematical pendulum. Let T = the period or time of one complete or double (to and fro) oscillation, k = the radius of gyration of the pendulum with respect to the axis of suspension, b = distance from the center of gravity of the pendulum to that axis, and 2β = the angle swept out by the pendulum in a single oscillation. Then, as shown presently, the period is given closely by

$$T = 2\pi\sqrt{\frac{k^2}{bg}} \qquad (1)$$

provided that β is small.* Since β does not appear in this formula the period of any pendulum is independent of β; that is, all small oscillations of a pendulum have equal periods, or, as we say, they are isochronous. When g is expressed in feet per second per second then k and b should be expressed in feet; T will be in seconds.

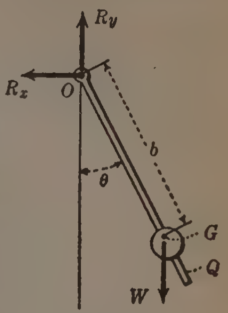

FIG. 213.

For the derivation of Eq. 1 let OG (Fig. 213) be a pendulum in any swinging position, O the center of suspension, G the center of gravity; let W = the weight of the pendulum, $b = OG$, and θ the (varying) angle which OG makes with the vertical, regarded as positive when the pendulum is on the right side of the vertical, as shown. The forces on the pendulum when swinging are its own weight W, the air resistance F, the reaction at the support comprising the components R_y and R_x through the center of the shaft, and a frictional couple C. If F and C are negligible, as we shall assume, then W alone has a moment about O, and $\Sigma M_0 = Wb \sin \theta$.

Hence, according to the equation of rotation,

$$Wb \sin \theta = -\frac{W}{g} k^2 \frac{d^2\theta}{dt^2}$$

the negative sign being introduced because $\sin \theta$ and $d^2\theta/dt^2$ are always opposite in sign. It follows readily from the preceding equation that

$$\frac{d^2\theta}{dt^2} = -\frac{bg}{k^2} \sin \theta = -A \sin \theta$$

where A is an abbreviation for bg/k^2. We will assume that the greatest value of θ, that is β, is so small that $\sin \theta$ and θ are nearly equal; then as a good approximation we may substitute θ for $\sin \theta$, and have

$$\frac{d^2\theta}{dt^2} = -A\theta$$

To integrate this simply, let $u = d\theta/dt$; then $d^2\theta/dt^2 = du/dt = (du/d\theta)(d\theta/dt) = (du/d\theta)u$, and hence

$$\frac{du}{d\theta} u = -A\theta \qquad \text{or} \qquad u\, du = -A\theta\, d\theta$$

* The exact value of the period is given by

$$T = 2\pi \sqrt{\frac{k^2}{bg}} \left[1 + \left(\frac{1}{2}\right)^2 \sin^2 \frac{\beta}{2} + \left(\frac{1}{2} \cdot \frac{3}{4}\right)^2 \sin^4 \frac{\beta}{2} + \cdots \right]$$

If $\beta = 8°$, the bracket above = 1.00122; and for smaller values of β the value of the bracket is still nearer unity. Hence the error in the approximate formula is less than $\frac{1}{8}$ per cent if β does not exceed 8°.

Now, integrating and replacing u by $d\theta/dt$, we get

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = -A\frac{\theta^2}{2} + C_1$$

where C_1 is a constant of integration. Remembering that $d\theta/dt =$ the angular velocity of the pendulum, we note that, where $\theta = \beta$, there $d\theta/dt = 0$; therefore for these (simultaneous) values the preceding equation becomes $0 = -\frac{1}{2}A\beta^2 + C_1$, or $C_1 = \frac{1}{2}A\beta^2$, and finally

$$\frac{d\theta}{dt} = \pm A^{\frac{1}{2}}\sqrt{\beta^2 - \theta^2}$$

The positive sign is to be used when $d\theta/dt$ is positive, that is, when the pendulum is swinging in the positive direction. Now let $\tau =$ the time required for the pendulum to swing out from its lowest to its highest position on the right, that is, while θ changes from 0 to β. To get a value of this time we integrate the preceding equation as follows:

$$\sqrt{A}\int_0^{\tau} dt = +\int_0^{\beta}\frac{d\theta}{\sqrt{\beta^2 - \theta^2}} \quad \text{or} \quad \tau = \sqrt{\frac{1}{A}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_0^{\beta} = \frac{\pi}{2}\sqrt{\frac{k^2}{bg}}$$

Let $\tau' =$ the time required for a swing from the extreme right position to the lowest position, that is, while θ changes from β to 0. To get this time we integrate as follows:

$$\sqrt{A}\int_0^{\tau'} dt = -\int_{\beta}^{0}\frac{d\theta}{\sqrt{\beta^2 - \theta^2}} \quad \text{or} \quad \tau' = -\sqrt{\frac{1}{A}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_{\beta}^{0} = \frac{\pi}{2}\sqrt{\frac{k^2}{bg}}$$

Hence τ and τ' are equal, as was to be expected. Finally, the time of one complete oscillation $= 4\tau = 2\pi\sqrt{k^2/bg}$, as was to be shown.

Period of a simple pendulum. Let $L =$ the length of the pendulum, that is, the distance from its point of suspension to the suspended particle or very small bob. Then $b = L$ and $k = L$; hence according to Eq. 1

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{2}$$

Equivalent simple pendulum. A simple pendulum whose period is the same as that of any particular or given physical pendulum is said to be equivalent to the physical pendulum. The length of such a simple pendulum is called the length of the physical pendulum.

As before let k and b refer to any physical pendulum; also let L' be the length of the equivalent simple pendulum. Equating the formulas for periods and simplifying gives

$$L' = \frac{k^2}{b} \tag{3}$$

Center of oscillation. That point of an actual pendulum into which its entire mass might (in imagination) be concentrated without changing the period is called the center of oscillation of the pendulum. If k and b refer as before to the real pendulum, and Q (Fig. 213) is its center of oscillation,

$$OQ = L' = \frac{k^2}{b} \tag{4}$$

Note that this is identical with the expression for d, the distance from the point of suspension to the center of percussion, obtained in Art. 115. Therefore the center of oscillation and center of percussion are coincident, and the centers of suspension and oscillation are interchangeable. It follows from the property of interchangeability that the periods of a pendulum when suspended from O and Q are equal.

Determination of g. Pendulums are used to determine local values of g. One has only to determine the period T and length k^2/b of a pendulum, and then compute g from Eq. 1.

The length k^2/b cannot be determined accurately for an ordinary pendulum, so a special form (shown in principle in Fig. 214) has been designed for the purpose. O_1 and O_2 are two knife-edges as shown at a known distance apart; W is a weight which can be slid along the stem and clamped where desired. By repeated trials, shifting the weight as necessary, the periods of oscillation for O_1 and O_2 suspension are made equal; then the length k^2/b is equal to O_1O_2.

FIG. 214.

By means of the pendulum described, the value of g for Washington was determined to be 980.100 cm/sec². Values of g at many other places have been determined more simply by comparing the periods of an ordinary pendulum at Washington and at the other places. This comparison is based on the principle that the squares of the periods of any pendulum at two different places are inversely proportional to the values of g at those places; hence, if T_w and T = the periods at Washington and some other station and g = the acceleration at the other station, then $g = 980.1\ (T_w/T)^2$.

CHAPTER XI

PLANE MOTION OF A RIGID BODY; RELATIVE MOTION

117. Definitions. **Plane motion** is motion in which each point of the moving body remains at a constant distance from a fixed plane. Each point of the body moves in a plane, and that plane in which the center of gravity of the body moves is called the *plane of the motion.* The wheels of a locomotive running on a straight track have plane motion, as has also a book which is slid about in any way on the top of a table. A translation (Art. 105) may or may not be a plane motion; a rotation about a fixed axis (Art. 108) is always a plane motion.

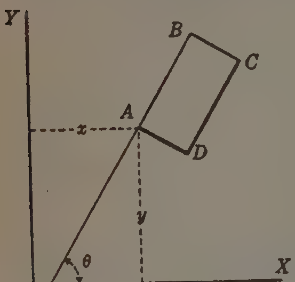

FIG. 215.

The moving body may be represented conveniently by a section of it parallel to the plane of the motion, and in general discussions it will be assumed, for simplicity, that the section is a rectangle. Thus the body may be thought of as a stiff card being slid about on the page of the book, and the rectangle as representing the card. The position of the card $ABCD$ (Fig. 215) is sufficiently specified by the coordinates x and y of one corner, as A, and the direction-angle θ of one edge, as AB.

The angle described in any interval of time by any line of the body parallel to the plane of the motion is called the *angular displacement* of the body for that interval. Obviously all such lines describe equal angles in the same interval of time. As in rotation, these angular displacements are regarded as positive or negative according as they are due to counterclockwise or clockwise turning of the body. If θ_1 and θ_2 denote initial and final values of θ corresponding to any motion of the body, then the angular displacement $= \theta_2 - \theta_1 = \Delta\theta$.

The *angular velocity* of the body is the time rate at which its angular displacement occurs, and its *angular acceleration* is the time rate at which its angular velocity changes. These definitions are precisely like those of the angular velocity and acceleration of a rotation about a fixed axis (Arts. 110 and 111); hence the expressions, units, and rules of sign given in those articles for rotation hold also for plane motion. The expressions are

$$\omega = \frac{d\theta}{dt} \qquad \text{and} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

where ω and α denote angular velocity and angular acceleration of the moving body, respectively.

It should be particularly noted that the angular velocity and acceleration of a body in plane motion are not "about" or "with respect to" any particular axis, and that the direction-angle θ can be measured between any fixed line parallel to the plane of motion and any coplanar line in the body.

118. Plane Motion as Combined Translation and Rotation. It will now be shown that any plane motion can be regarded as a combined translation and rotation. From this point of view we discuss subsequently the motion of any particle of the moving body.

Any displacement of a body in plane motion can be accomplished by means of a translation of the body followed by a rotation, or vice versa. Let $A_1B_1C_1D_1$ (Fig. 216a) be one position of a body $ABCD$ and $A_2B_2C_2D_2$ a later position,

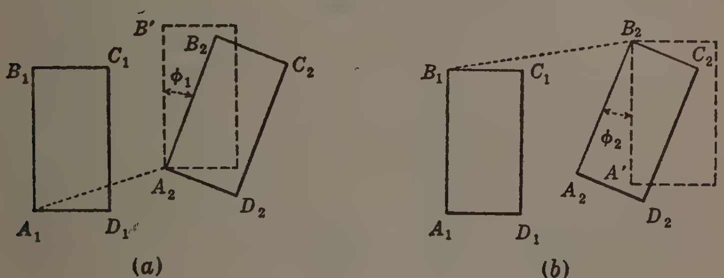

FIG. 216.

By means of a translation, one point of the body could be put into its final position, and by a rotation about that point the body could be put into its final position. Thus a translation from A_1B_1 to A_2B' puts A into its final position, and a rotation about A through an angle ϕ_1 puts the body into its final position. Figure 216b represents the displacement accomplished by a translation that puts B into its final position B_2 and a subsequent rotation about B through an angle $\phi_2 = \phi_1$ which puts the body into its final position. (You should draw figures corresponding to Fig. 216a and b showing the displacement accomplished by a rotation followed by a translation.)

The point of the body chosen for the translation and for center of rotation is called the **base point**; it is used as a moving origin of reference as explained in the next article. It is obvious from the foregoing that the translation (magnitude and direction) depends on the base point chosen and that the rotation does not. It is obvious also that the translations and rotations might be made simultaneously, and hence the actual displacement may be regarded as a combined translation and rotation, with any chosen base point.

Any actual continuous plane motion as of AB from A_1B_1 to A_2B_2, in which A and B describe smooth curves, can be closely duplicated by a succession of small combined translations and rotations of AB from A_1B_1 into successive intermediate positions $A'B'$, $A''B''$, etc., until A_2B_2 is reached. The closer

these intermediate positions, the more closely do the many successive combined translations and rotations reproduce the actual continuous motion. "In the limit" the actual motion is reproduced. Hence one may regard any plane motion as a combined translation and rotation, for any chosen base point.

119. Motion of Any Particle of the Body; Relative Motion. In order to determine the velocity and acceleration of any particle of a body having plane motion, it is necessary to understand what is meant by the motion of a particle relative to a moving point.

In all foregoing discussions of the motions of a particle, we have assumed fixed* axes of reference or a fixed origin. A path, displacement, velocity, or

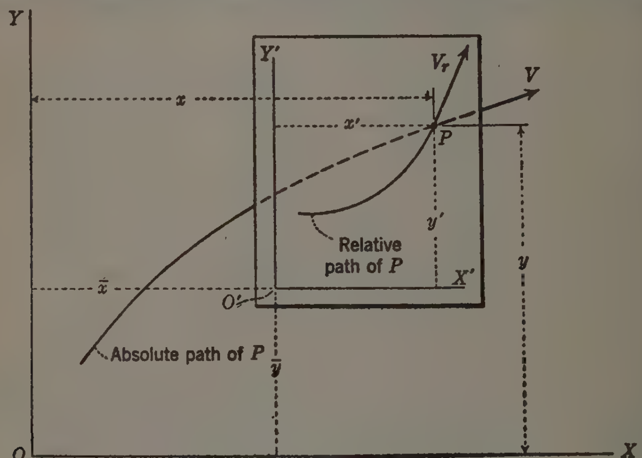

FIG. 217.

acceleration referred to fixed axes or a fixed origin may be called an *absolute* path, displacement, velocity, or acceleration. But the motion of a particle can also be referred to a moving origin, the base point, or to axes moving with the base point but fixed in direction. We then speak of the *relative* path, displacement, velocity, or acceleration of the particle, meaning the path, etc., relative to the base point.

As an illustration, imagine a card (Fig. 217) lying on a stationary page of this book and a bug† P on the card, and suppose that the bug can punch small

* Fixed, that is, with respect to the earth's surface at the place in question. Actually, such an origin is moving, because the earth itself moves, but in most engineering problems this fact can be disregarded, and we shall here call any set of axes or origin which is at rest with respect to the earth's surface fixed or stationary.

† This device, conceiving of the moving particle as a bug which can move of its own volition and mark out its own path, is very helpful in the discussion of relative motion. In spite of its ludicrous aspects, therefore, we shall employ it freely, and in order to avoid ambiguity we shall violate usage by referring to the bug as "he" rather than as "it."

holes rapidly through the card and through the page. Suppose also that the card is made to slide about on the page with motion of translation (no turning), and that while it thus slides the bug runs about on the card, punching holes through it and the page beneath. The running of the bug is his motion relative to any base point O' (or axes $O'X'$, $O'Y'$) on the card; the holes punched in the card mark out his path relative to O', and the holes punched in the page mark out his absolute path. The change in the bug's position on the card during any interval of time is his *relative displacement* for that interval. The *relative velocity* V_r is the time rate at which this displacement occurs; its magnitude is the speed with which the bug runs on the card, and its direction is along the tangent to the relative path. The *relative acceleration* A_r is the time rate at which V_r changes. These quantities — relative displacement, relative velocity, relative acceleration — are determined just as though the card itself were stationary.

The absolute displacement (velocity, acceleration) of a particle is equal to the vector sum of the displacement (velocity, acceleration) of the particle relative to any chosen base point and the absolute displacement (velocity, acceleration) of that base point. This is easily proved by showing it to be true for rectangular components of the motion. Thus in Fig. 217 it is obvious that, if $\bar{x}$, $\bar{y}$ are the coordinates* of the base point O' with respect to fixed axes O–XY, x', y' are the coordinates of P with respect to moving axes O'–$X'Y'$, and x, y are the coordinates of P with respect to the fixed axes, then

$$x = \bar{x} + x'$$

hence $dx/dt = (d\bar{x}/dt) + (dx'/dt)$, that is,

x velocity of P = (x velocity of O') + (x component of the relative velocity of P)

and $d^2x/dt^2 = (d^2\bar{x}/dt^2) + (d^2x'/dt^2)$, that is,

x acceleration of P = (x acceleration of O') + (x component of the relative acceleration of P)

Obviously, similar relationships hold for the y components of the velocities and accelerations.

It is to be noted that, though the proof here given assumes uniplanar motion for both particle and base point, it can readily be extended to three-dimensional motion by taking three rectangular axes instead of two.

As is implied in the footnote on p. 178, absolute motion is really motion relative to an origin that is arbitrarily assumed to be fixed. We have, in the above discussion, defined " fixed " as meaning at rest with respect to a coin-

* The notation $\bar{x}$, $\bar{y}$ is usually reserved for coordinates of the center of gravity of a body, but its use here for coordinates of a moving origin is convenient and should not prove confusing.

cident point of the earth's surface. If we wished to take the earth's *center* as our fixed origin we should have to add, to the velocity or acceleration of the particle relative to the earth's surface (heretofore called absolute), the velocity or acceleration of the coincident point of the earth's surface relative to its center, due to rotation. If we wished to take the sun as the fixed origin we should have further to add the velocity or acceleration of the earth's center relative to the sun. This procedure can be continued indefinitely, and so, assuming a moving particle P, any number of base points O', O'', $\cdots O^n$, and a fixed origin O, and, for brevity connoting by the word "motion" displacement, velocity, and acceleration, we may say that the absolute motion of P is its motion relative to O', plus (vectorially) the motion of O' relative to O'', plus the motion of O'' relative to O''', etc., plus, finally, the absolute motion of O^n.

It should be noted that the actual distance traveled by a moving particle (length of the absolute path) cannot be found by simply adding the distance traveled by the base point and the length of the relative path. (*Query.* Why not, and how could the actual distance traveled be found?)

120. Motion of Any Particle of the Body (Continued). The facts and relationships explained in Arts. 118 and 119 suggest the following method of determining the absolute velocity V and absolute acceleration A of any particle of

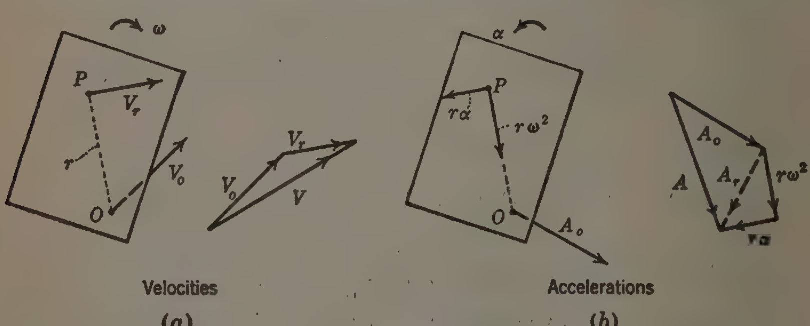

FIG. 218.

a body that has plane motion: Choose as base point some point O of the body (Fig. 218) whose velocity V_0 and acceleration A_0 are known; then determine the velocity V_r and acceleration A_r of the particle P relative to O exactly as if O were fixed and the body rotated about it with the angular velocity ω and the angular acceleration α it actually has; then add vectorially V_0 and V_r to get V (Fig. 218*a*), and add vectorially A_0 and A_r to get A (Fig. 218*b*). V_0 and A_0 are assumed to be known, also ω and α. V_r is equal to $r\omega$ and is directed perpendicularly to the radius r as shown in Fig. 218*a*; A_r is made up of a normal component $r\omega^2$ directed towards O and a tangential component $r\alpha$ directed perpendicularly to r as shown in Fig. 218*b*.

It is obvious that these same relationships can be used to determine the (unknown) angular velocity and acceleration of the body if enough is known about the motion of two points thereof; thus, in Fig. 218, ω can be determined if V and V_0 are known, and α can be determined if A and A_0 are known.

In some problems it is advantageous to use one base point for part of the solution and then to change to another in order to complete the solution. For this reason it is impracticable always to designate the base point by the letter O; any designation may be used as long as the identity of the base point is made perfectly clear. This change of base points is illustrated in Ex. 1 below.

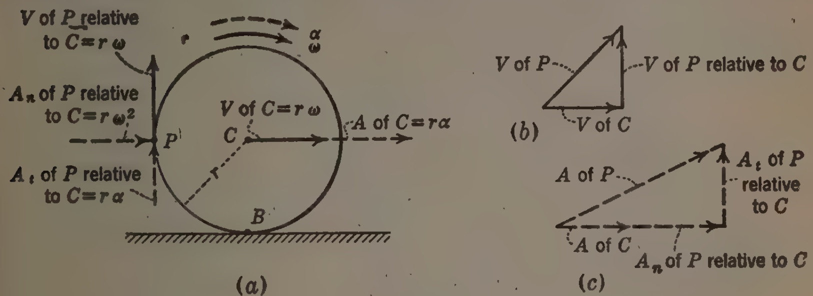

FIG. 219.

EXAMPLE 1. Figure 219a represents a circular disk of radius r which rolls without slipping along a horizontal floor. At a particular instant it has an angular velocity ω and an angular acceleration α, both clockwise. It is required to derive expressions for the velocity and acceleration at that instant of the center C, and also of a particle P at the extreme left edge of the disk. It is also required to determine these velocities and accelerations for the numerical data: $r = 2$ ft, $\omega = 5$ rad/sec, $\alpha = 10$ rad/sec^2.

Solution. We first choose a base point. There is no point of the wheel whose motion is completely known, but it is obvious that the velocity of B, which is in contact with the floor, is zero. Taking B as base point, the velocity of C is the vector sum of the velocity of B and the velocity of C relative to B, or: v (of C) $= 0 + r\omega = r\omega$, directed to the right. Since the motion of C is rectilinear, a (of C) $= d(r\omega)/dt = r\alpha$. The motion of C is now completely known, and it will be convenient to use C instead of B as base point in finding the velocity and acceleration of P.

The velocity of P is the vector sum of the velocity of C and the velocity of P relative to C. The former, as shown above, is equal to $r\omega$ and is directed to the right; the latter is equal to $r\omega$ and is directed vertically upward; the absolute velocity of P is therefore equal to $\sqrt{2}r\omega$, and is directed up and to the right at 45° to the horizontal, as shown in Fig. 219b, the vector diagram for velocities.

The acceleration of P is the vector sum of the acceleration of C and the acceleration of P relative to C. The former, as shown above, is equal to $r\alpha$ and is directed to the right. The latter consists of the normal component $r\omega^2$ directed toward C — that is, to the right — and the tangential component $r\alpha$, directed vertically upward. The absolute acceleration of P is therefore equal to $[(r\alpha)^2 + (r\alpha + r\omega^2)^2]^{\frac{1}{2}}$ and is directed up and to the right at an angle to the horizontal $\tan^{-1}\left(\dfrac{r\alpha}{r\alpha + r\omega^2}\right)$, as shown in Fig. 219$c$, the vector diagram for accelerations.

When $r = 2$, $\omega = 5$, $\alpha = 10$, substitution in the expressions derived above gives:

(for C) $V = 10$ ft/sec, $A = 20$ ft/sec² (to the right)
(for P) $V = 14.14$ ft/sec, up and to the right at 45° to the horizontal
$A = 72.8$ ft/sec², up and to the right at 15.9° to the horizontal

EXAMPLE 2. The lower end B of the straight bar (Fig. 220a) slides out along the horizontal floor with uniform velocity v while the upper end C slides down along the vertical wall. It is required to derive expressions for the angular velocity and acceleration of the rod, and for the velocity and acceleration of C, in terms of v, the slope of the bar θ, and the length of the bar L.

Solution. Here it is necessary to select two points of the body whose motion is known or partially known. For point B, the velocity and acceleration are both known. For point C, it is known that the motion is rectilinear and vertical, and that therefore the horizontal

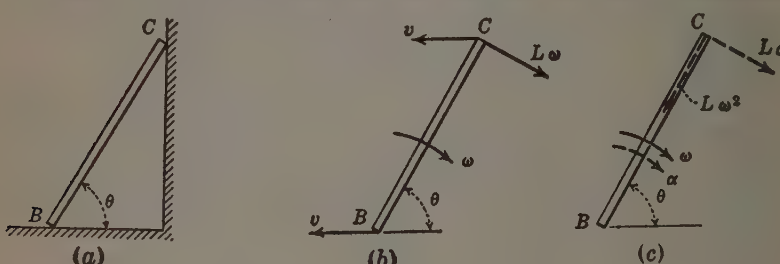

FIG. 220.

components of the velocity and acceleration are both zero. We use these facts to determine ω and α. Obviously ω is clockwise. The sense of α is not apparent; it will be assumed clockwise, and if this assumption is wrong the fact will be indicated by a negative sign for the computed value.

Taking B as base point, the velocity of C is the vector sum of v to the left (velocity of base point) and $L\omega$ (velocity of C relative to base point) directed as shown in Fig. 220b. The x component of the velocity of C is $v - L\omega \sin\theta = 0$, hence

$$\omega = \frac{v}{L \sin\theta} \quad \text{(clockwise as shown)}$$

The acceleration of C is the vector sum of zero (acceleration of base point), $L\omega^2$ (normal component of relative acceleration), and $L\alpha$ (tangential component of relative acceleration), directed as shown in Fig. 220c. The x component of the acceleration of C is

$$L\omega^2 \cos\theta - L\alpha \sin\theta = \frac{v^2}{L \sin^2\theta} \cos\theta - L\alpha \sin\theta = 0$$

hence

$$\alpha = \frac{v^2}{L^2 \sin^2\theta} \cot\theta = \omega^2 \cot\theta$$

The positive result shows that α is clockwise, as assumed.

The velocity of C is found by adding the vertical components of v and $L\omega$, thus

$$v \text{ (of } C) = L\omega \cos\theta = v \cot\theta$$

The acceleration of C is found by adding the vertical components of $L\omega^2$ and $L\alpha$, thus

$$a \text{ (of } C) = L\omega^2 \sin\theta + L\alpha\cos\theta = \frac{v^2}{L\sin\theta}(1 + \cot^2\theta)$$

(Note that, as θ approaches zero, the angular velocity and acceleration of the bar, and the velocity and acceleration of C, all approach infinity, and that it would therefore be impossible to reproduce physically the described motion for very small values of θ.)

121. Forces on a Body in Plane Motion. The principle of motion of the mass-center applies to a body in plane motion, as to any other body. It is necessary also to establish a relationship between the forces and the angular acceleration of a body in plane motion. It will now be shown that the angular acceleration of a body in plane motion is directly proportional to the moment of the external forces about an axis through the mass-center perpendicular to the plane of the motion, and inversely proportional to the moment of inertia about that axis. That is, if ΣM_G denotes the moment of the external forces about the axis described, $\bar{I}$ the moment of inertia of the body about that axis, and α the angular acceleration, and if consistent units are used, then

$$\alpha = \frac{\Sigma M_G}{\bar{I}} \qquad \text{or} \qquad \Sigma M_G = \bar{I}\alpha = m\bar{k}^2\alpha \tag{1}$$

Let Fig. 221 represent a body having plane motion in the plane of the page, with angular velocity ω and counterclockwise angular acceleration α. The external forces on the body are not shown. G is the mass-center of the body; $\bar{A}$ is the acceleration of G; P is any elementary particle of the body, of mass dm, and at a distance r from an axis through G perpendicular to the page. Now, if G is taken as base point, the acceleration of P is completely represented by its components $\bar{A}$, $r\omega^2$, and $r\alpha$, directed as shown. The resultant of all forces acting on P is therefore completely represented by the components $dm\bar{a}$, $dmr\omega^2$, and $dmr\alpha$. The moment of all forces on all the particles is the sum of the moments of all such components. Obviously the components $dmr\omega^2$ have no moment about the axis through G. The resultant of all the components $dm\bar{a}$ passes through the mass-center (Art. 105), and so these components have no moment about the axis. The moment of the components $dmr\alpha$ is

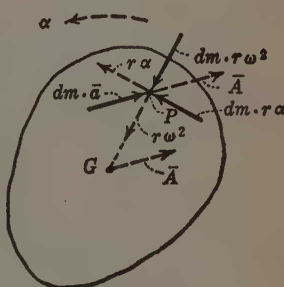

Fig. 221.

$$\int dmr\alpha \cdot r = \alpha\int dmr^2 = \bar{I}\alpha$$

Now the system of forces acting on all the particles consists of internal and external forces. The internal forces jointly have no moment, since they con-

sist of pairs of equal and opposite actions and reactions. Hence, the moment of the external forces equals $\bar{I}\alpha$, as stated.

The equation $\Sigma M_G = \bar{I}\alpha$ contains no term depending on the motion of the mass-center, and the equation of the motion of the mass-center contains no term depending on the rotation of the body about that point; therefore in plane motion the rotation about the mass-center is completely independent of the motion of that point, and the angular acceleration of the body is exactly the same as though it were rotating on a fixed axis through the mass-center perpendicular to the plane of the motion.

Equation 1 applies also to a body that has motion of rotation, since rotation is simply a special case of plane motion. But for a rotating body it is usually more convenient to use $\Sigma M_0 = I_0\alpha$ than to use $\Sigma M_G = \bar{I}\alpha$. The two expressions are, of course, identical when the axis of rotation passes through the mass-center.

Moment about an axis in the plane of the motion. If the body is homogeneous and symmetrical with respect to the plane of the motion, it is obvious that for every particle on one side of that plane there is a corresponding particle on the other side and that the accelerations of these particles are the same. The forces on these particles are therefore equal and parallel, and, since they are equidistant from the plane of the motion, their moments about any axis in that plane cancel. Since all the particles of the body can be thus grouped in pairs, the moment of all forces on all particles, and hence the moment of all external forces on the body, is zero with respect to any axis in the plane of the motion.

122. Typical Problems; Examples. The common application of the relations established in the preceding article is to problems that involve the determining of forces that produce a given plane motion. Thus it may be required to determine the forces that act on a rolling wheel, or on the connecting rod of a stationary engine, or on the side rod of a locomotive. If the body is symmetrical with respect to the plane of motion, the relation $\Sigma M_G = \bar{I}\alpha$, together with the principle of motion of the mass-center and the relations between the motions of the parts, usually makes possible the writing of a sufficient number of equations, though it may be necessary to use the additional condition that the moment of the external forces about a line in the plane of the motion is zero. For bodies not symmetrical with respect to the plane of the motion, a partial solution can often be reached by means of the relation $\Sigma M_G = \bar{I}\alpha$ and $\Sigma F_x = m\bar{a}_x$, but for a complete solution the methods explained in Chapter XII are usually necessary.

EXAMPLE 1. A pair of car wheels and attached axle are pushed along a level track by a horizontal force of 150 lb applied through the center of the axle as shown (Fig. 222). The coefficient of friction between the wheels and the rails is sufficient to prevent slipping, and rolling resistance is negligible. The weight of the entire body is 600 lb, the radius of gyration with respect to the axis of the axle is 9 in., and the radius of the wheels is 16.5 in.

It is required to determine the linear acceleration of the axis and the least value of the coefficient of friction between wheels and rails that will prevent slipping.

Solution. The body in question (wheel-and-axle system) has plane motion, which will be regarded as a translation like that of the mass-center G, and a rotation about G. The external forces are the weight, the push of 150 lb, and the reactions of the rails on the wheels, regarded for convenience as a single force consisting of the vertical component N and the horizontal or friction component F, which acts to the left since ΣM_G must be clockwise when α is clockwise.

For rotation:

$$\Sigma M_G = F \times \frac{16.5}{12} = \frac{600}{32.2}\left(\frac{9}{12}\right)^2 \alpha$$

For translation:

$$\Sigma F_x = 150 - F = \frac{600}{32.2}\bar{a}$$

and (Art. 120)

$$\alpha = \frac{\bar{a}}{(16.5 \div 12)}$$

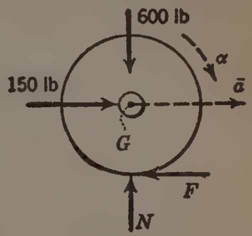

FIG. 222

Solution by means of these three equations gives $\bar{a} = 6.21$ ft/sec² and $F = 34.4$ lb.

From $\Sigma F_y = 0$, N is obviously 600 lb; therefore the coefficient of friction must be at least $34.4 \div 600 = 0.0573$. If it were less, the limiting friction would not be great enough to give the body an angular acceleration corresponding to the linear acceleration of its mass-center, and the motion would be a combination of slipping and rolling instead of pure rolling.

(It is easy to show that, for any circular cylindrical body whose mass-center is at the geometrical center and which is made to roll by a horizontal propelling force P applied at the mass-center, the frictional force $F = P/(1 + r^2/k^2)$, where r = the radius and k = the radius of gyration about the central axis. Do this, and test the formula by substituting the numerical values in this example.)

EXAMPLE 2. The rod BC (Fig. 223*a*) has transverse pins at the ends, which slide in smooth slots so that B moves horizontally and C vertically. The rod is uniform, weighs 60 lb, and is 12 ft long. By means of a vertical force P applied at C, the bar is made to move so that B slides to the left with uniform velocity v. It is required to determine all forces acting on the bar at an instant when $v = 20$ ft/sec and $\theta = 40°$.

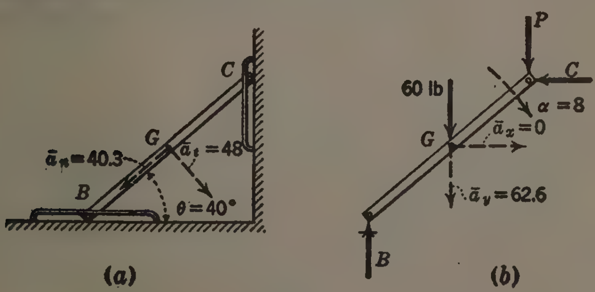

FIG. 223.

Solution. In Ex. 2 of Art. 120, where this motion was discussed, it was found that $\omega = v/(L \sin\theta)$ and that $\alpha = \omega^2 \cot\theta$. For $L = 12$ ft, $v = 20$ ft/sec, and $\theta = 40°$, these formulas give $\omega = 2.59$ rad/sec and $\alpha = 8$ rad/sec². Taking B as base point, the relative acceleration of G, the mass-center, is found to be as indicated by the two vectors (Fig. 223*a*),

and since B has no acceleration these vectors represent the absolute acceleration of G. From them it is found that $a_x = 0$ and $a_y = 62.6$ ft/sec² downward. (That $a_x = 0$ is also apparent from the fact that, no matter how B moves, the x coordinate of G is always half the x coordinate of B, and so v_x is always half the velocity of B and a_x half the acceleration of B.)

The fbd of the rod is shown in Fig. 223*b*; the external forces are the weight, the vertical force P (assumed downward), the pressure of the lower guide (denoted by B and assumed upward), and the pressure of the upper guide (denoted by C and assumed leftward). For the bar, $\bar{I} = \frac{1}{12}(60 \div 32.2)12^2 = 22.35$ slug-ft².

For rotation:

$$\Sigma M_G = B(6 \cos 40°) + P(6 \cos 40°) - C(6 \sin 40°) = (22.35)(8)$$

For translation:

$$\Sigma F_x = -C = 0 \qquad \Sigma F_y = P + 60 - B = (60 \div 32.2)62.6$$

Solution gives $P = 47.7$ lb, $B = -8.8$ lb, $C = 0$. Therefore the guide at B presses down, not up, and there is no pressure from the guide at C.

EXAMPLE 3. Figure 224 represents a flat body of mass m which rests on a smooth horizontal floor. If a force F, whose line of action does not pass through the mass-center G but does lie in the horizontal plane containing G, is applied to the body, the body will be given plane motion. Particles of the body will in general be given acceleration, but there is some vertical line O in the body which will not take on acceleration at the instant F is applied. It is required to determine the location of this line.

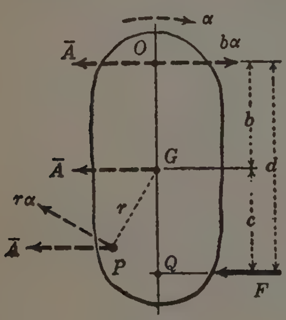

FIG. 224.

Solution. The effect of F is to give G an acceleration $\bar{A} = F/m$ and to give the body an angular acceleration $\alpha = Fc/\bar{I}$. Because of this angular acceleration, any point P not in the plane QG (normal to F and passing through G) will have some acceleration normal to $\bar{A}$ (see Fig. 224), and so O must lie in the plane QG. Let b denote the distance of O from G. Then the acceleration of O is the vector sum of $b\alpha$ (its acceleration relative to G) and $\bar{A}$, and, since by definition the acceleration of O is zero,

$$\bar{a} - b\alpha = \frac{F}{m} - b\frac{Fc}{\bar{I}} = 0 \qquad \text{or} \qquad b = \frac{\bar{I}}{mc} = \frac{\bar{k}^2}{c}$$

Since O does not take on acceleration when F is applied through Q, no instantaneous reaction would be produced at O if the body were pinned there. Since the acceleration produced by F is independent of any motion the body may have when F is applied, you would not feel any stinging sensation if you were grasping the body, which may be thought of as a bat, at the point O when a force F acting through Q was applied as by a baseball. Thus Q is seen to be the center of percussion for the body with respect to O. In Art. 115, where center of percussion was first discussed, it was shown that $d = k_0^2/b$, where k_0 is the radius of gyration of the body with respect to O. Show that this formula for d is in agreement with the one obtained above for b.

123. Plane Motion as Rotation about Successive Axes; Instantaneous Axis. It will now be shown that for certain purposes any plane motion can be regarded as a rotation about successive axes each fixed in the body. This view of plane

motion is useful for determining the relation between the velocity of any particle of the body and the angular velocity of the body.

We show first that any plane displacement of a body can be accomplished by a rotation about some line perpendicular to the plane of the motion and fixed in space. Let A_1 and B_1 (Fig. 225) be the initial and A_2 and B_2 the final positions of points A and B in any arbitrary displacement of the body (A and B are both in a plane parallel to the plane of motion). MO and NO are normal to lines A_1A_2 and B_1B_2 respectively at the midpoints of those lines. Obviously OA_1 and OA_2 are equal; hence a rotation of the body about O through the angle A_1OA_2 would displace A from A_1 to A_2. Similarly OB_1 and OB_2 are equal, and, since the angles A_1OA_2 and B_1OB_2 are equal, a single rotation of the body about O, through an angle equal to A_1OA_2, would displace both A and B, and hence the body, from their initial to their final positions. (There are two contingencies for which supplementary proofs are required, namely: (i) the bisecting normals MO and NO coincide, and (ii) they are parallel. You should make a figure for each of these cases, showing AB in its initial and final positions, and indicate the necessary constructions for finding the centers of rotation.)

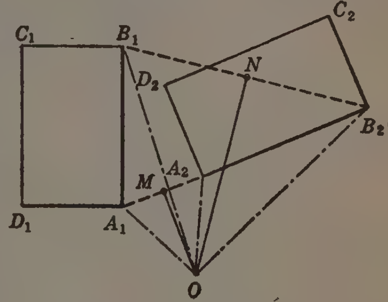

Fig. 225.

Next, imagine any motion of the body which would displace A and B from A_1B_1 to A_2B_2 in any time interval $t_2 - t_1$, and consider the assumed displacement of the body to be made up of many small displacements corresponding to many small intervals Δt_1, Δt_2, Δt_3, etc., which comprise the whole time interval $t_2 - t_1$. As just proved, each small displacement could be accomplished by a rotation about some definite line. Imagine all such rotations to be made in continuous succession and each in its proper interval Δt. Obviously, such a succession of rotations would approximately duplicate the imagined motion. "In the limit," the succession of rotations would duplicate the imagined motion. Therefore any plane motion may be looked upon as a continuous rotation about a continuous succession of axes, all perpendicular to the plane of the motion. Each such axis is the **instantaneous axis** of the motion at a particular instant, and the point where the instantaneous axis pierces the plane of the motion is called the **instantaneous center.** We shall use a single letter, as O, C, etc., to denote both the instantaneous axis and the instantaneous center.

Velocity of any particle of the body. At any instant the velocity of any particle of the body is exactly the same as though the body were rotating about a fixed axis coincident with the instantaneous axis, and with the angular

velocity it actually has. Thus the velocity of P (Fig. 226), which represents any particle of a body having plane motion with O as instantaneous axis, is equal to $OP \times \omega$, and is perpendicular to OP as shown.

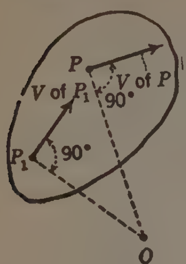

FIG. 226.

Location of the instantaneous axis. Since the velocity of any point of the body is perpendicular to a line joining the point and the instantaneous axis O, O can readily be found if the velocities of two points such as P and P_1 (Fig. 226) are known. It is only necessary to draw from P a line normal to its velocity and from P_1 a line normal to its velocity, and to extend these lines until they intersect. The intersection locates O. Obviously P and P_1 must be points whose velocities are not parallel. Sometimes the instantaneous axis is apparent from inspection, there being one line of the body which, at a given instant, obviously has zero velocity. Thus for a rolling wheel the line of contact between the wheel and the surface on which it rolls is the instantaneous axis.

Acceleration of any particle of the body. In general the acceleration of P cannot be found as though the motion were a rotation about a fixed axis O. For, if the speed $v = OP \times \omega$ is differentiated to get a_t, the result is not $OP \times \alpha$, because in general OP is changing as well as ω. Also, a_n is not equal to $v^2 \div OP$, because OP is not the radius of curvature of the path of P. But if, as sometimes happens, the distance OP to the instantaneous axis is not changing, then a_t does equal $OP \times \alpha$. This is true when OP is at a maximum or minimum, for then $d(OP)/dt = 0$ and $d(OP \times \omega)/dt = OP \times \alpha$. And, if the instantaneous center O has zero acceleration along some particular line, then the component of the acceleration of P along that line can be found as though the motion of the body were a rotation about a fixed axis through O. This is because that component depends wholly on the acceleration of P relative to O, since O (now regarded as a base point) has itself zero acceleration along the line in question.

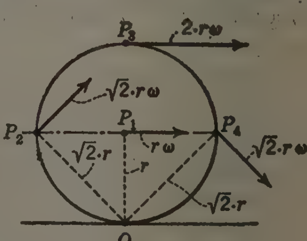

FIG. 227.

EXAMPLE 1. A circular disk of radius r rolls without slipping along a horizontal floor. At a particular instant it has a clockwise angular velocity ω. It is required to determine the velocity of each of the four particles P_1, P_2, P_3, P_4, located as shown in Fig. 227.

Solution. The line O, the line of the disk in contact with the floor, has zero velocity and so is the instantaneous axis. The velocity of any particle P is therefore equal to $OP \times \omega$, and is perpendicular to OP. The velocities of P_1, P_2, P_3, and P_4 therefore have the magnitudes and directions indicated in the figure. (Compare this solution for velocity with that made in Ex. 1 of Art. 120.)

Suppose now that the wheel has an angular acceleration α. It is not correct to say that P_2 has a tangential acceleration a_t equal to $OP_2 \times \alpha$, for, if we undertake to find a_t by differentiating $v = \omega \times OP_2$, we must take into account the fact that OP_2 is changing as well as

ω. Neither is it correct to say that P_2 has a normal acceleration a_n equal to $OP_2 \times \omega^2$, or $v^2 \div OP_2$, because OP_2 is not the radius of curvature of the path of P_2. (It is suggested that you thus erroneously calculate the components of the acceleration of P_2 by regarding the motion as a rotation about a fixed axis O, and compare the results with the correct results obtained in Ex. 1 of Art. 120.) But, for P_1, a_t is equal to $OP_1 \times \alpha$, because OP_1 is constant, and for P_3, a_t is equal to $OP_3 \times \alpha$, because, although OP_3 is not constant, at the instant in question it has its maximum value and $d(OP_3)/dt = 0$. On the other hand, P_1 does not have a normal acceleration $a_n = OP_1 \times \omega^2$; it has no normal acceleration at all because its motion is rectilinear. Furthermore the normal acceleration of P_3 is not equal to $OP_3 \times \omega^2$, because OP_3 is not the radius of curvature of the path of P_3. Another explanation is this: The horizontal acceleration of either P_1 or P_3 can be found by regarding the motion of the wheel as a rotation about O, because O has no horizontal acceleration. But the vertical accelerations of P_1 and P_2 cannot be thus found, because O (on the wheel) does have vertical acceleration. Its vertical acceleration is its normal acceleration relative to P_1 and is equal to $r\omega^2$.

EXAMPLE 2. The lower end of B of the straight bar (Fig. 228) slides along the horizontal floor with uniform velocity v while the upper end C slides down along the vertical wall. It is required to derive expressions for the angular velocity of the rod, and for the velocity of C, in terms of v, the slope of the bar θ, and the length of the bar L.

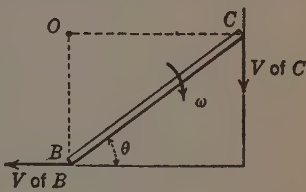

FIG. 228.

Solution. The instantaneous center O is located by extending lines drawn perpendicular to the paths of B and C until they intersect. At the instant in question, the motion of the bar BC is a rotation about O. The velocity of B equals $OB \times \omega$, or $v = L \sin \theta \omega$; hence $\omega = v/L \sin \theta$. The velocity of C equals $OC \times \omega = L \cos \theta \times (v/L \sin \theta) = v \cot \theta$. (Compare this solution with that in Ex. 2 of Art. 120.)

(It is now easy to determine the angular acceleration by differentiating the expression for ω, and to determine the linear acceleration of C by differentiating the expression for the velocity of C. Do this, and check your results by comparing them with the previous solution of this example. Note that particular attention must be paid to the signs of the several quantities when α and a of C are found by differentiation, whereas in the previous solution we ascertained all directions by inspection and took all quantities as positive.)

124. Motion of a Particle Relative to a Moving Body. In Art. 119 we discussed the motion of a particle relative to a moving point. We now explain what is meant by the motion of a particle relative to a moving *body*, and show how to determine the velocity and acceleration of a particle that has such motion. The methods discussed are useful in problems of plane motion where there is no available base point whose motion is known, and in certain other problems.

In the bug-card-page illustration of Art. 119, it was supposed that the bug ran about on a card while the card moved with motion of translation on the page. Under these circumstances the running of the bug is his motion relative to any base point O on the card (since all such points move exactly alike) and his motion relative to any such base point is also his motion relative to the moving card. Now suppose that the card has any plane motion instead of motion of translation, that is, suppose that it turns as it moves. The running

of the bug is, as before, his motion relative to the card, but not his motion relative to any base point on the card. The difference can be clearly seen from a consideration of Fig. 229. This represents two cards, I and II, I being on top. These cards are pinned together at some point O so that at that common point they must move together, while being free to turn independently. Now imagine card II to slide on the (stationary) page with motion of translation, card I to move with any plane motion, and the bug to run about on card I

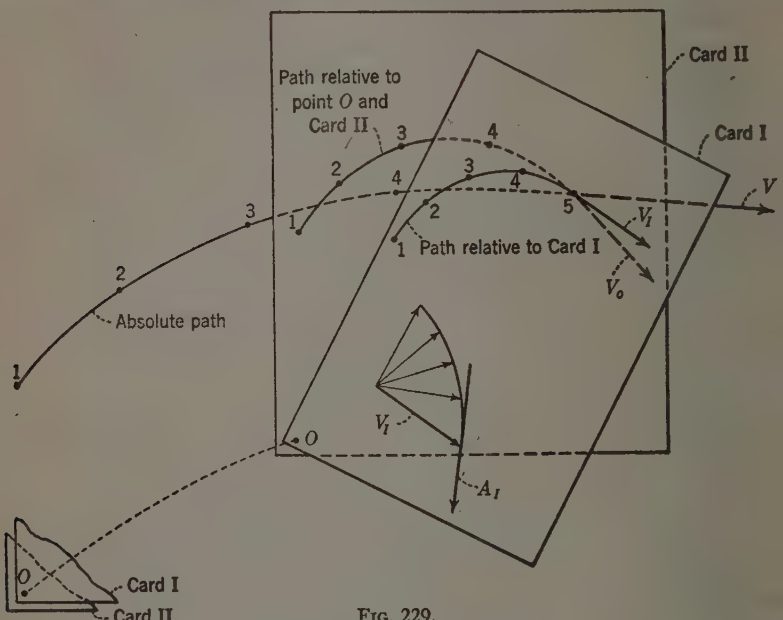

FIG. 229.

and to punch holes through both cards and through the page, thus marking out three paths. The path in the page is the absolute path of the bug; the path in card II is the path relative to point O; the path in card I is the path relative to card I. (The three holes numbered 1 were punched simultaneously, as were the three holes numbered 2, etc.) At any instant (as at the instant represented by Fig. 229) the velocity of the bug relative to card I (vector V_{I} drawn on card I) is tangent to the path in card I and its magnitude is equal to the distance on card I per unit time the bug is running. The velocity relative to point O (vector V_0 drawn on card II) is tangent to the path in card II and its magnitude is equal to length of that path punched out per unit time. The absolute velocity of the bug (vector V drawn on the page) is tangent to the absolute path and its magnitude is equal to length of absolute path punched out per unit time. Obviously these three velocities V_{I}, V_0, and V are all different, in both magnitude and direction.

The relative acceleration of a particle that moves relative to a moving body

is the time rate at which its relative velocity changes, as viewed by an observer moving with the moving body. For the bug-card illustration, the relative acceleration A_{I} of the bug is represented by the velocity on card I with which the tip of the vector representing V_{I} moves along the hodograph drawn on card I, and so is directed along the tangent to that hodograph as shown. (If the bug runs along a straight line on card I with uniform relative speed — say 1 in. on the card per second — his relative acceleration is zero. If he runs in a circle of radius r on card I, with uniform relative speed v_{I}, then his relative acceleration is equal to v_{I}^2/r and is directed toward the center of the circle.)

125. Relative Motion Referred to the Moving Body and to the Earth. When a particle is moving relative to a moving body, its relative displacement, velocity, and acceleration are its displacement, velocity, and acceleration as viewed by an observer moving with the body and having as axes of reference for position and direction only axes fixed in the body. Therefore relative displacement, velocity, and acceleration do not have absolute direction but only direction with reference to the moving body. The relative displacement, velocity, and direction *referred to the earth* have respectively the same magnitudes as the relative displacement, velocity, and acceleration, but they have the absolute directions of the corresponding vectors transferred or projected from the moving body onto the earth. These directions depend on the orientation of the moving body, and they change as the moving body turns. Thus suppose that in the illustration used in Art. 124 the bug runs with constant relative speed along a straight line drawn on card I, so as to travel say 3 in./sec along that line. Then his relative velocity is constant in both magnitude and direction; his relative velocity referred to the earth is constant in magnitude (3 in./sec) but changes continuously in direction as the card turns. Again suppose that the bug runs faster and faster along the line so that his relative speed increases at the rate of 1 in./sec^2. His relative acceleration is constant in both magnitude and direction; his relative acceleration referred to the earth is constant in magnitude (1 in./sec^2) but changes continuously in direction as the card turns.

It is essential that the distinction between relative displacement, velocity, and acceleration (understood as referred to the moving body) and relative displacement, velocity, and acceleration referred to the earth be clearly understood, because it is only the latter that can be compounded with absolute displacements, velocities, and accelerations. In the following discussion, relative velocity and acceleration will be denoted by V_r and A_r, respectively; relative velocity referred to the earth and relative acceleration referred to the earth will be denoted by V_{re} and A_{re}, respectively.*

* We have put more emphasis perhaps on this distinction than is usual; in most discussions of these principles, and especially of their applications, it is taken for granted that, when relative velocity or acceleration is to be compounded with absolute velocity or acceleration, the relative velocity or acceleration referred to the earth is to be used.

126. Absolute Velocity of the Moving Particle. If a particle moves relative to a moving body, *the absolute velocity V of the particle is the vector sum of its relative velocity referred to the earth V_{re} and the absolute velocity V_0 of the coincident point of the moving body.*

Proof. As pointed out in Art. 124, the velocity of a particle relative to a moving body is in general quite different from its velocity relative to a chosen base point in that body. At any instant, however, these relative velocities are identical if the chosen base point is coincident with the moving particle. Thus, in Fig. 229, if the bug runs so as to pass through O, then at the instant he is at O his velocity relative to O and his velocity relative to card I are identical. This is obvious from the fact that at their common point O cards I and II are moving exactly alike, and so the velocity of the bug is the same relative to both cards at the instant when the bug is at O. But it has already been proved (Art. 119) that the absolute velocity of a moving particle is the vector sum of its velocity relative to a moving base point and the absolute velocity of that base point, and so $V = V_{re} \nrightarrow V_0$, as stated.

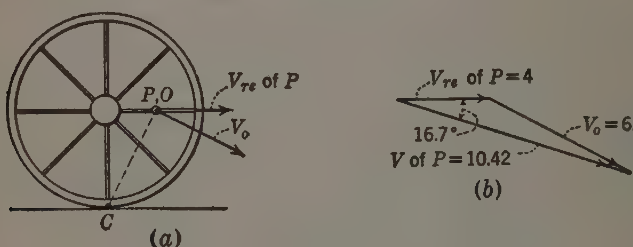

FIG. 230.

EXAMPLE. Figure 230*a* represents a wheel 6 ft in diameter with wire spokes which rolls to the right with constant angular velocity $\omega = 2$ rad/sec. A small pierced ball P slides out along one spoke with a constant relative speed of 4 ft/sec. It is required to determine the absolute velocity of P when it reaches a point on the spoke halfway between center and perimeter of wheel, the spoke being assumed to be horizontal and in front of the center at that instant.

Solution. V_0, the velocity of the coincident point O, is found by the use of the instantaneous center C to be equal to $CP \times 2 = 3.35 \times 2 = 6.70$ ft/sec, directed as shown. V_{re}, the relative velocity referred to the earth of P, is 4 ft/sec to the right. V, the absolute velocity of P, is the vector sum of V_0 and V_{re}, and is readily found to have the magnitude and direction shown in Fig. 230*b*.

127. Absolute Acceleration of the Moving Particle. Suppose the body to have plane motion, and the particle to move relative to the body in a plane parallel to the plane of motion of the body. Then *the absolute acceleration of the particle is the vector sum of its relative acceleration referred to the earth A_{re}, the absolute acceleration A_0 of the coincident point of the body, and an additional acceleration which is equal to twice the product of the magnitude v_r of the relative*

velocity of the particle and the angular velocity of the body and which has the same direction as that in which the tip of the V_{re} vector, drawn from a fixed point, is made to move by the turning of the body.

Proof. Since the absolute velocity of the particle is the vector sum of its relative velocity referred to the earth V_{re} and the absolute velocity V_0 of the coincident point of the body, the absolute acceleration of the particle is the vector sum of the rate of change of V_{re} and the rate of change of V_0. We determine these rates of change separately, under (*a*) and (*b*).

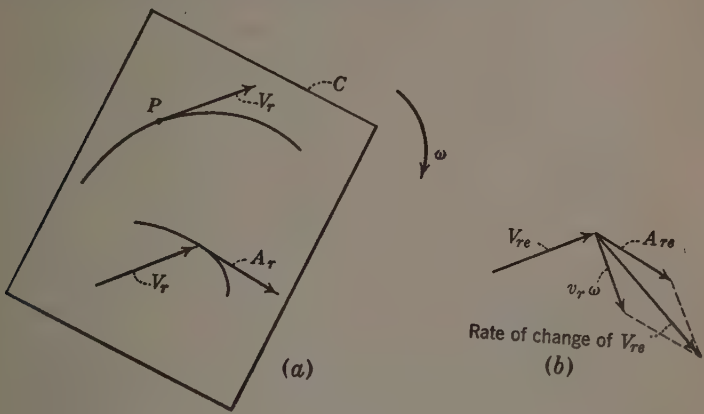

FIG. 231.

(*a*) *Rate of change of V_{re}.* In Fig. 231*a*, C represents the moving body (say a card which slides on the page with plane motion) and P represents the particle (say a bug which runs along the relative path marked on the card). The hodograph on the card shows the variation of the relative velocity; the "velocity" on the card of the tip of the V_r vector is the relative acceleration A_r. The vector V_{re} on the page, drawn from a fixed point (Fig. 231*b*), represents the relative velocity referred to the earth. Now the tip of this V_{re} vector moves (i) with a "velocity" A_{re} because of change in V_r, a change independent of the motion of the card, and (ii) with a "velocity" $v_r\omega$ perpendicular to V_{re}, due solely to the turning of the card and independent of the relative acceleration. The rate of change of V_{re} is the vector sum of these two "velocities," or $A_{re} \nrightarrow v_r\omega$.

(*b*) *Rate of change of velocity of the coincident point.* This velocity also changes in two ways: (i) the coincident point of the body is itself changing velocity at the rate A_0, its absolute acceleration; (ii) the moving particle is continuously shifting to a new coincident point of the moving body; it thus shifts or changes its relative position at the rate V_r, and so in time dt it moves

from O (Fig. 232) to O'. This new coincident point O' has a velocity which differs from that of O by the amount $v_r\,dt \cdot \omega$ in the direction shown, and this change has taken place at the rate $v_r\,dt \cdot \omega \div dt = v_r\omega$. The total rate of change of velocity of the coincident point is therefore $A_0 \nrightarrow v_r\omega$.

The total rate of change of the absolute velocity of P, that is the absolute acceleration of P, is therefore $A = A_{re} \nrightarrow v_r\omega \nrightarrow A_0 \nrightarrow v_r\omega$ or $A_{re} \nrightarrow A_0 \nrightarrow 2v_r\omega$, as was to be proved.

To clarify the physical meaning of these several components of the absolute acceleration we will consider some examples of motion, starting with cases in which one or more of the component rates are zero, and passing to the general case where each has a value.

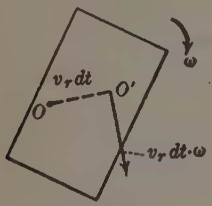

FIG. 232.

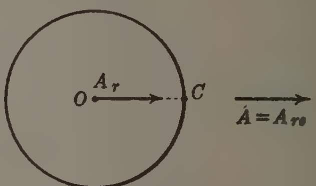

FIG. 233.

1. Suppose the moving body to be a horizontal disk (Fig. 233) which rotates in any way about a central vertical axis, and suppose that a bug starts from rest at the center and begins to run east along the radius OC with a relative acceleration A_r. Then, at the instant of starting, $v_r = 0$, so both rates $v_r\omega = 0$. Also $A_0 = 0$, because the coincident point O of the disk has no acceleration. Therefore the absolute acceleration A of the bug is his relative acceleration referred to the earth A_{re}, equal to the initial rate at which he " speeds up " and directed east.

2. Suppose that the disk (Fig. 234) is rotating uniformly with clockwise angular velocity ω and that the bug runs along the diameter BOC in the direction B to C with constant relative velocity V_r, passing through O when BOC points west-east. Then $A_0 = 0$, and $A_{re} = 0$, and the absolute acceleration of the bug consists of a component $v_r\omega$ due to the fact that V_{re} is changing direction and a component $v_r\omega$ due to the fact that the bug is continuously changing his position to a new coincident point of the disk that has a greater absolute velocity. Both these components $v_r\omega$ are southward; hence the absolute acceleration of the bug is $A = 2v_r\omega$ southward as indicated in the figure.

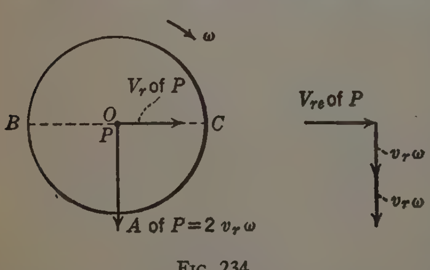

FIG. 234.

It should be noted that, though these two acceleration components were separately determined, each is due to the combination of a relative velocity of the particle with a rotation of the body, and so neither can exist without the other's being present. It is convenient to regard their sum, $2v_r\omega$, as a single acceleration component, which is called the **complementary acceleration** or **Coriolis' acceleration,** after Coriolis, who first pointed out the existence of the rates that compose it.

3. Suppose that the disk (Fig. 235a) is rotating with a clockwise angular velocity ω and a counterclockwise angular acceleration α, and that the bug runs along the diameter BOC from B toward C and reaches the point Q on the disk at the instant BOC points west-east. Suppose further that when the bug reaches Q his relative speed is v_r and is increasing at the rate a_r. To find the absolute acceleration of the bug we must determine A_{re}, A_Q, and $v_r\omega$. Since the relative path of the bug is rectilinear, A_{re} is equal to a_r and its direction is eastward. A_Q has a tangential component $e\alpha$ northward and a normal component $e\omega^2$ westward. The rates $v_r\omega$ are as shown, southward. The absolute acceleration of the bug, A, is the vector sum of A_{re}, A_Q, and $2v_r\omega$ as shown in Fig. 235b.

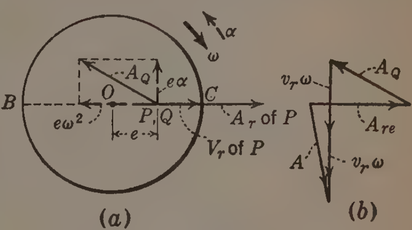

FIG. 235.

It is not necessary that the particle whose velocity and acceleration are to be found should be contiguous to the moving body; we can imagine the particle in or on a rigid extension of the body, or on an axis fixed to and moving with the body, as in Ex. 2 below. Or we can conceive the particle as moving, not relative to any actual body, but relative to a chosen set of moving axes or frame of reference.

EXAMPLE 1. Figure 236a represents a horizontal disk 4 ft in diameter, in the upper face of which there is a groove forming a circular track 1.5 ft in diameter; the center of the track is 1 ft from the center O of the disk. The disk rotates counterclockwise (as viewed from above) about a vertical axis through O with an angular acceleration of 15 rad/sec², while a small sphere travels uniformly clockwise around the track, making 80 circuits per minute. When the disk has attained an angular velocity of 10 rad/sec, it and the sphere (regarded as a particle and denoted by P) are in the positions shown. It is required to determine the absolute velocity and absolute acceleration of the sphere at that instant.

Solution. Let C denote the point of the disk immediately under P. Then the absolute velocity of P is

$$V = V_{re} \leftrightarrow V_C$$

V_r, the relative velocity of P, is computed just as though the disk were stationary. Its magnitude is $(80 \times 1.5 \times \pi) \div 60 = 6.28$ ft/sec, and it is directed along the tangent to the track as shown in Fig. 236a. Therefore $V_{re} = 6.28$ ft/sec and is directed to the left, as shown in Fig. 236b. V_C, the absolute velocity of C, has a magnitude equal to $OC \times \omega =$

$1.25 \times 10 = 12.5$ ft/sec and is directed perpendicularly to OC as shown in Fig. 236a and Fig. 236b. The vector sum of V_{re} and V_C is determined graphically as shown in Fig. 236b and is found to have the magnitude and direction indicated.

The absolute acceleration of P is

$$A = A_{re} \leftrightarrow\!\!\!\!+ A_C \leftrightarrow\!\!\!\!+ 2v_r\omega$$

A_r, the relative acceleration of P, is computed just as though the disk were stationary; it consists simply of the normal acceleration of P when traveling at a constant speed of 6.28 ft/sec (its relative speed) in a circle 1.5 ft in diameter. Its magnitude is therefore $6.28^2 \div 0.75 = 52.6$ ft/sec², and it is directed toward the center of the track as shown in Fig. 236a.

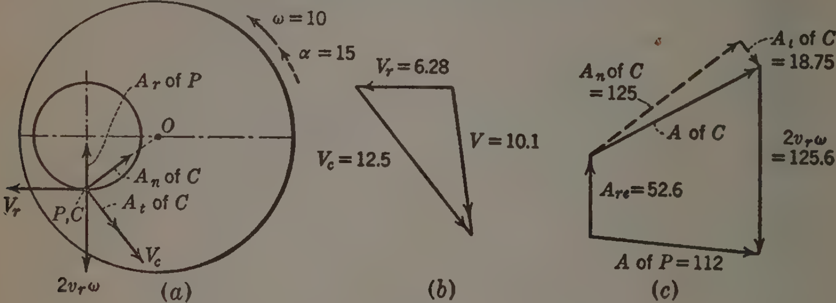

FIG. 236.

Therefore $A_{re} = 52.6$ ft/sec² directed as shown in Fig. 236c. A_C, the absolute acceleration of C, has a tangential component A_t equal to $OC \times \alpha = 1.25 \times 15 = 18.75$ ft/sec² and a normal component a_n equal to $OC \times \omega^2 = 1.25 \times 10^2 = 125$ ft/sec²; the vector sum of these gives A_C, as shown in Fig. 236c. The complementary acceleration of P is $2v_r\omega = 2 \times 6.28 \times 10 = 125.6$ ft/sec², and its direction is perpendicular to V_{re} as shown in Fig. 236a. The accelerations A_{re}, A_C, and $2v_r\omega$ can, of course, be added or compounded either algebraically or graphically. The graphical solution is indicated in Fig. 236c, and A is found to have the magnitude and direction there given.

EXAMPLE 2. Through a diametral hole in a rotating vertical shaft (Fig. 237a) a slender rod AB of length L and weight W is made to slide, the sliding being controlled by a mechanism which is not shown but which may be thought of as integral with the shaft. It is required to determine the forces exerted on the rod by the shaft at the instant the end A of the

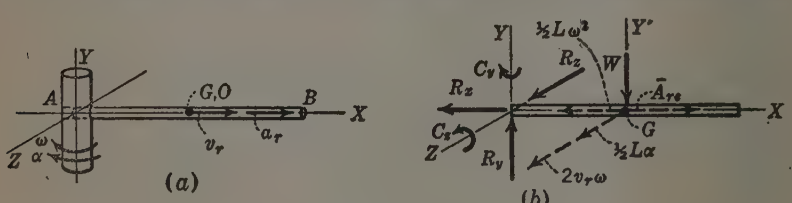

FIG 237.

rod reaches the axis of the shaft, it being assumed that at that instant the shaft is rotating with angular velocity ω and angular acceleration α, and that the rod (i.e., each particle of the rod) is sliding with relative velocity v_r and relative acceleration a_r. (Note that the relative motion of each particle of the rod is rectilinear.)

Solution. Obviously the rod has plane motion. The forces on it can be found by the equations of Art. 121 when the acceleration $\bar{A}$ of the mass-center G of the rod is known. To find $\bar{A}$, we conceive of G as moving along an axis X collinear with the axis of the rod, but fixed to and moving with the shaft. Then $\bar{A}$ is the vector sum of the relative acceleration referred to the earth A_{re} of G, the acceleration A_o of the coincident point O of the axis, and the complementary acceleration $2v_r\omega$. A_o in turn consists of a tangential component $\frac{1}{2}L\alpha$ and a normal component $\frac{1}{2}L\omega^2$. These several components of $\bar{A}$ are directed as shown in the fbd of the rod, Fig. 237*b*. The external forces on the rod are its own weight W and the following forces and couples exerted on it by the shaft at A: a component R_x normal to the path of G; a component R_z parallel to the path of G; a vertical component R_y; a couple C_y in the horizontal plane; a couple C_z in the vertical plane containing W and R_y. These forces and couples are solved for as follows:

$$\Sigma F_x = R_x = \frac{W}{g}\left(\tfrac{1}{2}L\omega^2 - a_r\right)$$

$$\Sigma F_y = R_y - W = 0 \qquad R_y = W$$

$$\Sigma F_z = R_z = \frac{W}{g}\left(\tfrac{1}{2}L\alpha + 2v_r\omega\right)$$

$$\Sigma M_y' = C_y - R_z(\tfrac{1}{2}L) = \bar{I}\alpha \qquad C_y = \frac{W}{g}\left(\tfrac{1}{2}L\alpha + 2v_r\omega\right)(\tfrac{1}{2}L) + \left(\frac{1}{12}\frac{W}{g}L^2\alpha\right)$$

$\Sigma M_z = C_z - W(\frac{1}{2}L) = 0$ (the rod is symmetrical with respect to the plane of the motion; see Art. 121) $C_z = W(\frac{1}{2}L)$

128. Acceleration of the Particle in Any Motion. The relations stated in Art. 127 and their proof presuppose plane motion on the part of the body and relative motion of the particle in a plane parallel to the plane of motion of the body. But the relation $A = A_{re} \nrightarrow A_0 \nrightarrow$ (complementary acceleration) holds for a particle moving in any way relative to a body that has any motion, provided that the complementary acceleration is defined as twice the "velocity" with which the tip of the V_{re} vector, drawn from a fixed point, is made to move by the turning of the body.

For the special case discussed in Art. 127 the "velocity" of the tip of the V_{re} vector is equal to $v_r\omega$. For the general case, it is equal to the angular velocity of the body (as defined in Art. 171) times the length of the projection of the V_{re} vector on a plane normal to the instantaneous axis of rotation (as defined in Art. 171). The complementary acceleration is therefore equal to $2v_r \sin\phi\omega$, where ϕ is the angle between the direction of V_{re} and the instantaneous axis of rotation, and the general equation for the absolute acceleration of the particle becomes $A = A_{re} \nrightarrow A_0 \nrightarrow 2v_r \sin\phi\omega$.

CHAPTER XII

D'ALEMBERT'S PRINCIPLE

129. Preliminary. In the foregoing chapters the relationships between the motion of a body and the actual external forces that act upon it have been developed and expressed by equations such as $\Sigma F_x = m\bar{a}_x$, $\Sigma M = I\alpha$, etc., which can be used in solving various types of problems in dynamics.

This chapter deals with an alternative method of solving such problems, which has certain advantages. This method is based on relationships which constitute what is commonly known as d'Alembert's principle, developed in Art. 131.*

130. Definitions. The terms defined in this article are used mainly in connection with d'Alembert's principle. Their meaning and import should be clearly comprehended for a working understanding of the principle.

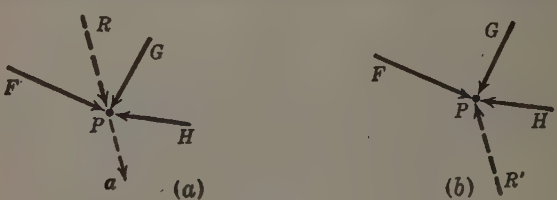

FIG. 238.

By **effective force** for a particle is meant the resultant of all forces acting on it. This resultant is a single force whose magnitude is equal to the product of the mass and acceleration of the particle, whose direction is the same as the direction of the acceleration, and whose line of action passes through the particle (see Art. 89). Thus let F, G, and H (Fig. 238a) be three forces acting on the particle P, a the acceleration of P due to these forces, and dm the mass of P; then the resultant of these forces, or the effective force for the particle, is indicated by the vector marked R, and $R = dm \cdot a$. Like all resultant forces, the effective force is an imaginary force that is equivalent to the real forces; the effective force R, acting alone on the particle, would give it a motion just like the motion given it by the real forces F, G, and H. By **reversed effective force** for a particle is meant a force equal to and collinear with, but opposite to, the effective force; it is an imaginary force which would balance the real

* In his own writings d'Alembert does not give a succinct and definite statement of this principle and method, but both are implicit in the relationships which he points out in his *Traité de dynamique.*

forces. In Fig. 238*b*, R' is the reversed effective force; if R' were actually applied to the particle along with F, G, and H, the particle would have no acceleration.

Consider now a body, regarded as an aggregate of particles. The effective forces for all the particles constitute a system of imaginary forces called the **effective system** for the body. Similarly, the reversed effective forces for all the particles constitute a system of imaginary forces called the **reversed effective system.** Simple equivalents of these systems will be called resultants, for convenience, even if they are not the simplest equivalents.

Some writers and most engineers call reversed effective forces **inertia forces.** We adopt this usage at times, especially in the discussion of balancing (Art. 140) and of general motion (Chapter XV).

131. D'Alembert's Principle. Since the reversed effective force for each particle of a body would balance all the real forces that act on the particle, the reversed effective system would balance all the real forces, external and internal, that act on the body. And, since the internal forces mutually balance (Art. 106), the reversed effective system for the body would balance the external system. Thus, let a body (Fig. 239) be composed of particles 1, 2, 3, etc., and subjected to forces F, G, H, and W; and let the accelerations of the particles respectively be a_1, a_2, a_3, etc. The reversed effective forces for the particles are R_1', R_2', R_3', etc.; this reversed effective system would, if actually applied to the body, be in equilibrium with the real external forces F, G, H, and W.

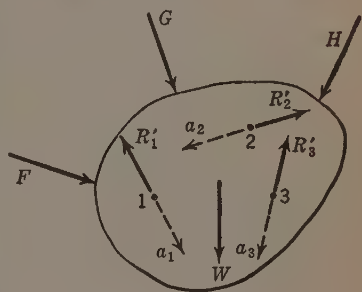

Fig. 239.

This concept of equilibrium as between the actual external forces and the reversed effective system is what is called d'Alembert's principle in this book. It is convenient and appropriate to speak of the two systems jointly as the **d'Alembert system,** and to express the principle by saying that the d'Alembert system, for any body, is in equilibrium.

It might appear that the obvious way to identify the d'Alembert system for a body would be to make a fbd showing all forces of the system, but, as these forces are infinite in number, the reversed effective system must be replaced or represented by its resultant. In the next few articles we compound or reduce the effective systems for some bodies and motions, commonly met in practice, to resultants. These resultants reversed are, of course, the desired resultants of the reversed effective systems, respectively.

D'Alembert's principle is the basis of a special plan of analysis of dynamic problems which may be indicated as follows: (i) identify the d'Alembert system of forces for the moving body under consideration; (ii) write the

appropriate equations of equilibrium for this system; (iii) solve the equations, if possible, for the desired unknowns. Thus a problem in dynamics is transformed, so to speak, into a problem in statics. This transformation constitutes the essential feature of the d'Alembert plan. Many workers in dynamics prefer this plan, and most books on dynamics of machinery and on stress analyses of moving bodies employ it.

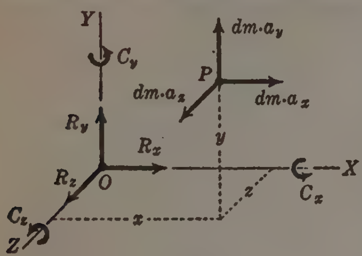

FIG. 240.

132. Resultant Effective System. As explained in Art. 130 we will call any simple equivalent of the effective system the "resultant system" even though such equivalent may not be the simplest. In general, any effective system can be reduced to a single force R "at" any chosen point and a single couple C, called for brevity the resultant RC. See Fig. 240; the chosen point O is used as origin O of a set of coordinate axes, and R and C are shown resolved into axial components.

It can be proved that

$$R_x = m\bar{a}_x \qquad R_y = m\bar{a}_y \qquad R_z = m\bar{a}_z \tag{1}$$

Thus let P (Fig. 240) be any particle of the moving body, not shown, dm the mass of P, and a the acceleration of P. Then the axial components of the effective force for P are as indicated. Since the x component of R is R_x, R_x is equal to the sum of such components as $dm \cdot a_x$ for all the particles. This sum is found in Art. 106 to be $m\bar{a}_x$; hence $R_x = m\bar{a}_x$. Similar proofs apply for R_y and R_z. It follows readily from (1) that $R = m\bar{a}$ and that the directions of R and $\bar{a}$ are identical. Obviously the magnitude and direction of R are independent of the position of the chosen point at which R acts.

It can be proved also that

$$\begin{aligned} C_x &= -\int (dm \cdot a_y)z + \int (dm \cdot a_z)y \\ C_y &= -\int (dm \cdot a_z)x + \int (dm \cdot a_x)z \\ C_z &= -\int (dm \cdot a_x)y + \int (dm \cdot a_y)x \end{aligned} \tag{2}$$

Thus, since the combined moment of R and C about the x axis, say, is C_x, C_x is equal to the sum of the moments about that axis of such forces as those shown at P for all the particles; and this sum is represented by the first of the three equations above. Obviously C_x, C_y, and C_z, and therefore C, depend on the position of the chosen point at which R acts. Equations 2 are rather complex*

* In Chapter XV they are used in the form here given in developing some simple and practical formulas for three-dimensional motion of rigid bodies.

but may be simplified for some problems commonly met in engineering practice. In the following articles we derive such simplified expressions for the resultant effective system.

133. Translation. The resultant of the effective system is a single force, R. The magnitude of R is ma, and the direction of R is the same as that of A where A is the acceleration of the body; the line of action of R passes through the center of gravity of the body. (See Fig. 241, where G is the center of gravity and the acceleration is as indicated.) The foregoing statements are proved in Art. 92 but without use of the term effective force.

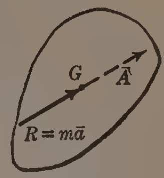

FIG. 241.

EXAMPLE. *BCD*, Fig. 242*a*, represents a uniform triangular plate which weighs 100 lb and is suspended by two ropes each 10 ft long. The plate is raised to the right and allowed to swing down in its own plane. Just before it reaches the bottom of its swing the corner D strikes the spring buffer E. When the plate reaches its lowest position it is moving with a speed of 20 ft/sec and is being slowed down by the resistance of the buffer at the rate of 50 ft/sec². It is required to determine all forces on the plate at that instant.

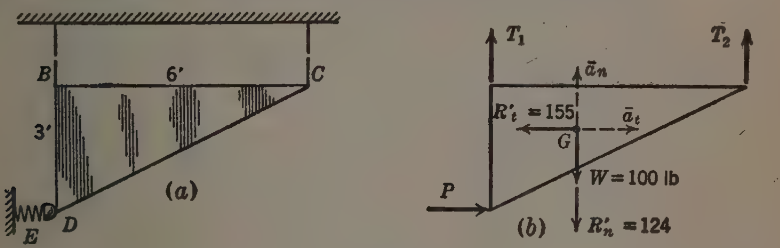

FIG. 242.

Solution. In Fig. 242*b* are shown the following: the plate; the external forces; the acceleration of the mass-center G represented by its normal component $\bar{a}_n = 20^2 \div 10 = 40$ ft/sec² and its tangential component $\bar{a}_t = 50$ ft/sec²; and the resultant of the reversed effective forces R' represented by its normal component $R_n' = (100 \div 32.2)40 = 124$ lb and its tangential component $R_t' = (100 \div 32.2)50 = 155$ lb. The d'Alembert system, comprising the external forces and R', being in equilibrium, the unknown forces P, T_1, and T_2 are solved for by applying appropriate equilibrium equations, thus:

$$\Sigma F_x = P - 155 = 0 \quad \text{whence } P = 155 \text{ lb}$$

$$\Sigma M_D = -(100)(2) - (124)(2) + T_2(6) + (155)(2) = 0 \quad \text{whence } T_2 = 23 \text{ lb}$$

$$\Sigma F_y = T_1 + 23 - 124 - 100 = 0 \quad \text{whence } T_1 = 201 \text{ lb}$$

(You should solve this problem also by the methods of Art. 105 and compare the two solutions.)

134. Rotation, General Case; Resultant *RC*. Here, as in Chapter X, "plane of rotation" means the plane that is perpendicular to the axis of rotation and contains the center of gravity of the moving body; and "center of

rotation" means the intersection of the axis of rotation and the plane of rotation (see Fig. 243). In Arts. 134–137 the center of rotation O is taken as an origin of coordinates and the axis of rotation as z axis.

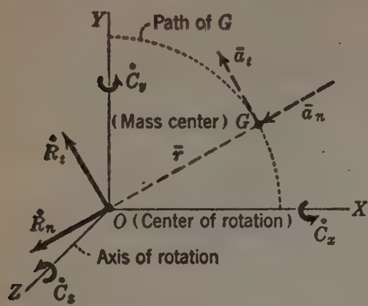

FIG. 243.

By $\mathring{R}\mathring{C}$ (read *ROCO*) we mean the resultant force R and couple C of Art. 132, when it is understood that R is at the center of rotation O.

Since the center of gravity of the rotating body has no z motion, $\bar{a}_z = 0$ and $\mathring{R}$ lies in the plane of rotation. And, since $\mathring{R} = m\bar{a}$ and is directed like $\bar{a}$, the components of $\mathring{R}$ along and perpendicular to OG are $m\bar{a}_n$ and $m\bar{a}_t$, or

$$\mathring{R}_n = m\bar{r}\omega^2 \qquad \text{and} \qquad \mathring{R}_t = m\bar{r}\alpha \tag{1}$$

where as heretofore ω and α denote the angular velocity and angular acceleration of the rotating body, and $\bar{r}$ the length OG. The directions of $\mathring{R}_n$ and $\mathring{R}_t$ are respectively like those of $\bar{a}_n$ and $\bar{a}_t$; the former is always from G toward O; the latter may be as shown or opposite.

It is proved below that the components of $\mathring{C}$ are given by

$$\mathring{C}_z = I_z\alpha \tag{2}$$

$$\mathring{C}_x = \omega^2\int dm \cdot zy - \alpha\int dm \cdot zx \qquad \mathring{C}_y = -\omega^2\int dm \cdot zx - \alpha\int dm \cdot zy \tag{3}$$

The sense of $\mathring{C}_z$ is always like the sense of α. $\mathring{C}_x$ and $\mathring{C}_y$ are represented with positive sense in Fig. 243; in any particular case the sense of either is given by the sign of its computed value. Obviously Eqs. 1 and 2 do not depend on the position of any coordinate axis.

Proof. Let P (Fig. 244) represent any elementary particle of the body; the components of the effective force for P are as indicated. Since $\mathring{R}$, $\mathring{C}_x$, and $\mathring{C}_y$ have no moment about the z axis, $\mathring{C}_z$ is equal to the sum of the moments about that axis of the indicated components for all the particles. That is,

$$\mathring{C}_z = \int (dm \cdot r\alpha)r = \alpha\int dm \cdot r^2 = I_z\alpha$$

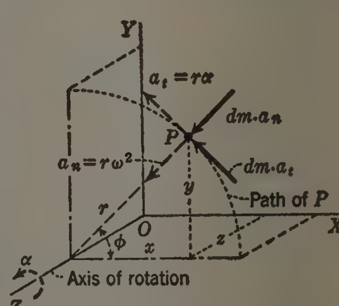

FIG. 244.

(This sum was found in Art. 113 without use of the term effective force.)

$\mathring{C}_x$ is equal to the sum of the moments about the x axis of the indicated

components for all the particles. That is,

$$\mathring{C}_x = \int (dm \cdot r\omega^2) \sin\phi \cdot z - \int (dm \cdot r\alpha) \cos\phi \cdot z$$
$$= \omega^2 \int dm \cdot zy - \alpha \int dm \cdot zx$$

and similarly

$$\mathring{C}_y = -\int (dm \cdot r\omega^2) \cos\phi \cdot z - \int (dm \cdot r\alpha) \sin\phi \cdot z$$
$$= -\omega^2 \int dm \cdot zx - \alpha \int dm \cdot zy$$

Equations 1, 2, and 3 may be used in determining the bearing reactions and other external forces that act on a rotating body. In general it is not easy to evaluate the integrals of Eqs. 3, and indeed for a body of irregular form it may be impossible to do so except approximately.* But fortunately most practical problems come under one or another of the special cases discussed in the next three articles, and for these special cases Eqs. 3 become much simpler.

135. Rotation; Special Case I. The rotating body is homogeneous and has at least one plane of symmetry; the axis of rotation is perpendicular to that plane. For explanation, we suppose the body to be an oblique cone with a circular base. See Fig. 245 where the triangle ABC represents the plane of symmetry; A is the apex of the cone, BC indicates the base, and G is the center of gravity. (Half of the cone is in front of the paper and half behind.) The axis of rotation pierces the paper at O. GO is taken as a coordinate axis for clearness only.

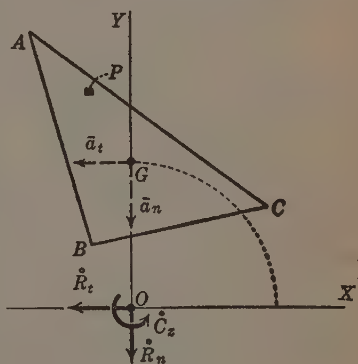

FIG. 245.

The resultant of the effective system consists of a force and a couple, as indicated in Fig. 245, where the force is represented by two components

$$\mathring{R}_t = m\bar{r}\alpha \qquad \mathring{R}_n = m\bar{r}\omega^2 \qquad \mathring{C}_z = I_z\alpha$$

(Notation as in Art. 134.)

Proof that $\mathring{C}_x = \mathring{C}_y = 0$. Consider the rotating body (cone) as consisting of elementary rods perpendicular to the plane of symmetry and of a cross section $dx\,dy$. Consider a rod as consisting of equal parts $dx\,dy\,dz$. The effective forces for all such parts are equal and parallel; hence the resultant of

* The expression $\int dm \cdot zy$ represents the *product of inertia* of the body with respect to the planes from which the z and y coordinates of dm are measured; similar definitions hold for $\int dm \cdot zx$ and $\int dm \cdot xy$. See Appendix B, Art. B5.

these forces lies in the plane of symmetry. Since such resultants for all rods lie in that plane, the effective system for the rotating body has no moment about any line in that plane; hence $\mathring{C}_x$ and $\mathring{C}_y$ equal zero.

Note. If the rotation is uniform ($\alpha = 0$), $\mathring{R}_t$ and $\mathring{C}_z = 0$. If the center of gravity is in the axis of rotation ($\bar{r} = 0$), $\mathring{R}_t$ and $\mathring{R}_n = 0$. If both α and $\bar{r} = 0$, the resultant of the effective system is nil; the system is balanced or in equilibrium, and so is the external system.

EXAMPLE. A homogeneous triangular block (Fig. 246) has attached to it a light shaft or axle. The shaft can turn in bearings at A and B, and its axis is collinear with one edge of the block. The block weighs 10 lb, and the weight of the shaft is negligible. The system is made to rotate about the axis of the shaft z with uniform angular velocity $\omega = 12$ rad/sec by a driving couple T applied to the shaft as shown. It is required to determine T and the bearing reactions at A and B at an instant when the block is in the position shown, base horizontal ($AO = 3$ in.; $BO = 7$ in.).

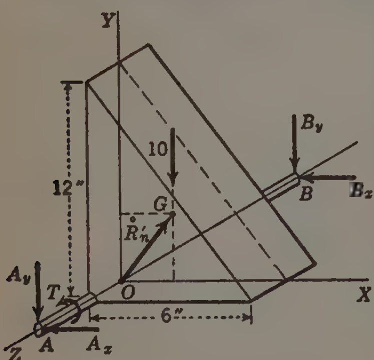

FIG. 246.

Solution. Since the block has a plane of symmetry, the xy plane, and rotates about the z axis which is perpendicular thereto, the problem comes under Case I. Since $\alpha = 0$, $\mathring{R}_t = 0$, $\mathring{C}_z = 0$, and the resultant of the effective system is $\mathring{R}_n = m\bar{r}\omega^2 = (10 \div 32.2) \times (4.47 \div 12) \times 12^2 = 16.7$ lb. The reversed effective system is represented in the figure by $\mathring{R}_n'$. The external system consists of the weight, the driving couple T, and the bearing reactions represented by the components A_x, A_y, B_x, B_y. The unknowns are readily determined by applying appropriate conditions of equilibrium to the d'Alembert system; thus $\Sigma M_z = 0$ gives $T = 1.67$ ft-lb; $\Sigma M_{Ax} = 0$ gives $B_y = 1.49$ lb; $\Sigma F_y = 0$ gives $A_y = 3.46$ lb; $\Sigma M_{Ay} = 0$ gives $B_x = 2.24$ lb; $\Sigma F_x = 0$ gives $A_x = 5.24$ lb.

CENTRIFUGAL FORCE. The normal component R_n' of the reversed effective force system is commonly called the *centrifugal force* of or for the body. It is equivalent to the actual outward force or forces that the rotating body exerts on its bearings by virtue of its rotation, but of course it is not identical therewith. These actual outward forces are defined as centrifugal forces in Art. 104.

136. Rotation; Special Case II. The rotating body is homogeneous and has at least one axis of symmetry; the axis of rotation is parallel to the axis of symmetry. For explanation we suppose the rotating body to be a symmetrical wedge, Fig. 247. It has two planes of symmetry, intersecting in line L, which is the axis of symmetry. GO was taken as a coordinate axis for simplicity only. The resultant of the effective system consists of a force and a couple as indicated in the figure, where the force is represented by two components.

$$\mathring{R}_t = m\bar{r}\alpha \qquad \mathring{R}_n = m\bar{r}\omega^2 \qquad \mathring{C}_z = I_z\alpha$$

(Notation as in Art. 134.)

Proof that $\mathring{C}_x = \mathring{C}_y = 0$. Consider the rotating body to consist of elementary laminas perpendicular to the axis of symmetry. One such lamina is represented in Fig. 247; its coordinate with respect to the xy plane is z. Since this lamina has a plane of symmetry perpendicular to the axis of rotation, it falls under Case I. Therefore the effective system for the lamina can be expressed by two forces and a couple as indicated in the figure, where dm denotes the mass and dI the moment of inertia of the lamina about the axis of rotation. Clearly the couple has no moment about the axis x or y; hence the moments about the axes x and y respectively of the effective system for the lamina are $+(dm \cdot \bar{r}\omega^2)z$ and $-(dm \cdot \bar{r}\alpha)z$. Therefore

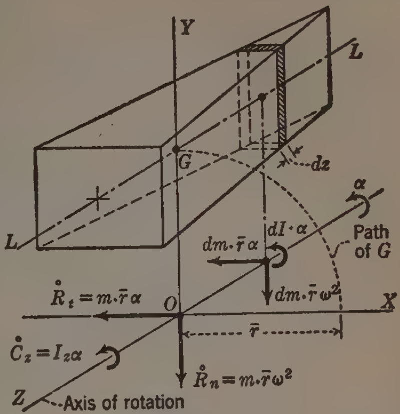

FIG. 247.

$$\mathring{C}_x = +\bar{r}\omega^2 \int z\,dm \qquad \text{and} \qquad \mathring{C}_y = -\bar{r}\alpha \int z\,dm$$

Since G is in the xy plane, $\int z\,dm = 0$; hence $\mathring{C}_x = \mathring{C}_y = 0$.

Since the resultant of the effective system consists in general of two forces in the xy plane and a couple parallel to that plane, the system has no moment about any line in that plane.

Note. If the rotation is uniform ($\alpha = 0$), $\mathring{R}_t$ and $\mathring{C}_z$ are zero. If the axis of rotation coincides with the line of symmetry ($\bar{r} = 0$), $\mathring{R}_t$ and $\mathring{R}_n$ are zero. If both α and $\bar{r} = 0$, the effective system is balanced; hence the external system also.

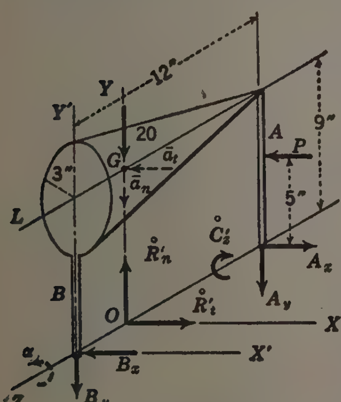

FIG. 248.

EXAMPLE. A solid homogeneous cone (Fig. 248) is mounted on a horizontal axle by means of two rigid extensions or arms A, B (of negligible weight) and is made to rotate about the axle by a force P applied perpendicularly to one of the arms as shown. The cone weighs 20 lb and is 6 in. in diameter at the base. At a certain instant, when it is in its highest position as shown, it is rotating with an angular velocity $\omega = 10$ rad/sec and an angular acceleration $\alpha = 20$ rad/sec². It is required to determine the magnitude of the force P and the axle reactions on arms A and B at that instant.

Solution. The cone has an axis of symmetry L and rotates about an axis z which is parallel thereto; therefore the problem comes under Case II. The acceleration of the mass-

center G has components $\bar{a}_t = (9 \div 12)(20) = 15$ ft/sec^2 and $\bar{a}_n = (9 \div 12)(10^2) = 75$ ft/sec^2. The moment of inertia $I_z = (20 \div 32.2)[(0.3)(\frac{1}{4})^2 + (\frac{3}{4})^2] = 0.361$ slug-ft^2. The effective system consists of $\mathring{R}_t = (20 \div 32.2)15 = 9.32$ lb; $\mathring{R}_n = (20 \div 32.2)75 = 46.6$ lb; $\mathring{C}_z = (0.361)20 = 7.22$ ft-lb. The reversed effective system $\mathring{R}_t'$, $\mathring{R}_n'$, $\mathring{C}_z'$ is represented in the figure. The external system consists of the weight, the force P, and the axle reactions at A and B. The unknowns are found by applying appropriate equilibrium equations to the d'Alembert system. Thus $\Sigma M_z = 0$ gives $P = 17.33$ lb; $\Sigma M_x' = 0$ gives $A_y = 6.65$ lb; $\Sigma F_y = 0$ gives $B_y = 19.95$ lb; $\Sigma M_y' = 0$ gives $A_x = 15$ lb; $\Sigma F_x = 0$ gives $B_x = 6.99$ lb.

Query. Would the cone rotate as described if the arms A and B were simply pinned to the cone instead of being rigidly attached to it? In order to answer, analyze the forces that act on each arm at the end where it is attached to the cone.

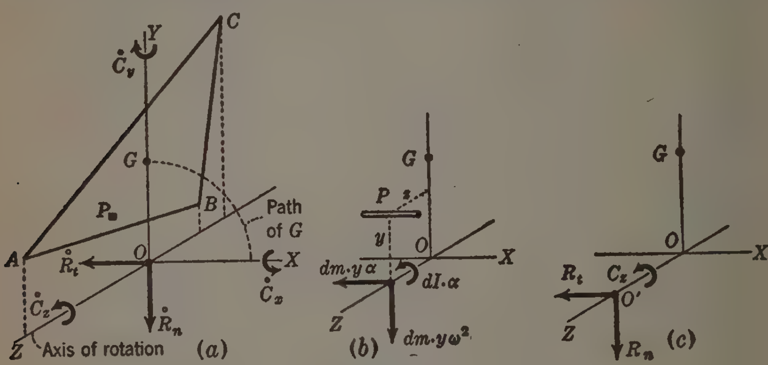

FIG. 249.

137. Rotation; Special Case III. The rotating body is homogeneous and has a plane of symmetry; the axis of rotation is in that plane. For explanation we suppose the body to be an oblique cone with a circular base. See Fig. 249*a*, where the triangle ABC represents the plane of symmetry; A is the apex of the cone, BC indicates the base; and G is the center of gravity of the cone. The x axis is taken perpendicular to the plane of symmetry.

The resultant of the effective system consists in general of a force and a couple as indicated in Fig. 249*a*, where the force is represented by two components and the couple by three components.

$$\mathring{R}_t = m\bar{r}\alpha \qquad \mathring{R}_n = m\bar{r}\omega^2 \qquad \mathring{C}_z = I_z\alpha$$

as in all cases, and as proved below

$$\mathring{C}_x = +\omega^2\int dm \cdot yz \qquad \mathring{C}_y = -\alpha\int dm \cdot yz$$

(Notation as in Art. 134.)

Derivation for $\mathring{C}_x$ *and* $\mathring{C}_y$. Consider the rotor as consisting of elementary rods perpendicular to the plane of symmetry. One such rod, its center located at P in Fig. 249*a*, is represented in Fig. 249*b*. The coordinates of its center are

0, y, and z. Since this rod has a plane of symmetry perpendicular to the axis of rotation, it falls under Case I. Therefore the effective system for the rod can be expressed by two forces and a couple as indicated in Fig. 249b, where dm denotes the mass and dI the moment of inertia of the lamina about the axis of rotation. Clearly the couple has no moment about the axis x or y; and the moments about those axes respectively of the effective system for the rod are $+(dm \cdot y\omega^2)z$ and $-(dm \cdot y\alpha)z$. Hence

$$\mathring{C}_x = +\omega^2 \int dm \cdot yz \qquad \text{and} \qquad \mathring{C}_y = -\alpha \int dm \cdot yz$$

The resultant effective system represented in Fig. 249a can be reduced or simplified to one force and one couple; see Fig. 249c, where the force is represented by two components. This force and couple are just like those of Cases I and II, except that here the force acts through a point O' whose z coordinate is

$$z' = \frac{\int dm \cdot yz}{m\bar{r}}$$

Note. If the body is rotating uniformly ($\alpha = 0$), $\mathring{R}_t$, $\mathring{C}_z$, and $\mathring{C}_y$ vanish. If the center of gravity is in the axis of rotation ($\bar{r} = 0$), $\mathring{R}_t$ and $\mathring{R}_n$ vanish. If both α and $\bar{r} = 0$, the effective system is balanced; and the external system also.

Example 1. Figure 250a represents a solid homogeneous block which is half of a parallelepiped $ABCD$ of dimensions a, b, c; the triangle ABC is the plane of symmetry of this block. By means of short projecting lugs the block is mounted in bearings at B and C, and rotates with uniform angular velocity about the z axis, which is in the plane of symmetry. It is required to determine the reactions of the bearings due to rotation, that is, the reactions that balance the reversed effective system for the block.

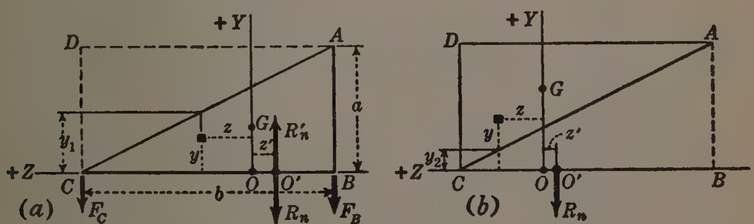

Fig. 250.

Solution. Obviously this problem comes under Case III. The resultant of the effective system is a force $R_n = m\bar{r}\omega^2$ acting at a point O' distant $z' = \int dm \cdot yz/m\bar{r}$ from the center of rotation O. We first determine R_n and z'.

Obviously $\bar{r} = OG = \frac{1}{3}a$, hence $R_n = \frac{1}{3}ma\omega^2$. Consider the block as made up of elementary rods parallel to the x axis (perpendicular to the paper) and of cross section $dy \cdot dz$. Then the mass of each such elementary rod is $dm = \rho c\, dy\, dz$ (where ρ = density of block), and

$$\int dm \cdot yz = \rho c \int_{-\frac{1}{2}b}^{+\frac{1}{2}b} \int_0^{y_1} yz\, dy\, dz$$

where the upper limit of the y integration is $y_1 = (\frac{2}{3} - z/b)a$. Performing the indicated operations, you find that

$$\int dm \cdot yz = -\tfrac{1}{36}mab \qquad \text{Hence } z' = -\tfrac{1}{12}b$$

The minus sign means that R_n is on the negative side of the y axis as indicated in the figure. The reversed effective resultant R_n' is in equilibrium with the reactions F_B and F_C, and solving in the usual way gives $F_B = \frac{1}{4}ma\omega^2$ and $F_C = \frac{1}{12}ma\omega^2$.

EXAMPLE 2. It is required to locate the resultant of the effective force system for the block shown in Fig. 250*b* (the other half of the parallelepiped of Ex. 1), which rotates about the axis z with uniform angular velocity.

Solution. Referring to the figure, $r = \frac{2}{3}a$; $R_n = \frac{2}{3}ma\omega^2$, and

$$\int dm \cdot yz = \rho c \int_{-\frac{2}{3}b}^{+\frac{1}{3}b} \int_{y_2}^{a} yz\, dy\, dz = -\tfrac{1}{36}mab \qquad \text{Hence } z' = -\tfrac{1}{24}b$$

(The resultant of the two forces R_n for Exs. 1 and 2 is, you will find, midway between AB and CD, so that R_n for the entire parallelepiped acts through its center of gravity, as stated under Cases I and II.)

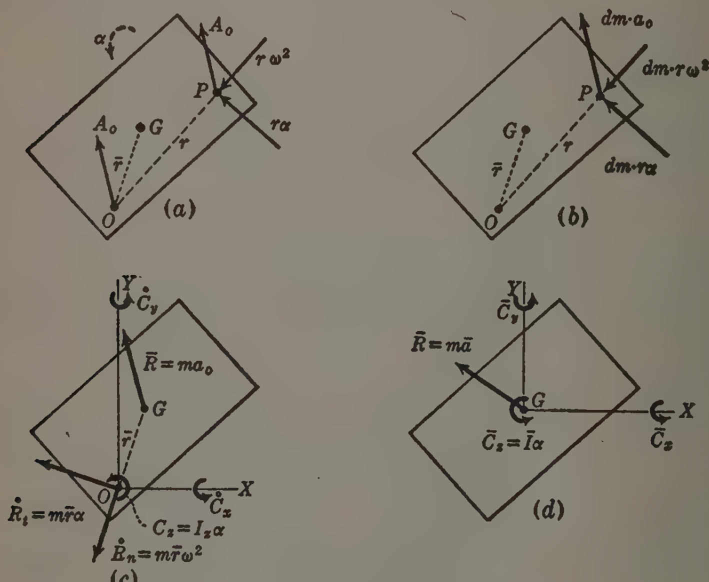

FIG. 251.

138. Plane Motion. It is advantageous here to look upon a plane motion as a combined rotation and translation, as explained in Art. 120. Figure 218*b* of that article, with slight addition, is reproduced here as Fig. 251*a*. The supposed acceleration of the base point O is represented by the vector at O marked A_0. The acceleration of any point P in the translation is A_0; the acceleration in the rotation is represented by two components, $r\alpha$ and $r\omega^2$. The effective force for P is represented in Fig. 251*b* by three components corresponding to the three accelerations just mentioned.

Components $dm \cdot a_0$ for all the particles of the body make up the effective system for the *translation*. The resultant of such a system is described in Art. 133; here we call it $\bar{R}$. Components $dm \cdot r\omega^2$ and $dm \cdot r\alpha$ for all the particles make up the effective system for the *rotation*. The resultant of such a system is described in Art. 134; we continue to call it the resultant $\mathring{R}\mathring{C}$. The resultant of the effective system for the combined motion consists of $\bar{R}$, $\mathring{R}$, and $\mathring{C}$; we may call it the resultant $\mathring{R}\mathring{C}\bar{R}$. This resultant is indicated in Fig. 251c, where $\mathring{R}$ and $\mathring{C}$ are represented, as in Art. 134, by components.

Since $\mathring{R}$ and $\mathring{C}$ are independent of the translation, they are exactly the same as they would be for a rotation of the body about the base point. Therefore, from Arts. 135, 136, and 137, it follows that: (i) if the body has a plane of symmetry and moves in that plane, $\mathring{C}_x = \mathring{C}_y = 0$ and $\mathring{C}_z = I_z\alpha$; (ii) if the body has an axis of symmetry and moves in a plane normal to that axis, $\mathring{C}_x = \mathring{C}_y = 0$ and $\mathring{C}_z = I_z\alpha$; (iii) if the body has a plane of symmetry and moves in a plane perpendicular thereto, $\mathring{C}_x = +\omega^2\int dm \cdot yz$, $\mathring{C}_y = -\alpha\int dm \cdot yz$, and $\mathring{C}_z = I_z\alpha$.

The rotational part of a plane motion may be looked upon as taking place about an axis through the mass-center; that is, the mass-center may be taken as base point. Then a_0 becomes $\bar{a}$, and so $\bar{R} = m\bar{a}$; $\bar{r} = 0$, and so $\mathring{R}_n$ and $\mathring{R}_t = 0$. Hence the resultant effective system reduces to $\bar{C}$ (or $\bar{C}_x$, $\bar{C}_y$, $\bar{C}_z$) and $\bar{R}$, where $\bar{C}$ denotes the value of C when R acts through the mass-center (see Fig. 251d).

EXAMPLE. It is required to determine the forces on the rod of Ex. 2, Art. 122, by d'Alembert's principle.

Solution. It was shown in the example referred to that, for the rod moving as described, $\bar{a}_x = 0$, $\bar{a}_y = 62.6$ ft/sec² downward, $\alpha = 8$ rad/sec² clockwise, and $\bar{I} = 22.35$ slug-ft². Since the rod has a plane of symmetry and moves in that plane, the effective system (taking G as base point) consists of $\bar{R}$, with components $\bar{R}_x = 0$ and $\bar{R}_y = (60 \div 32.2)(62.6) = 116.5$ lb downward, and the couple $\bar{C} = \bar{I}\alpha = (22.35)(8) = 178.8$ ft-lb, in the plane of motion and clockwise.

In Fig. 252 are shown the rod and the d'Alembert system; this comprises the external forces C, B, P, and the weight, and the reversed effective system $\bar{R}'$ and $\bar{C}'$. Applying appropriate equilibrium equations we find $C = 0$; $P = 47.7$ lb, $B = 8.8$ lb.

You should solve this example by taking some other point, as B or C, for base point.

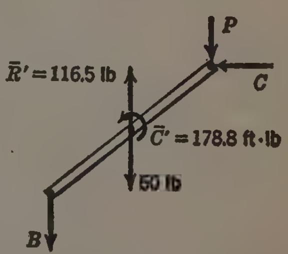

FIG. 252.

139. Group of Moving Bodies. D'Alembert's principle is especially useful when it is desired to determine the external forces that act on a group or aggregate of bodies each of which has a different motion. If the d'Alembert system for the entire group, collectively considered, can be set up and solved, the external forces can be found without considering the internal forces at all. This method greatly facilitates the solution of some problems.

EXAMPLE. In Fig. 253*a*, *A* weighs 200 lb; *B* weighs 100 lb; the pulley weighs 300 lb, is 3 ft in diameter, and has a radius of gyration of 1.2 ft. The rope may be considered weightless and flexible, and the bearings of the pulley as frictionless. It is required to determine the force *P* that will give *B* a downward acceleration of 30 ft/sec² and to determine the reaction *Q* of the bearings on the pulley.

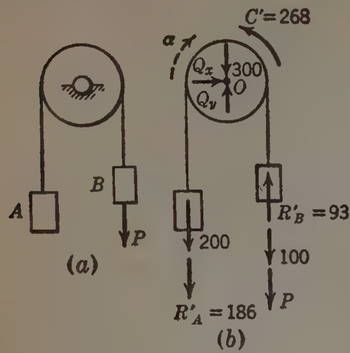

FIG. 253.

Solution. We first establish the reversed effective system for each individual body. *A* has motion of translation with upward acceleration of 30 ft/sec²; the reversed effective system is $R_A' = (200 \div 32.2)30 = 186$ lb, directed downward. Similarly, for *B*, $R_B' = 93$ lb directed upward. The pulley has motion of rotation about an axis of symmetry, with clockwise angular acceleration $\alpha = 30 \div 1.5 = 20$ rad/sec²; hence the reversed effective system is a couple $C' = \bar{I}\alpha = (300 \div 32.2)(1.2^2)(20) = 268$ ft-lb, counterclockwise.

Figure 253*b* represents the entire group of bodies and the d'Alembert system. There are three unknowns, P, Q_x, Q_y; and, since the system is coplanar, nonconcurrent, nonparallel, solution is possible. Writing appropriate conditions of equilibrium, we have

$$\Sigma F_x = Q_x = 0 \qquad \text{whence } Q_x = 0$$

$$\Sigma M_0 = +(186)(1.5) + (200)(1.5) + 268 + (93)(1.5) - (100)(1.5) - P(1.5) = 0$$
$$\text{whence } P = 558 \text{ lb}$$

$$\Sigma F_y = -200 - 186 + 93 - 100 - 558 - 300 + Q_y = 0 \qquad \text{whence } Q_y = 1251 \text{ lb.}$$

(For the sake of comparison, you should solve this example by the methods of Art. 114.)

140. The Balancing of Rotors. The pressures a running rotor exerts on its bearings depend on the weight of the rotor, the applied forces, and its reversed effective or inertia forces. The pressures due to the inertia forces are sometimes called *dynamic forces;* they change direction through 360° every time the rotor makes one revolution, and their components parallel to any line vary periodically from a positive maximum to a negative maximum. They tend to shake the supports on which the rotor is mounted, and, if the number of revolutions per unit time happens to correspond to the natural frequency of the supporting structure, they are likely to cause large vibrations that may result in the failure of some of the parts. (See Appendix C.) In any event, the dynamic forces are more or less harmful, and it is generally desirable that a rotor be so designed as to avoid them. For this, it is necessary that the reversed effective system should vanish, or have no resultant. This is equivalent to saying that the effective system should vanish, or that $R = 0$ and $C = 0$. A body for which this condition obtains is said to be *in balance* or *balanced;* one for which it does not obtain is said to be *out of balance* or *unbalanced.*

For a rotor having constant speed of rotation, the resultant effective system

is shown by Eqs. 1, 2, and 3 of Art. 134 to be

$$\mathring{R} = m\bar{r}\omega^2 \qquad \mathring{C}_x = \omega^2 \int zy \cdot dm \qquad \text{and} \qquad \mathring{C}_y = -\omega^2 \int zx \cdot dm$$

Hence $\mathring{R} = 0$ if $\bar{r} = 0$, and $\mathring{C} = 0$ if the foregoing integrals equal zero.

Designing rotors for balance. Many rotors are just naturally designed to be symmetrical in form with respect to the axis of rotation and homogeneous or with symmetrical density variation. Flywheels, electric motor and generator armatures, and turbine wheels are common examples. For any such rotor, if manufactured exactly and precisely as designed, it is obvious that $\bar{r} = 0$ and (see Appendix B, Art. B5) $\int zy \cdot dm = 0$ and $\int zx \cdot dm = 0$ also. Hence such a rotor is in balance.

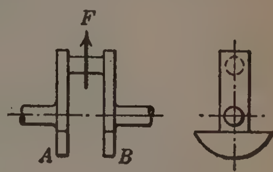

Fig. 254.

But not all rotors can be made thus symmetrical; crankshafts are examples. For balance such a shaft is designed with counterweights whose function is to put the running shaft into balance. The obvious counterweights for a single-throw crankshaft (Fig. 254) consist of two like extensions A and B of the cheeks such that the centrifugal forces of A and B exactly balance the centrifugal force F of the throw. Multi-throw crankshafts are designed with two counterweights for each throw, thus putting the whole shaft in balance. Such a crankshaft could be balanced by means of two suitable counterweights properly placed on the shaft, as shown below, but the common practice of balancing each throw separately has the advantage of reducing shear and bending in the shaft to a minimum (see Art. 141).

Any rotor consisting of simple parts whose centrifugal forces can be calculated can be designed readily with two counterweights that would put the rotor in balance. For example, let OZ (Fig. 255) be the axis of rotation of a rotor consisting of three parts whose centrifugal forces are $10\omega^2$, $30\omega^2$, and $20\omega^2$, acting at a given instant as indicated. Let A and B be chosen convenient centers of rotation of the counterweights and P and Q their centrifugal forces represented by components as indicated. Now this system (seven forces) is in equilibrium; it has four conditions of equilibrium, $\Sigma F_x = \Sigma F_y = \Sigma M_x = \Sigma M_y = 0$; and it is solvable. A solution gives

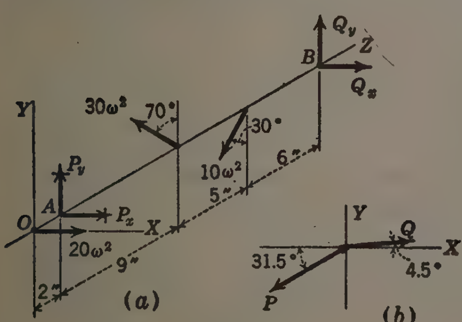

Fig. 255.

$$P_x = -4.98\omega^2 \qquad P_y = -3.05\omega^2 \qquad Q_x = +18.18\omega^2 \qquad Q_y = +1.45\omega^2$$

Hence $P = 5.84\omega^2$ and $Q = 18.25\omega^2$, and their angles with the x axis are 31.5° and 4.5° respectively, as shown in Fig. 255*b*. Suppose that the given forces were calculated from $m\bar{r}\omega^2$, m in slugs and $\bar{r}$ in feet. Then for counterweights at A and B respectively

$$m\bar{r} = 5.84 \qquad \text{and} \qquad m\bar{r} = 18.25$$

For the counterweight at A, one chooses a value of m (or $\bar{r}$) and calculates $\bar{r}$ (or m); similarly for the counterweight at B.

CORRECTING RESIDUAL UNBALANCE. Even a rotor that is nominally symmetrical, or one that is designed for balance in the way described above, will nearly always have some residual unbalance due to imperfections of form and lack of perfect homogeneity. Its effective system does not vanish entirely but has a small resultant RC. The existence of this residual unbalance can be detected by a "balancing machine," which also determines R and C. The unbalance can then be corrected, and it will now be shown that this can be done by the addition or subtraction of a mass (piece of material) in each of two arbitrarily selected transverse planes.

Proof. Figure 256*a* represents a rotating body; z is the axis of rotation. The inertia forces are represented by R' and C' opposite to R and C. (C' is sometimes called the inertia couple.) Planes A and B are the arbitrarily selected transverse planes in which the added masses are to be placed.

Now C' can be thought of as consisting of equal and opposite forces P_A, P_B, acting in planes A and B and through z, as shown in Fig. 256*b*. Also R' can be replaced by parallel components Q_A and Q_B acting in planes A and B and through z, as shown in Fig. 256*b*. Finally, P_A and Q_A can be compounded into a force F_A, and P_B and Q_B into a force F_B, as shown in Fig. 256*c*. Thus the reversed effective system is completely represented by the two radial forces F_A, F_B acting in the selected planes. If now a mass m_A be added in plane A and a mass m_B in plane B, so that the inertia forces for these masses will be equal and opposite to and collinear with F_A and F_B respectively, the entire reversed effective system will vanish. For this, it is necessary that $m_Ar_A\omega^2 = F_A$ and $m_Br_B\omega^2 = F_B$, and that m_A and m_B be located on the lines of action of F_A and F_B and on the side of z from which F_A and F_B respectively point. The proper location of m_A and m_B is indicated in Fig. 256*c*. Obviously the removal of masses m_A and m_B from corresponding points on the opposite side of z would have the same effect.

It should be noted that the inertia force of each mass is assumed to be a single force $mr\omega^2$, acting through the mass-center. This is true if the mass is so small that it may be regarded as a particle, and it is also true if the mass is symmetrical about the plane A or B in which it is placed. But it would not be true in general for an unsymmetrical mass of considerable size, for then the reversed effective force system for the mass would have as resultant both a force and a couple.

Modern balancing machines not only detect and measure unbalance; they also automatically indicate the necessary correction. Planes A and B having been selected so that the addition or removal of small masses in those planes will not weaken the part or affect its functioning, the machine indicates the longitudinal planes in which m_A and m_B should be placed and the distances

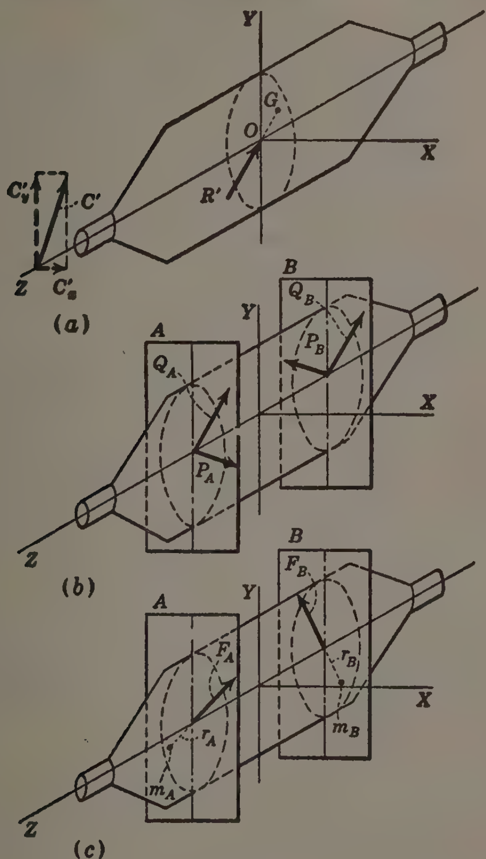

FIG. 256.

r_A and r_B (if m_A and m_B are arbitrarily chosen) or the masses m_A and m_B (if r_A and r_B are arbitrarily chosen). Material may be added by attaching small lumps of solder, or removed by drilling.

The scheme of balancing just described requires that it be possible to add or remove small masses in any longitudinal plane containing the axis of rotation. The form or function of the rotor may be such that these alterations can be

made only in a limited number of such planes, and if this is so some other scheme must be adopted. It can be proved that it is possible to correct unbalance by the addition or subtraction of a mass at each of four arbitrarily selected points, provided that not more than two of these points are in the same transverse plane and not more than two of them are in the same longitudinal plane.

KINDS OF UNBALANCE. A distinction is usually made between unbalance due to the mass-center not being on the axis of rotation (which causes R') and unbalance due to $\int dm \cdot zy$ and $\int dm \cdot zx$ not being zero (which causes C'). The former is called *static unbalance*, the latter *dynamic unbalance*. Either kind of unbalance results in dynamic forces on the bearings only when the rotor is running, but static unbalance may be made apparent when the rotor is not running. Thus a cylinder not in static balance, if laid on smooth horizontal tracks, will roll until the center of gravity is in the lowest possible position. Dynamic unbalance, on the other hand, is apparent only when the rotor is running. The correction of unbalance is called static balancing or dynamic balancing according to whether it eliminates R or C. Most balancing machines measure both kinds of unbalance; some, only one.

141. Dynamic Stresses. In rapidly moving machine elements such as flywheels and connecting rods, any part of the element has, in general, an acceleration. Some of the forces that produce this acceleration may be exerted on the part by outside bodies (e.g., pressure of bearings, earth-pull, electromagnetic force), but some are exerted on it by adjacent parts of the same element. These last are internal forces with respect to the element, and, as explained in Art. 49, such internal forces are stresses. If these stresses become great enough they may cause rupture; thus a flywheel, if rotated at excessive speed, may fly apart. Those stresses in a moving body that account for the acceleration of its parts are called *dynamic stresses*. In determining these dynamic stresses d'Alembert's principle can be used to advantage, as will be shown presently.

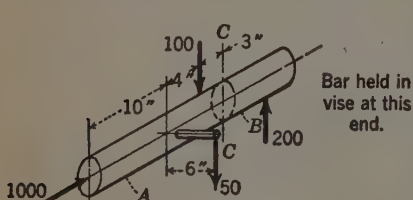

FIG. 257.

In this very limited discussion of dynamic stresses we deal only with elements that may be regarded as bars or rods, relatively slender and uniform in section but not necessarily straight. If such a bar is at rest and is subjected to loads, as shown in Fig. 257, the system of external forces that acts on that part A of the bar that lies on one side of any section C–C is balanced by internal forces exerted on A by the other part of the bar B. These internal forces comprise the stresses in the bar at the section C–C. When the external system is simply an axial pull or push, the stress is an axial tension or compression of equal magnitude, as explained in Art. 49. But in general the external system may be of any kind, and the stresses may be very complex. The determination of the

distribution and intensity of the stresses does not come within the scope of this book, but it may be said that the stresses at a given section of a stationary bar depend wholly on the system of external forces that acts on the bar to one side of that section, and in general the effect of that force system depends on the following: (i) The component of the system normal to the section; this component we call the *axial component* and denote by P. It is a measure of the tendency of the forces to pull the bar in two or to crush it longitudinally. For section C–C of the bar shown in Fig. 257, $P = 1000$ lb. (ii) The component of the system parallel to the section; this component we call the *shearing component* and denote by V. It is a measure of the tendency of the forces to shear the bar in two transversely, as a rivet or pin might be sheared by the members it connects. For section C–C in Fig. 257, $V = 150$ lb. (iii) The moment of the system about an axis in the section and passing through its centroid; this moment we call the *bending moment** and denote by M. It is a measure of the tendency of the forces to bend the bar, as a beam is bent. For section C–C in Fig. 257, $M = 650$ in.-lb. (iv) The moment of the system about an axis normal to the section and passing through its centroid; this moment we call the *twisting moment** and denote by T. It is a measure of the tendency of the forces to twist the bar, as a transmission shaft is twisted. For section C–C in Fig. 257, $T = 300$ in.-lb.

For a stationary bar, then, P, V, M, and T at any given cross section are components and moments of the actual external force system that acts on the bar to one side of that section. For a moving bar, however, they are the components and moments of the combined external and reversed effective force systems for that part of the bar to one side of the section, and even if there are no external forces acting the inertia forces must be taken into account. How this is done will now be shown by several simple examples in which P, V, M, and T are calculated.

EXAMPLE 1. A round steel rod 1 in. in diameter is fixed at right angles to a vertical shaft (Fig. 258). The length of the rod is such that it projects 4 ft from the axis of the shaft Y. The system is made to rotate about Y by a torque applied to the shaft. It is required to determine, for section C–C, 1 ft from Y, the values of P, V, M, and T due to inertia forces at the instant the shaft is rotating with an angular velocity $\omega = 100$ rad/sec and an angular acceleration $\alpha = 5$ rad/sec².

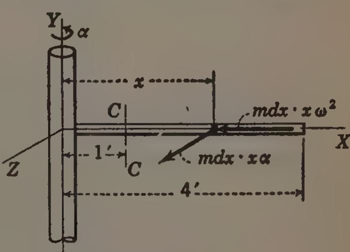

FIG. 258.

Solution. The desired quantities will first be solved for in terms of ω, α, and m (the mass per unit length of rod). Then the numerical values $\omega = 100$, $\alpha = 5$, and $m = 0.083$ (slugs per linear foot) will be substituted.

* Definitions iii and iv require some modification if the cross section of the bar is other than symmetrical about its centroid (round, square, octagonal, etc.), but for present purposes they will suffice.

It will be assumed that an elementary segment of the bar dx feet in length can be regarded as a particle; then the acceleration of any such particle distant x from Y is represented by a tangential component $x\alpha$ and a normal or radial component $x\omega^2$. The inertia forces on the particle are therefore $m\,dx \cdot x\alpha$ (tangential) and $m\,dx \cdot x\omega^2$ (radial) directed as shown. The reversed effective system for that part of the bar to the right of C–C is made up of the inertia forces for all the particles between $x = 1$ and $x = 4$; its radial component is P and is equal to the summation of the forces $m\,dx \cdot x\omega^2$; its tangential component is V and is equal to the summation of the forces $m\,dx \cdot x\alpha$; its moment about the line C–C is M and is equal to the summation of the moments $m\,dx \cdot x\alpha\,(x - 1)$. Thus

$$P = \int_1^4 m\omega^2 x\,dx = 7.5m\omega^2 = 6225 \text{ lb}$$

$$V = \int_1^4 m\alpha x\,dx = 7.5m\alpha = 3.11 \text{ lb}$$

$$M = \int_1^4 m\alpha x(x - 1)\,dx = 13.5m\alpha = 5.6 \text{ ft-lb}$$

None of the inertia forces has a moment about the axis of the bar; therefore $T = 0$.

(In this problem an elementary segment of the bar can be regarded as a particle because, excepting segments very near the shaft, all points of it are practically equidistant from Y and so have almost identical accelerations. If the diameter of the bar were not small compared with its length it would be necessary to base the solution upon consideration of an elementary particle all of whose dimensions were infinitesimal.)

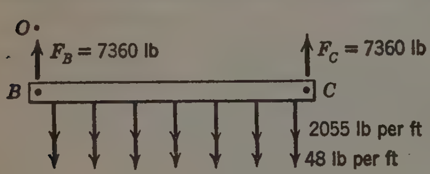

FIG. 259.

EXAMPLE 2. Figure 259 represents the side rod of a locomotive. The rod is 7 ft long between pins B and C and weighs 48 lb/ft; the drivers are 6 ft in diameter, and pins B and C are 1.6 ft from the centers of the drivers to which they connect the rod. It is required to determine the bending moment in the rod at a section halfway between B and C due to its own weight and to the inertia forces acting when the rod is in its lowest position and the train is running at a uniform speed of 60 mi/hr.

Solution. The side rod has translational plane motion. All constituent particles have the same acceleration A, which is the vector sum of the acceleration A_0 of a wheel center O (taken as base point) and the acceleration A_r of B relative to O. Since $A_0 = 0$, $A = A_r = r\omega^2 = 1.6(88 \div 3)^2 = 1380$ ft/sec², and is directed upward when the rod is in its lowest position. The reversed effective force system is therefore equivalent to a uniformly distributed downward load of $(48 \div 32.2)1380 = 2055$ lb/ft, as shown. There is in addition the weight, a uniform downward load of 48 lb/ft, and so the total load is 2103 lb/ft. The rod can be regarded as in equilibrium under this load and the upward reactions F_B, F_C of the pins, each equal to $\frac{1}{2}(2103)(7) = 7360$ lb. The bending moment at mid-span is $(7360)(3.5) - (2103)(3.5)(1.75) = 12{,}880$ ft-lb.

CHAPTER XIII

WORK; POWER; ENERGY

A. Work

142. Work Defined. **Work** is done by a force when the point of application of the force moves so that the force has a component along the path of the point of application. This component we call the *working component* of the force, and the length of the path of the point of application we call the *distance through which the force acts*. If the working component is constant, the amount of work done is equal to the product of the magnitude of the working component and the distance through which the force acts. When the working component acts in the direction of the motion, the work of the force is positive; when the working component acts oppositely to the direction of motion, the work of the force is negative. Forces which do positive work are sometimes called *efforts;* those which do negative work, *resistances*. When a horse draws a cart, the forward pull of the horse is an effort, which does positive work on the cart; the rearward pull of the cart is a resistance, which does negative work on the horse.

If the working component of a force is not constant, the work done is calculated by a process of integration as explained in Art. 143.

We denote work by W_k. ~~Since~~ It is the product of two ~~scalar~~ quantities, work is a scalar quantity. It can be expressed in any units of force and distance; two commonly used units are the foot-pound and the centimeter-dyne (also called erg). Two other important units of work frequently used by engineers, the horsepower-hour and the kilowatt-hour, are defined in Art. 147.

In the above discussion we have spoken of work as being done by a force, but, since the force which does work must be exerted by some body on some other body, it is also correct to say that the work is done by the one body on the other body. Thus a horse does the work of drawing a cart and the work is done on the cart; a spring does the work of closing a door and the work is done on the door, etc. The amount of work done in any given case is usually determined by separately calculating the work done by each of the forces that act, and so in this discussion we usually speak of the work done by a force, rather than of the work done by a body.

143. Calculation of Work Done by a Force. The calculation of the work done by a force is simple when the working component of the force is constant. For example, suppose that the block represented in Fig. 260 is moved in a

straight line along the horizontal floor from A to B by a number of forces, two of which, F_1 and F_2, are constant in magnitude and direction. The work done by F_1 is $+(F_1)(AB)$, the product of the working component, F_1, and the distance AB through which it acts; this work is positive because F_1 acts in the direction of motion. The work done by F_2 is $-(F_2 \cos \phi)(AB)$, the product

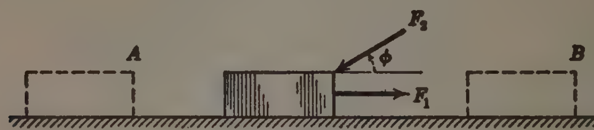

FIG. 260.

of the working component $F_2 \cos \phi$ and the distance AB through which it acts; this work is negative because $F_2 \cos \phi$ acts oppositely to the direction of the motion. The expression for the work of F_2 can also be written $-(F_2)(AB \cos \phi)$, the product of the force and the component of the displacement parallel to the line of action of the force; this definition of amount of work done by a force is sometimes more convenient than the other.

When the working component is not constant in magnitude, the work done can be expressed by an integral, as will now be shown. Let AB (Fig. 261) be the path of the point of application P of a variable force F; P moves from A to B. Let ϕ be the angle between F and the tangent to the path, and ds an elementary portion of the path. Then the work done by F during the elementary displacement is $F \cos \phi \cdot ds$ or $F_t\, ds$, where F_t denotes the working or tangential component of F, and the work done by F in the displacement from A to B is

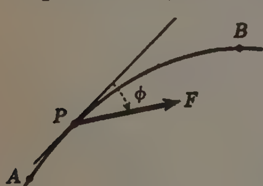

FIG. 261.

$W_k = \int_{s_1}^{s_2} F \cos \phi\, ds$, where s_1 and s_2 respectively are the s-coordinates of P when at A and at B. This integral can be evaluated in the usual way if F and ϕ can be expressed in terms of s. Its evaluation gives not only the magnitude of the work but also the sign, provided that s is measured positive in the direction of the motion and ϕ is measured from the "positive tangent" around to the line of action of the force as indicated in Fig. 261. Sometimes it is convenient to express F and ds in terms of ϕ, or ϕ and ds in terms of F, and then integrate. In any case the limits of integration must be such as to include all elementary works $F \cos \phi\, ds$ in the described motion.

EXAMPLE 1. The block shown in Fig. 262 moves up the inclined plane under the action of certain forces, three of which are shown. F_1 is horizontal and equal to 20 lb; F_2 is normal to the plane and equal to 10 lb; F_3 is parallel to the plane and equal to 30 lb. It is required to determine the work done by each of these forces while the block and the point of application of each force move 100 ft up the plane.

Solution. The working component of F_1 is constant and is $20 \times \cos 30° = 17.3$ lb; it acts in the direction of the motion, and so the work F_1 does is $+(17.3 \times 100) = 1730$ ft-lb. The working component of F_2 is zero; and so F_2 does no work. The working component of F_3 is 30 lb; it acts opposite to the direction of the motion and so the work F_3 does is $-(30 \times 100) = -3000$ ft-lb.

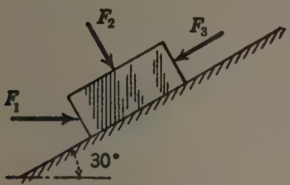

FIG. 262.

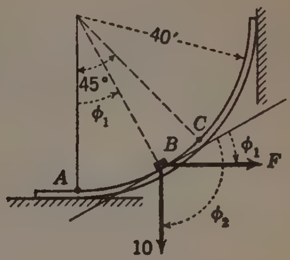

FIG. 263.

EXAMPLE 2. Figure 263 represents a side view of a smooth track which is curved upward to form the arc of a circle of 40-ft radius. A small body B weighing 10 lb is pulled along this track by a horizontal force F, the magnitude of which is at all times just sufficient to keep the body moving. It is required to determine the work done by F in pulling the body from A to C, and the work done by gravity during this displacement. (Note that the reaction of the track, being always normal to the direction of motion, does no work.)

Solution. The work done by F is given by $\int F \cos \phi_1 \, ds$. Both F and ϕ_1 are variable; if they are expressed in terms of s the integral may be evaluated. To find F, the fbd for B is drawn and the forces acting solved for. Since B moves with uniform speed, $a_t = 0$ and so

$$\Sigma F_t = F \cos \phi_1 - 10 \sin \phi_1 = 0 \qquad \text{whence } F = 10 \tan \phi_1$$

Now from the figure it is evident that $\phi_1 = s/40$; therefore the work done by F is

$$\int_0^{31.4} \left(10 \tan \frac{s}{40}\right)\left(\cos \frac{s}{40}\right) ds = 10 \int_0^{31.4} \sin \frac{s}{40} \, ds = +117.2 \text{ ft-lb}$$

The work done by gravity is $\int_0^{31.4} 10 \cos \phi_2 \, ds$. It is obvious from the figure that

$$\phi_2 = \phi_1 + 90° = \frac{s}{40} + \frac{1}{2}\pi \qquad \text{also} \qquad \cos\left(\frac{s}{40} + \frac{1}{2}\pi\right) = -\sin \frac{s}{40}$$

Therefore the work of gravity is

$$\int_0^{31.4} 10\left(-\sin \frac{s}{40}\right) ds = -10 \int_0^{31.4} \sin \frac{s}{40} \, ds = -117.2 \text{ ft-lb}$$

(This result could have been obtained more easily by the special method explained in Arts. 144 and 145.)

Note that the work done on B by gravity is equal in magnitude but opposite in sign to the work done by F, and, as shown in Art. 153, this is consistent with the fact that the block B neither gains nor loses velocity during the displacement.

In the above solution for the work done by the force F we chose to express F and ϕ in terms of s in order to integrate. We could just as well have expressed F and ds in terms of

ϕ. Thus $F = 10 \tan \phi_1$; and $\phi_1 = s/40$, whence $s = 40\phi_1$ and $ds = 40\, d\phi_1$. Therefore the work is

$$\int_0^{31.4} F \cos \phi_1 \, ds = \int_0^{\frac{31.4}{40}} 10 \tan \phi_1 \cdot \cos \phi_1 \cdot 40\, d\phi_1 = 400 \int_0^{0.785} \sin \phi_1 \, d\phi_1 = +117.2 \text{ ft-lb}$$

It should be noted that in this example the sign of the work is given in each case by the evaluation of the integral. Here, and in most problems, one can tell by inspection whether a force is an effort, helping to move the body and so doing positive work, or a resistance, tending to prevent the motion and so doing negative work. But sometimes this cannot be done if the working component is an effort during one part of the motion and a resistance during the other part.

Graphical solution. When the expression $\int F \cos \phi \, ds$ or $\int F_t \, ds$ cannot be evaluated mathematically, graphical integration can be resorted to if corresponding values of the working component F_t and of s are available or can be determined. A curve plotted with s as abscissa and F_t as ordinate is called a work diagram; the area under such a diagram between verticals at s_1 and s_2 represents $\int_{s_1}^{s_2} F_t \, ds$, and hence, to some scale, is equal to the work done by the force F while its point of application moves from the position s_1 to the position s_2. The scale used in interpreting this area depends upon the scales used in plotting s and F_t.

The average ordinate of the work diagram between s_1 and s_2 is the space average value of the working component F_t for the corresponding displacement. It is evidently equal to the work done during the displacement divided by Δs. This definition assumes that neither the direction of the motion nor the sense of F_t is reversed during the displacement.

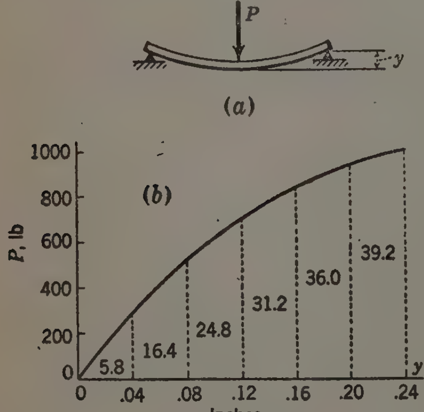

FIG. 264.

EXAMPLE 3. A small beam of cast iron was tested in bending as indicated in Fig. 264*a*. The beam, supported at each end, was loaded at the center with a progressively increasing load P until it broke, and the deflection at the center, y, was measured for each load. The data obtained were used to plot the load-deflection (P–y) curve shown in Fig. 264*b*. It is required to determine the work done by P on the beam in breaking it.

Solution. Since y represents the displacement of the point of application of P, and since P acts in the direction of the displacement, the load-deflection curve is also the work diagram for the test. The area under it is determined by summing the area of vertical strips, these areas being evaluated directly in inch-pounds of work by multiplying the mid-ordinates by the widths. The results for the several strips are recorded on the figure, and the work is the sum of these, or 153.4 in.-lb.

144. Calculation of Work Done by a Force: Special Cases. Many engineering problems that involve the calculation of work come under one or another of the following special cases:

1. WORK DONE BY A FORCE CONSTANT IN DIRECTION AND MAGNITUDE. The work done by a force that is constant in magnitude and direction is equal to the product of the force and the projection of the displacement of its point of application upon a line parallel to the force.

Proof. Let F (Fig. 265) be the force, AB the path of its point of application P, and ϕ the variable angle between F and the direction of the motion of P. Then the work done by F is

$$\int F \cos \phi \, ds = F \int \cos \phi \, ds$$

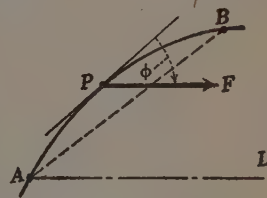

FIG. 265.

Now $\cos \phi \, ds$ is the projection of the elementary portion of the path ds upon a line (such as AL) parallel to F, and $\int \cos \phi \, ds$ is the sum of the projections of all the elements of the path upon that line. But the sum of the projections = the projection of the path AB = the projection of the displacement AB.

The weight of a particle is an example of a force constant in magnitude and direction and having a definite point of application, and so the work done by gravity on a particle in any displacement is equal to the product of the weight and the vertical component of the displacement of the particle.

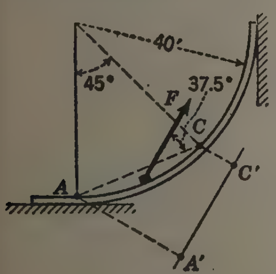

FIG. 266.

EXAMPLE 1. The small body (Fig. 266) is drawn up the circular track by a force F, which is always equal to 12 lb and always acts at an angle of 60° to the horizontal. It is required to determine the work done by F while the body moves from A to C.

Solution. Since F is constant in direction and magnitude, the work it does is equal to the product of its magnitude and the projection of the displacement AC on a line parallel to F. $AC = 2 \times 40 \sin 22.5° = 30.6$ ft; the projection of AC on a line parallel to F is $A'C' = 30.6 \times \cos 37.5° = 24.3$ ft; therefore the work done by F is $12 \times 24.3 = 292$ ft-lb. Since the working component of F is directed like the motion, the work done is positive.

2. WORK DONE BY A FORCE ALWAYS DIRECTED THROUGH A FIXED POINT. The work done by a force always directed through a fixed point is equal to

$$+\int_{r_1}^{r_2} F \, dr \quad \text{or} \quad -\int_{r_1}^{r_2} F \, dr$$

according as the force tends to move its point of application away from or towards the fixed point; here F = magnitude of the force (not necessarily

constant), r = the distance between its point of application and the fixed point, and r_1 and r_2 are respectively the initial and final values of r.

Proof. Let O (Fig. 267) be the fixed point and F the force, whose point of application P moves along the path shown from A to B. The work done by F during an elementary displacement ds is $F \cos \phi \, ds$. But it is obvious from the figure that $\cos \phi \, ds = dr$; hence the work done by F in any elementary displacement is $F \, dr$, and the total work is given by $\int_{r_1}^{r_2} F \, dr$. Obviously changing the sense of F (so that it tends to move P towards instead of away from O) changes the sign of the work done.

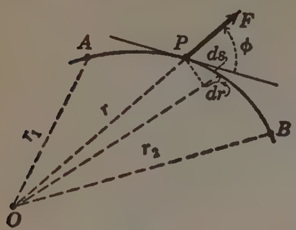

FIG. 267.

An illustration of a force acting through a fixed point is afforded by a fisherman who stands in one spot and pulls in a fish on a hand line. The work the fisherman does in drawing in the line depends only on the tension in the line and the amount of line pulled in; whether the fish comes straight in or zigzags or circles makes no difference.

EXAMPLE 2. A small box C is dragged up an irregular slope from A to B (Fig. 268) by means of a cable and winch as indicated. A constant tension of 250 lb is maintained in the cable. It is required to determine the work done by the cable on the box.

Solution. The pull of the cable acts (practically) through the fixed point O at the top of the winch drum. The initial distance r_1 is AO = 110 ft; the final distance r_2 is BO = 15.6 ft; the pull tends to move C toward O. Therefore

$$W_k = - \int_{r_1}^{r_2} F \, dr = - \int_{110}^{15.6} 250 \, dr = +23{,}600 \text{ ft-lb}$$

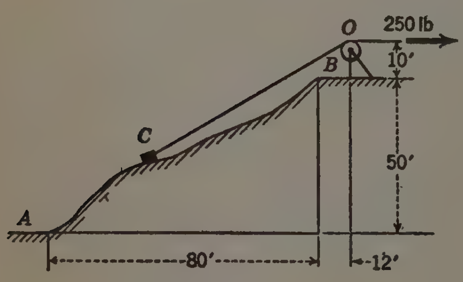

FIG. 268.

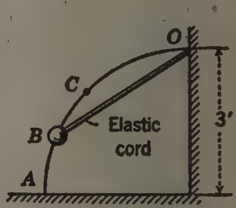

FIG. 269.

EXAMPLE 3. Figure 269 represents a wire bent into a circular arc and a small pierced sphere B which can slide on the wire. The sphere is connected to the fixed point O by a rubber cord, which has a normal length of 1 ft and which stretches $\frac{1}{2}$ in. for each pound of tension. The sphere is released at A and is pulled up along the wire by the rubber cord. It is required to determine the work done on the sphere by the cord during the displacement from A to C, a point halfway between A and O.

Solution. The cord exerts on the sphere a variable force F which always acts through O and tends to move B towards O. The magnitude of F is 2 lb for every inch or 24 lb for every foot the cord is stretched; therefore, if r denotes the distance in feet from B to O, $F = (r - 1)24$. The initial value of r is 4.24 ft; the final value is 2.30 ft. Therefore

$$W_k = -\int_{r_1}^{r_2} F\,dr = -\int_{4.24}^{2.30} (r-1)\,24\,dr = +105 \text{ ft-lb}$$

145. Work Done by a System of Forces. The work done by a system of forces during any displacement of the body or bodies on which they act can be found by determining the work done by each of the forces and then adding these works algebraically. There are, however, a number of important special cases in which the work done by a system of forces can be found more expeditiously than by this general method, and these special cases will now be discussed.

1. Work done by a number of forces with a common point of application. The work done by a system of forces having a common point of application during any displacement is equal to the work that would be done during that displacement by a single force having the same point of application and equal at all times to the resultant of the system. Conversely, the work done by a force in any displacement is equal to the work that would be done by the components of that force acting at its point of application.

Proof. Let F', F'', etc. = the forces; R = their resultant; F_t', F_t'' the tangential components of the forces; and R_t, the tangential component of the resultant. Now $F_t' + F_t'' \cdots = R_t$; hence $F_t'\,ds + F_t''\,ds \cdots = R_t\,ds$, and
50

$$W_k = \int F_t'\,ds + \int F_t''\,ds \cdots = \int R_t\,ds$$

Example 1. It is required to determine the work done by the force F of Ex. 1 of Art. 144 from the work done by its x and y components.

Solution. Since $F = 12$ lb and is directed up and to the right at 60° to the horizontal, $F_x = 6$ lb to the right and $F_y = 10.4$ lb up. Both these components are constant, and both are taken to act at the point of application of F. The projection of the displacement on the horizontal is $30.6 \cos 22.5° = 28.25$ ft, and its projection on the vertical is $30.6 \sin 22.5° = 11.7$ ft; therefore the work done by F_x is $6 \times 28.25 = 170$ ft-lb, and the work done by F_y is $10.4 \times 11.7 = 122$ ft-lb. Both these works are positive, and so the work done by the two components, which is the work done by F, is +292 ft-lb.

2. Work done by gravity on any body. The work done by gravity upon any body in any displacement is equal to the product of the weight of the body and the vertical displacement of its center of gravity; this work is positive when the center of gravity descends and negative when it ascends.

Proof. Let Fig. 270 represent any body (here assumed nonrigid), of weight $= W$, in the initial and final positions indicated. Particle 1 of the body, of weight w_1, moves from A to A'; particle 2, of weight w_2, moves from B to B'; etc. The initial heights of the particles from the assumed datum plane are

denoted by y_1, y_2, etc., and their final heights by y_1', y_2', etc., while $\bar{y}$ and $\bar{y}'$ respectively denote the initial and final heights of the center of gravity G. Now according to Art. 144 the works done by gravity on the individual particles

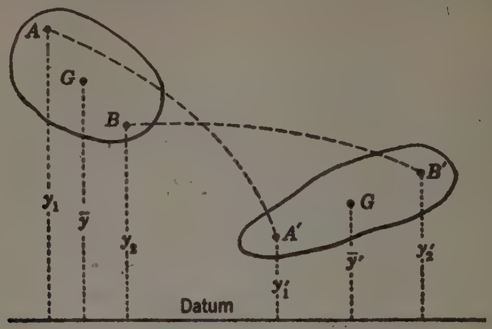

FIG. 270.

are $w_1(y_1 - y_1')$, $w_2(y_2 - y_2')$, etc., and the sum of these works can be written

$$(w_1y_1 + w_2y_2 \cdots) - (w_1y_1' + w_2y_2' \cdots)$$

But

$$(w_1y_1 + w_2y_2 \cdots) = W\bar{y} \qquad \text{and} \qquad (w_1y_1' + w_2y_2' \cdots) = W\bar{y}' \quad \text{(Art. 73)}$$

Hence the total work done on all the particles is $W(\bar{y} - \bar{y}')$, or the product of the weight of the body and the vertical displacement of its center of gravity. Obviously if $\bar{y} > \bar{y}'$ (G descends) the work is positive, and if $\bar{y} < \bar{y}'$ (G ascends) the work is negative.

EXAMPLE 2. A homogeneous rectangular block of stone 2 ft by 2 ft by 6 ft standing on end is pushed over so that it falls to a horizontal position on the ground. It is required to determine the work done by gravity on the block during this displacement. The weight of the block is 3600 lb.

Solution. The initial height of the center of gravity is 3 ft, the final height is 1 ft; the vertical displacement of the center of gravity is therefore 2 ft downward, and the work of gravity is $3600 \times 2 = 7200$ ft-lb, positive because the center of gravity has descended. (During the first part of the motion, while the block is being tipped up on edge, the center of gravity rises and gravity does negative work; during the rest of the motion gravity does positive work, and the algebraic sum of these works is 7200 ft-lb.)

EXAMPLE 3. A cylindrical cistern 20 ft deep and 6 ft in diameter, containing 10 ft of water, is pumped dry, the water being discharged at the ground level. It is required to determine the work done by gravity on the water.

Solution. The weight of the water is 17,650 lb; the vertical displacement of its center of gravity is 15 ft; therefore the work done by gravity is $-(17{,}650 \times 15) = -265{,}000$ ft-lb, negative because the center of gravity has ascended. (The work done "against gravity" is $+265{,}000$ ft-lb.)

3. WORK DONE BY TWO EQUAL AND OPPOSITE COLLINEAR FORCES. The work done by a pair of equal and opposite collinear forces in any displacement

of their points of application is equal to

$$+\int_{r_1}^{r_2} F\,dr \qquad \text{or} \qquad -\int_{r_1}^{r_2} F\,dr$$

according as the forces tend to separate or to move the points of application closer together; here F = the common magnitude of the two forces (not constant necessarily), r = the varying distance between their points of application, and r_1 and r_2 are respectively the initial and final values of r.

Proof. Let F_1 and F_2 (Fig. 271) represent the equal forces, whose points of application P_1 and P_2 move respectively from A_1 to B_1 and from A_2 to B_2 along the paths shown. The work done by F_1 in an elementary displacement ds_1 is $F_1 \cos\phi_1\, ds$; the work done by F_2 in an elementary displacement ds_2 is $F_2 \cos\phi_2\, ds_2$; the work done by both forces is the sum of these works or $F(\cos\phi_1\, ds_1 + \cos\phi_2\, ds_2)$. But it is obvious from the figure that $\cos\phi_1\, ds_1 + \cos\phi_2\, ds_2 = dr$; hence the work done by both forces in the elementary displacement is $F\,dr$ and the total work is given by $\int_{r_1}^{r_2} F\,dr$. Obviously changing the senses of F_1 and F_2 reverses the sign of the work.

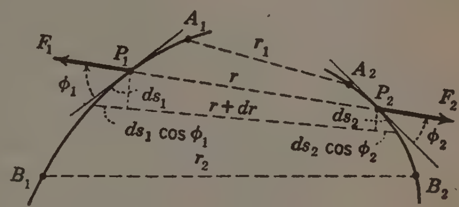

FIG. 271.

If r is constant, then no work is done, and so two equal and opposite collinear forces whose points of application have fixed positions in a rigid body do no work in any displacement of that body. Since all internal forces consist of pairs of equal, opposite, and collinear actions and reactions, the total work done by internal forces in any displacement of a rigid body is zero.

EXAMPLE 4. Two balls are connected by a light spiral spring, the normal length of which is 2 ft and which shortens 2 in. for each 1 lb compression. The balls are pressed to within 6 in. of each other and the whole then tossed into the air. It is required to determine the work done by the spring on the balls while expanding to its normal length.

Solution. The spring exerts equal and opposite forces F on the two balls. Let r = length of spring, in inches, at any instant; then $F = \frac{1}{2}(24 - r)$ lb, and the work done is

$$W_k = +\int_{r_1}^{r_2} F\,dr = +\int_6^{24} \tfrac{1}{2}(24 - r)\,dr = 81 \text{ in.-lb}$$

Solution may be effected more simply by using the average value of F. Thus the maximum compression in the spring is $(24 - 6) \times \frac{1}{2} = 9$ lb; the average force it exerts is half this or 4.5 lb; the value of Δr is 18, and the work done is $4.5 \times 18 = 81$ in.-lb.

4. WORK DONE BY A COUPLE ACTING ON A RIGID BODY. If a couple acts on a rigid body, with points of application of the forces fixed in that body, then during any displacement of the body in the plane of the couple the couple does

work equal to $\int_{\theta_1}^{\theta_2} C\, d\theta$, where $C =$ the moment of the couple, $\theta =$ the angle between any fixed line in the plane of the couple and a coplanar line in the body, and θ_1 and θ_2 are the initial and final values of θ. The work is positive when the couple tends to produce the angular displacement and negative when it tends to prevent it.

Proof. Let Fig. 272 represent the rigid body, acted on by the couple C, lying in the plane of the paper and comprising the forces F_1, F_2 with arm a and points of application P_1, P_2. Any elementary uniplanar displacement of the body can be accomplished by an elementary translation of any chosen base point followed by an elementary rotation about an axis through that base point (Art. 118). Suppose that P_2 is chosen as base point. During the elementary translation no work is done by the couple because the works done by the forces are equal and opposite in sign. During the elementary rotation $d\theta$ about P_2 the couple does work; this work is done by F_1 (since P_2 does not move) and is equal to the product of the working component $F \cos \phi$ and the displacement $r\, d\theta$, or $F \cos \phi r\, d\theta$. But $r \cos \phi = a$; hence the work done is $Fa\, d\theta$ or $C\, d\theta$. And in any finite displacement $W_k = \int_{\theta_1}^{\theta_2} C\, d\theta$.

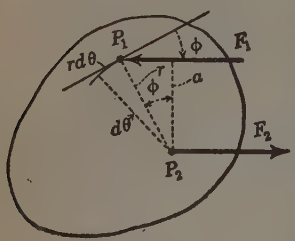

FIG. 272.

This integral can, of course, be evaluated in the usual way if C can be expressed as a function of θ, as in the example below; otherwise graphical integration can be employed. If C is constant, then the work done in any angular displacement $\Delta\theta$ is simply $C \cdot \Delta\theta$.

In the above discussion it is assumed that the couple acts on a rigid body and consists of two definite forces. But the analysis applies to any case in which the points of application P_1 and P_2 remain at a fixed distance apart if θ refers to the position of the line joining P_1 and P_2, and it also applies to any force system whose resultant is a couple if the points of application of the several forces have positions fixed in a rigid body.

EXAMPLE 5. A straight steel shaft is held in a vise at one end and twisted by a couple applied at the other end. The angle through which the shaft is twisted is proportional to the moment of the couple, and is given by the equation $\theta = C/400$, where θ is the angle of twist in degrees and C the moment of the couple in foot-pounds. It is required to determine the work done by the couple in twisting the end of the shaft through 90°.

Solution. The shaft as a whole is not a rigid body, but that part immediately adjacent to the end where the couple is applied may be regarded as rigid; that is, it may be assumed that the points of application of the forces making up the couple remain a constant distance apart. Under these circumstances the work of the couple is given by $\int C\, d\theta$. Since $\theta = C/400$, $C = 400\ (\theta \text{ deg}) = 22{,}920\ (\theta \text{ rad})$, and $W_k = \int_0^{\frac{1}{2}\pi} 22{,}920\theta\, d\theta = 28{,}240$ ft-lb.

146. Virtual Work. All the forces that act on any particle of a body have a common point of application, and in any displacement of that particle these forces do work equal to that which would be done by their resultant (Art. 145). If the forces on the particle constitute a balanced system, their resultant is nil, and in any displacement of the particle they collectively would do no work. Now if a rigid body is in equilibrium every particle of that body is in equilibrium, and if the body suffers a displacement (due, say, to some extra force or forces) the work done by all the original forces on all the particles is zero, and since in any case the work done by all the internal forces is zero (Art. 145) the work of the original external forces is zero, provided that they are always in equilibrium during the displacement.

When a body acted on by balanced forces suffers a finite displacement the magnitudes and relative positions of the forces may change so that they are no longer in equilibrium, but if the displacement is small these changes are small, and if the displacement is infinitesimal we may say that equilibrium persists throughout the displacement and that the forces collectively do no work. Therefore if a rigid body in equilibrium under balanced forces were to undergo an infinitesimal displacement the external forces on the body collectively would do no work, and conversely if in any supposed infinitesimal displacement of a rigid body the external forces do no work then those forces are in equilibrium. This is a statement of the **principle of virtual work** (sometimes called the principle of virtual displacements or virtual velocities). The word "virtual" here connotes that the work or displacement is supposed, or imaginary.

The principle of virtual work can be used for solving problems in statics, and for some problems it is preferable to the usual equilibrium equations. It can be applied not only to rigid bodies but also to assemblages of rigid bodies such as hinged frames and mechanisms whose parts can move relative to one another, provided that such motion involves no friction.

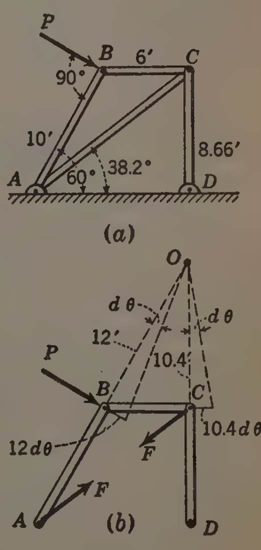

FIG. 273.

EXAMPLE 1. The framework shown in Fig. 273*a* is pinned to the floor at *A* and *D* and is pin-jointed at *B* and *C*. Member *BC* is horizontal. It is required to determine the stress in member *AC* due to a load *P* of 100 lb applied as shown.

Solution. The member *AC* is assumed to be removed, and the equal forces *F*, *F* which it exerts on joints *A* and *C* are represented as pulls (Fig. 273*b*). This amounts to assuming that the stress in *AC* is a tension of magnitude *F*. Now any infinitesimal displacement of the incomplete frame shown in Fig. 273*b* would result in a rotation $d\theta$ of member *BC* about its instantaneous center *O*, and as a result of this rotation joints *B* and *C* would respectively

suffer the displacements 12 $d\theta$ and 10.4 $d\theta$ as shown, while joints A and D would not move at all. The only work done during the displacement would therefore be that done by the working components of P and F, and this would be equal to zero; therefore

$$\text{Virtual work} = 100(12\,d\theta) - (F\cos 38.2°)(10.4\,d\theta) = 0$$

whence $F = 146.6$ lb. Since a positive result is obtained, F is tension as assumed.

(It should be noted that Fig. 273*b* is not a complete fbd for the part of the structure there shown, and the forces F and P are not mutually in equilibrium. A complete fbd would include the reactions from the supports at A and D, and these reactions, together with P, F, and F, would constitute a force system in equilibrium. But since the reactions at A and D do no work they need not be included in the virtual work equation.)

EXAMPLE 2. The jointed frame shown in Fig. 274 consists of four bars of length L and four bars of length $\frac{1}{2}L$, pinned together at points of contact and pinned to the floor at A. The frame is symmetrical about a vertical line through A. The joints permit the frame to be extended or compressed vertically. A load W suspended at B as shown can be balanced either by a vertical force P or by equal horizontal forces Q, Q. It is required to determine (*a*) the necessary force P, and (*b*) the necessary forces Q, Q.

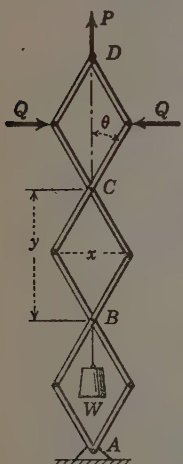

FIG. 274.

Solution. (*a*) An infinitesimal lengthening or shortening of the frame would be accompanied by a change $d\theta$ in the angle θ and by changes dx and dy in the dimensions x and y respectively. The only forces doing work are P and W. The point of application of W would move a vertical distance dy; the point of application of P would move a vertical distance $3dy$. Assuming upward displacement, the equation of virtual work is

$$-W(dy) + P(3dy) = 0 \qquad \text{whence } P = \tfrac{1}{3}W$$

(*b*) The only forces doing work are W and Q, Q. The point of application of W would move a vertical distance dy; the point of application of each force Q would suffer a horizontal displacement $\frac{1}{2}dx$. Now $x = L\sin\theta$, and $y = L\cos\theta$; therefore $dx = L\cos\theta\,d\theta$ and $dy = -L\sin\theta\,d\theta$. (The signs indicate that, as θ increases, x increases and y decreases.) Again assuming upward displacement, the equation of virtual work is

$$-W(L\sin\theta\,d\theta) + 2Q(\tfrac{1}{2}L\cos\theta\,d\theta) = 0 \qquad \text{whence } Q = W\tan\theta$$

Evidently it makes no difference to which panel of the frame forces Q, Q are applied, but if W were suspended at C, W would suffer twice as much displacement and Q would have to equal $2W\tan\theta$, and if W were suspended at D, Q would have to equal $3W\tan\theta$.

Most problems in statics can be solved just as easily by means of the ordinary equilibrium equations as by the method of virtual work, but occasionally the latter method will be found much easier. Thus Ex. 1 above could be readily solved by passing a section through AB, AC, and CD and considering that part of the structure above that section; by taking moments about O the stress F could be found by a single equation identical with the virtual work equation except that $d\theta$ would not appear in it. But to solve Ex. 2 by statics would prove much more laborious than to solve by virtual work. You should carry out a solution by statics for the sake of comparison.

B. Power

147. Power Defined. The time rate at which work is done is called power. In common parlance the word power is used with several somewhat different meanings; thus the adjective powerful may mean capable of exerting a great force, or capable of doing a great amount of work, or capable of working at a high rate. But in most engineering usage power refers specifically to one thing — time rate of doing work — and in this book the word is used in that sense alone.

Power, like work, is a scalar quantity. It can be expressed in any units of work and time, as foot-pounds per minute, ergs per second, etc. The units commonly used in engineering, however, are: the English and American horsepower, equal to 550 foot-pounds per second or 33,000 foot-pounds per minute; the continental horsepower, equal to 75 kilogram-meters per second or 4500 kilogram-meters per minute; the kilowatt, equal to 10^{10} ergs per second. (The kilowatt is equal to about 1.34 English horsepower.) The symbol for power is P.

From these units of power are derived convenient units of work, namely: the horsepower-hour and the kilowatt-hour; these are respectively the amounts of work done in 1 hour by an agent working at the rates of 1 horsepower and 1 kilowatt. The horsepower-hour is thus equal to 1,980,000 foot-pounds, and the kilowatt-hour is equal to 36×10^{12} ergs, or 36×10^5 joules, or about 2,654,800 foot-pounds.

148. Calculation of Power. At times it is required to determine the power developed by a force or forces that act on a body (as a car or other machine) and cause it to move in some particular manner.

The average power for a given time interval is the work done during that interval divided by the interval.

The power developed by a force at any instant is equal to the product of the working component of the force and the speed of its point of application, or

$$P = F \cos \phi \cdot v$$

Proof. In any elementary period of time dt the point of application of the force moves an elementary distance ds and the force does work equal to $F \cos \phi \, ds$. The average power for the infinitesimal period dt is $(F \cos \phi \, ds)/dt$, and this is the power at the instant in question. But $(F \cos \phi \, ds)/dt = F \cos \phi \, (ds/dt) = F \cos \phi v$. This power is positive or negative according as $F \cos \phi$ is directed like the motion or oppositely to the motion. The power developed at any instant by a couple acting on a rigid body is equal to the product of the magnitude of the couple C and the angular velocity $d\theta/dt$ of the body about an axis normal to the plane of the couple. For it is shown in paragraph 4 of Art. 145 that in any elementary displacement $d\theta$ of the body in the plane of the couple the work done is $C \, d\theta$; the average power for the in-

finitesimal period dt is therefore $C\, d\theta/dt$, and this is the power at the instant in question.

The power developed by any system of forces is simply the algebraic sum of the powers developed by the several constituent forces. In speaking of the power developed by a machine it is customary to use some qualifying term that defines the availability of the work in question. Thus the "brake horsepower" of an airplane engine is the rate at which the crankshaft torque can do work; it is proportional to the torque multiplied by the angular velocity of the shaft, or $C \cdot d\theta/dt$. On the other hand, the "thrust horsepower" of the engine-propeller combination is the rate at which the pull of the propeller does work on the airplane; it is proportional to the pull of the propeller multiplied by the velocity of the plane, or $F \cdot v$. The thrust horsepower is less than the brake horsepower because of the imperfect efficiency of the propeller, which wastes more or less energy in simply stirring up the air.

EXAMPLE 1. A load weighing 1000 lb is hoisted by a vertical cable, the pull of which gives the load a uniform upward acceleration of 5 ft/sec² until it has been raised 50 ft. The load is initially at rest. It is required to determine the average power developed by the pull of the cable and also the maximum power developed at any instant during the hoisting of the load.

Solution. The pull F required to give the load an upward acceleration of 5 ft/sec² is found from the equation $\Sigma F_y = F - 1000 = (1000 \div 32.2)5$ to be 1155 lb. The time required to raise the load 50 ft is found from the equation $s = \frac{1}{2}at^2$, or $50 = \frac{1}{2}(5)t^2$, to be 4.47 sec; and the maximum velocity is therefore $5 \times 4.47 = 22.35$ ft/sec. The average power is the work done divided by the time, or $(1155 \times 50) \div 4.47 = 12{,}900$ ft-lb/sec. The maximum power is the product of the working component and the maximum velocity of the load, or $1155 \times 22.35 = 25{,}800$ ft-lb/sec. Note that at the instant of starting, when the velocity is zero, the power is zero.

EXAMPLE 2. A hoisting rig consists of a drum mounted on a horizontal axle and a rope attached to and wound around the drum. The drum is 4 ft in diameter, and the load weighs 500 lb. By means of a driving couple C the shaft is made to rotate uniformly so as to raise the load 4 ft/sec. The frictional resistance at the bearing amounts to a resisting couple of 20 ft-lb. It is required to determine the power developed by the driving couple.

Solution. Since the load has no acceleration the tension in the hoisting rope is 500 lb. Since the drum has no angular acceleration the sum of the moments of the external forces about the axis of rotation is zero, or $C - 20 - (500 \times 2) = 0$; whence $C = 1020$ ft-lb. The angular velocity of the drum is $\omega = 4 \div 2 = 2$ rad/sec; therefore the power developed by C is $C\omega = 1020 \times 2 = 2040$ ft-lb/sec or 3.71 hp.

C. Energy

149. Definitions. When the state or condition of a body is such that it can do work, the body is said to possess **energy.** It is customary to distinguish several kinds of energy. Thus a body may have *kinetic* energy by virtue of its motion, *potential* energy by virtue of its position in a field of force* or by

* By *field of force* is meant a region anywhere within which a body is subjected to a force such as gravity, or electromagnetic attraction or repulsion. The potential energy ascribed to a body in such a field really belongs to both or all bodies involved; thus we speak of the potential energy of a heavy body raised above the earth's surface, but it is the earth-body

virtue of its state of internal stress, *thermal* energy by virtue of its temperature,† *chemical* energy by virtue of its chemical composition. The amount of energy, of any given kind, that a body possesses at a given instant is the amount of positive work the body can do in changing from the condition it is in at that instant to some other condition taken as standard. Thus we may reckon the kinetic energy of a rotating flywheel to be the work the flywheel can do in coming to rest relative to the earth, the potential energy of water behind a dam to be the work the water can do in descending to the level of the river below the dam, the potential energy of a stretched spring to be the work the spring can do in contracting to its normal length, the thermal energy of a hot body to be the work the hot body can do in cooling to the temperature of its surroundings, the chemical energy of a block of TNT to be the work the TNT can do in exploding and expanding to atmospheric pressure.

Energy is measured in the same units as work and, like work, is a scalar quantity. We denote energy by E.

In this book we are concerned only with kinetic and potential energy, the two forms of what is sometimes called mechanical energy in contradistinction to thermal and chemical energy. In the articles immediately following we derive expressions for the kinetic energy of a moving particle and of a rigid body having various types of motion.

150. Kinetic Energy of a Particle. It will now be shown that *the kinetic energy of a moving particle is equal to half the product of the mass of the particle and the square of its velocity.*

Proof. Consider the particle P (Fig. 275), which has, when at A, the speed v and which moves along any path AB under the action of forces F_1, F_2, etc., which bring it to rest at B. The forces acting on the particle have a common point of application, and so the work done by them on the particle is equal to $\int R_t\,ds$, where R_t is the tangential or working component of their resultant (Art. 145). The work done by the particle against the forces that stop it is therefore equal to $-\int R_t\,ds$. But $R_t = ma_t$, where m = mass of the particle and a_t = its tangential acceleration, and so the work done by the particle in coming to rest is

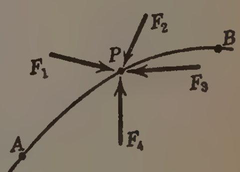

Fig. 275.

$$-\int_{s_1}^{s_2} ma_t\,ds = -\int_{s_1}^{s_2} m\left(\frac{dv}{dt}\right) ds = -\int_{v}^{0} m\left(\frac{ds}{dt}\right) dv = -\int_{v}^{0} mv\,dv = \tfrac{1}{2}mv^2$$

system that possesses this energy. If earth and body are allowed to do work in coming together, however, it is substantially correct to say that all this work is done by the body, since the earth does not move appreciably, and so it is customary to say that the body possesses potential energy equal to its weight times its height above the earth.

† Thermal energy is really molecular kinetic energy, but in this book we generally speak of kinetic energy only with reference to bodies of sensible size or elementary particles of a body that is assumed to be rigid.

and by definition this is the kinetic energy of the particle when at A.

In evaluating the expression for kinetic energy, consistent units must be used; thus, if m is in slugs and v is in feet per second, the kinetic energy will be given in foot-pounds. It should be remembered that kinetic energy is a scalar, not a vector, quantity, and that it is always positive.

151. Kinetic Energy of a Rigid Body. The kinetic energy of any body is equal to the sum of the kinetic energies of its constituent particles. By evaluating this sum, we now derive expressions for the kinetic energy of a rigid body having each of the kinds of motion that have been considered in preceding chapters.

TRANSLATION. In a motion of translation all particles of the body have at any instant equal velocities; hence the sum of the kinetic energies of the elementary particles, each of mass dm, is simply

$$E = \int (\tfrac{1}{2}dm \cdot v^2) = \tfrac{1}{2}mv^2$$

where m = mass of the body.

ROTATION. In motion of rotation the velocity of any particle of the body is equal to $r\omega$, where r denotes the distance of the particle from the axis of rotation and ω the angular velocity in radians per unit time. Therefore the sum of the kinetic energies of the constituent particles is

$$E = \int (\tfrac{1}{2}dm \cdot v^2) = \int (\tfrac{1}{2}dm \cdot r^2\omega^2) = \tfrac{1}{2}\omega^2 \int dmr^2 = \tfrac{1}{2}I\omega^2$$

where I denotes the moment of inertia of the body about the axis of rotation. The kinetic energy is given in foot-pounds if I is in slug-feet-squared and ω in radians per second.

EXAMPLE 1. A bar AB (Fig. 276) is pinned to two other bars AC and BD, which are mounted on axles at C and D so that the whole system can swing in the vertical plane containing the three bars. Each of the bars AC and BD is 6 ft long and weighs 30 lb; AB is 4 ft long and weighs 25 lb; all the bars are slender and uniform. It is required to determine the kinetic energy of the system when it is swinging with an angular velocity of 20 rev/min.

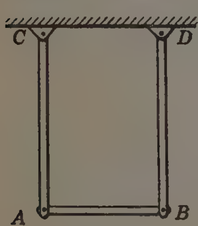

FIG. 276.

Solution. Bar AB has motion of translation; its velocity is equal to $6 \times \omega = 6 \times (20 \times 2\pi/60) = 12.56$ ft/sec. Its kinetic energy is equal to $\frac{1}{2}(25 \div 32.2)12.56^2 = 61.3$ ft-lb.

The other bars have motion of rotation. For each, the moment of inertia about the axis of rotation is $I = \frac{1}{3}mL^2 = \frac{1}{3}(30 \div 32.2)6^2 = 11.19$ slug-ft², and the angular velocity is $\omega = (20 \times 2\pi)/60 = 2.1$ rad/sec. The kinetic energy of each is therefore $\frac{1}{2}I\omega^2 = \frac{1}{2}(11.19)2.1^2 = 24.65$ ft-lb. The total kinetic energy of the system is therefore $61.3 + 2(24.65) = 110.6$ ft-lb.

PLANE MOTION. The velocity of any particle of a body in plane motion can be found either by regarding the motion as a combined translation and rotation

(Art. 119) or by regarding it as a rotation about an instantaneous center (Art. 123). The two methods lead to different but equal expressions for the kinetic energy of a rigid body in plane motion, as will now be shown.

Figure 277 represents any rigid body having plane motion in the plane of the paper. Let the motion be regarded as a translation of the mass-center G with velocity $\bar{V}$ accompanied by a rotation with angular velocity ω about an axis through G normal to the plane of motion. Let x, y, z axes be assumed with origin at G, the x axis being parallel to $\bar{V}$ and the z axis perpendicular to the plane of motion. Then the velocity V of any particle P is the vector sum of $\bar{V}$, the velocity of G, and the velocity of P relative to G, which is equal to $r\omega$ and is directed as shown. Now it is obvious from the figure that

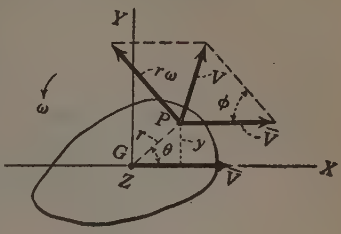

FIG. 277.

$$v^2 = \bar{v}^2 + (r\omega)^2 - 2(\bar{v})(r\omega)\cos\phi = \bar{v}^2 + (r\omega)^2 - 2(\bar{v})(r\omega)\sin\theta$$

The kinetic energy of any elementary particle P is half the product of v^2 and the mass dm of the particle, and the kinetic energy of the body is the summation of such products for all constituent elementary particles, or

$$\int \tfrac{1}{2}dm(\bar{v}^2 + r^2\omega^2 - 2\bar{v}r\omega\sin\theta) = \int \tfrac{1}{2}dm\bar{v}^2 + \int \tfrac{1}{2}dmr^2\omega^2 - \int dm\bar{v}r\omega\sin\theta$$

Now the first of these terms is $\frac{1}{2}\bar{v}^2\int dm = \frac{1}{2}m\bar{v}^2$, where m is the mass of the body. The second term is $\frac{1}{2}\omega^2\int dmr^2 = \frac{1}{2}\bar{I}\omega^2$, where $\bar{I}$ is the moment of inertia of the body about the axis through G normal to the plane of motion. The third term is $\bar{v}\omega\int r\sin\theta\,dm = \bar{v}\omega\int y\,dm$, where y is the distance of the particle from the xz plane. But $\int y\,dm$ represents the moment of the masses of all particles with respect to the xz plane, and, since that plane passes through the mass-center, $\int y\,dm = 0$. The kinetic energy of the body is therefore

$$E = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2$$

The first term of this expression equals the kinetic energy the body would have if its motion were one of translation with velocity equal to $\bar{v}$; the second term is equal to the kinetic energy it would have if its motion were one of rotation about an axis through the mass-center and perpendicular to the plane of motion, with angular velocity equal to the angular velocity the body actually has. Hence the kinetic energy of a body in plane motion may be regarded as consisting of a translational part and a rotational part, each quite independent of the other.

If the motion is regarded as a rotation about an instantaneous center O, the velocity of any particle has the magnitude $r\omega$, where r is the distance of the

given particle from O and ω is the angular velocity of the body. Therefore the kinetic energy of the body is

$$E = \int \tfrac{1}{2} dmr^2\omega^2 = \tfrac{1}{2}\omega^2 \int dmr^2 = \tfrac{1}{2} I_O \omega^2$$

where I_0 denotes the moment of inertia of the body about the instantaneous axis. That this expression for the kinetic energy is equal to the one first derived follows at once from the fact that $I_0 = \bar{I} + m\bar{r}^2$ and $\bar{v} = \bar{r}\omega$, where $\bar{r}$ denotes the distance from the instantaneous center to the mass-center.

EXAMPLE 2. A homogeneous cylindrical disk 6 ft in diameter and weighing 400 lb rolls so that its center has a velocity of 4 ft/sec. It is required to determine the kinetic energy of the disk.

Solution. We solve first regarding the motion as a combined translation and rotation. The moment of inertia of the disk about an axis through its mass-center perpendicular to the plane of motion is $\bar{I} = \frac{1}{2}(400 \div 32.2)3^2 = 55.9$ slug-ft², and its angular velocity is $\omega = v/r = 4/3$ rad/sec. Its kinetic energy is therefore

$$E = \tfrac{1}{2}(400/32.2)4^2 + \tfrac{1}{2}(55.9)(4/3)^2 = 99.4 + 49.7 = 149.1 \text{ ft-lb}$$

We solve next regarding the motion as a rotation about the instantaneous axis. The instantaneous axis O is the line of contact between disk and floor. $I_0 = \bar{I} + mr^2 = 55.9 + (400 \div 32.2)3^2 = 167.7$ slug-ft²; therefore

$$E = \tfrac{1}{2}(167.7)(4/3)^2 = 149.1 \text{ ft-lb}$$

152. Principle of Work and Kinetic Energy for a Particle. In general, in any displacement of a particle the forces acting on it do work; as a result of this work the velocity and kinetic energy of the particle change during the displacement, and it will now be shown that in any displacement of a particle the work done by all the forces acting on it is equal to the increment in the kinetic energy of the particle.

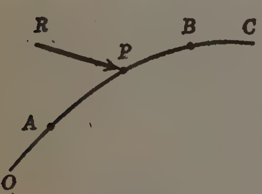

FIG. 278.

Proof. Let P (Fig. 278) be the particle; m = its mass; $OABC$ its path (not a plane curve necessarily) and AB its assumed displacement; v_1 = its velocity at A; v_2 = its velocity at B; R = the resultant of all the forces acting on P, and R_t the tangential component of R. Then the work done by all the forces during an elementary displacement ds is $R_t\,ds$. But $R_t = ma_t = m(dv/dt)$, where a_t is the tangential acceleration of P. Hence the work done on P in the displacement ds is $m(dv/dt)\,ds = m(ds/dt)\,dv = mv\,dv$, and the work done in the total displacement AB is

$$\int_{v_1}^{v_2} mv\,dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

Now $\frac{1}{2}mv_2^2$ is the kinetic energy of the particle when at B, and $\frac{1}{2}mv_1^2$ is its kinetic energy when at A. The difference between these quantities is there-

fore the increment in the kinetic energy of P, and so the equality stated above is proved.

If the total work done on the particle is positive, the increment in the kinetic energy is positive also, and there is a gain in velocity; if the total work done is negative, the increment in kinetic energy is negative also, and there is a loss in velocity.

153. Principle of Work and Kinetic Energy for a Body. In any displacement of any body the work done on each particle is equal to the increment in the kinetic energy of that particle. Therefore the sum of the works done on all particles is equal to the increment in their collective kinetic energy. But the sum of the works done on all particles is equal to the work done on the body by all forces, external and internal, and the increment in the collective kinetic energy of the particles is equal to the increment in the kinetic energy of the body, and so *in any displacement of any body the total work done upon it by all the external and internal forces acting equals the increment in the kinetic energy of the body.*

In any displacement of a rigid body the work done by the internal forces collectively is zero (Art. 145), and so *in any displacement of a rigid body the work done by the external forces is equal to the increment in kinetic energy.*

From the relations stated above it follows that the rate at which work is done upon a body equals the rate at which it gains kinetic energy. But the rate at which work is done is power, and so the combined power of all the forces doing work upon a body at any instant is equal to the rate at which the body is gaining kinetic energy at that instant.

154. Typical Problems; Examples. The principles of work and kinetic energy are especially well adapted for ascertaining the change in the velocity of a body that takes place during any displacement for which it is possible to calculate the work done on the body. By their application it is also possible to ascertain something concerning the forces and displacements that accompany any given change in the kinetic energy of a body.

EXAMPLE 1. A block weighing 50 lb, initially at rest, is pushed along a horizontal floor by a constant horizontal force P of 25 lb, which ceases to act when the block has moved 30 ft. The coefficient of friction between block and floor is 0.2. It is required to determine (*a*) the velocity of the block at the instant P ceases to act, and (*b*) how much farther the block slides before its velocity is reduced to 10 ft/sec by the frictional resistance.

Solution. (*a*) The forces acting on the block while the displacement is occurring are its own weight, the 25-lb push, the normal component of the floor reaction $N = 50$ lb, and the friction component of the floor reaction $0.2N = 10$ lb. The weight and the normal component of the floor reaction do no work; the work done by the applied push is equal to $25 \times 30 = 750$ ft-lb (positive); the work done by the friction component of the floor reaction is equal to $10 \times 30 = 300$ ft-lb (negative). Since the body is rigid the internal forces do no work, and so the total work done is $750 - 300 = 450$ ft-lb. This is equal to the increment in the kinetic energy of the block, or

$$450 = \frac{1}{2}\,\frac{50}{32.2}\,(v^2 - 0) \qquad \text{whence } v = 24.1 \text{ ft/sec}$$

To determine how far the block would slide before being slowed down to 10 ft/sec, we set the total work done during the entire displacement equal to the increment in the kinetic energy for the entire displacement. Let x = the additional distance the block slides; while the block undergoes the displacement x the only force that does work on it is friction, and this work is equal to $-10x$. Therefore, for the entire displacement the total work done is $450 - 10x$; the increment in kinetic energy is $\frac{1}{2}(50 \div 32.2)(10^2 - 0)$; and equating and solving for x gives $x = 37.2$ ft.

This problem can be readily solved by means of the equation $\Sigma F_x = ma_x$. Thus for the first stage of the motion $\Sigma F = 25 - 10 = (50 \div 32.2)a$, whence $a = 9.66$ ft/sec². Since both v and s are zero when $t = 0$, $s = \frac{1}{2}at^2$, and so, when the block has gone 30 ft, $t = 2.49$ sec and $v = 9.66 \times 2.49 = 24.1$ ft/sec. You should solve part (b) of the problem by this method and compare with the solution arrived at by work and energy. You should also solve part (b) by equating the work done during the second stage of the motion only to the corresponding increment in kinetic energy.

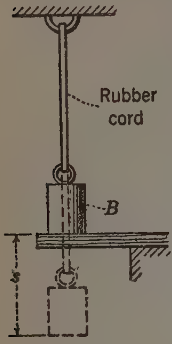

FIG. 279.

EXAMPLE 2. A rubber cord (or other elastic body) is attached at one end to a fixed support and at the other to a body B of weight W (Fig. 279). The cord is of such a nature that its stretch is proportional to the tension in it and is e when B hangs freely and at rest. It is required to determine the tension caused in the cord by suddenly removing the support under B and allowing it to fall until stopped by the tension in the cord.

Solution. Let s = the distance dropped through by B at any instant; then s is also the stretch in the cord, and the tension in the cord is $T = (W/e)s$. The forces that do work on B during its descent are T and its own weight W. The total amount of work done on B up to the time it is brought to rest is zero (increment in kinetic energy is zero); therefore, if x denotes the total stretch of the cord,

$$W_k = +Wx - \int_0^x \frac{W}{e} s\,ds = 0 \qquad \text{whence } Wx = \frac{1}{2}\frac{W}{e}x^2 \quad \text{or} \quad x = 2e$$

That is, the stretch of the cord when fully extended is twice the stretch when the load B hangs freely and at rest. Hence the tension in the cord when fully extended is $2W$. Thus, it is seen that the effect of applying the load suddenly (but without initial velocity) is to cause twice the stress in the supporting member that would exist under ordinary static conditions. (This problem, like Ex. 1, can be solved by means of the equation $\Sigma F_x = ma_x$, but not so easily. Try it.)

EXAMPLE 3. A slender uniform rod of length L is pinned at one end to a horizontal floor. The rod is raised to a vertical position and then released and allowed to fall over. It is required to determine the speed with which the free end will be moving when it strikes the floor.

Solution. Neglecting air resistance and friction on the pin, the only work done on the rod as it falls is that done by gravity, equal to $\frac{1}{2}WL$ (Art. 145). The bar has motion of rotation; therefore its kinetic energy when it strikes the floor is

$$E = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\left(\frac{1}{12}\frac{W}{g}L^2 + \frac{W}{g}\frac{1}{4}L^2\right)\omega^2 = \frac{1}{6}\frac{W}{g}L^2\omega^2$$

But $\omega = v/L$, where v = speed of the end of the rod, and so

$$\tfrac{1}{2}WL = \frac{1}{6}\frac{W}{g}v^2 \qquad \text{whence } v = \sqrt{3gL}$$

EXAMPLE 4. A (Fig. 280) is a sheave supported on a rough horizontal shaft. A is 3 ft in diameter, and its radius of gyration with respect to the axis of rotation = 9 in. The weights of A, B, and C are 100, 200, and 300 lb, respectively. The axle friction is equivalent to a couple of 10 in.-lb. The system is released and allowed to move under the influence of gravity and the resistances brought into action. It is required to determine the velocity of the suspended weights when they have moved through 10 ft.

Solution. The system moves under the action of the following forces: gravity, axle reaction, air resistance, and the internal reactions between sheave and rope and in the fibers of the rope. If the rope is quite flexible, the forces in the rope do little work; this will be neglected. If the rope does not slip on the sheave, no work is done by the reaction between rope and sheave. Thus, little or no work is done by the internal forces. The work done by air resistance is small unless the speeds of the moving bodies become high; it will be neglected. The work done by the axle friction is equal to the constant resisting couple times the angular displacement, or $-10 \times (10 \div 1.5) = -66.7$ in.-lb. or -5.55 ft-lb (Art. 145). Gravity does no work on A; on B and C its work $= (300 \times 10) - (200 \times 10) = 1000$ ft-lb; its work on the rope is negligible. Hence, the total work done on the system $= 1000 - 5.55 = 994.45$ ft-lb. Now let v = the required velocity in feet per second; then the angular velocity of the wheel $= v \div 1.5 = 0.667v$ rad/sec. The kinetic energy of the system is

FIG. 280.

$$E = \frac{1}{2}\frac{300}{32.2}v^2 + \frac{1}{2}\frac{200}{32.2}v^2 + \tfrac{1}{2}I(0.667v)^2$$

where I = moment of inertia of the sheave. Now $I = (100 \div 32.2) \times (9 \div 12)^2 = 1.75$ slug-ft²; hence $E = 8.16v^2$ ft-lb. Therefore the work-energy equation is

$$994.45 = 8.16v^2 \qquad \text{whence } v = 11 \text{ ft/sec}$$

EXAMPLE 5. As explained in Art. 148, the thrust horsepower means the rate at which work is done on the airplane by the thrust or pull of the propeller. (*a*) Suppose that a plane weighing 2000 lb is flying level and that at a certain instant the velocity is 150 ft/sec, the total air resistance or drag is 220 lb, and the thrust horsepower is 110. It is required to determine the acceleration of the plane at the instant in question. (*b*) Suppose the conditions to be as described in (*a*) except that instead of flying level the plane is climbing, the flight path being inclined at 3° to the horizontal. It is required to determine the acceleration under these conditions.

Solution. (*a*) The rate at which work is done on the plane is equal to the rate at which its kinetic energy is changing, or

$$P = \frac{d(\frac{1}{2}mv^2)}{dt} = mv\frac{dv}{dt} = mv \cdot a \qquad \text{or } a = \frac{P}{mv}$$

The total power is the sum of the positive power of the propeller thrust and the negative power of the drag, or $(110 \times 550) - (220 \times 150) = 27{,}500$ ft-lb/sec, and so $a = 27{,}500 \div (2000 \div 32.2)(150) = 2.95$ ft/sec².

(*b*) Gravity as well as drag is doing negative work on the plane. The center of gravity is rising at the rate of $150 \sin 3° = 7.84$ ft/sec, and so gravity is doing work at the rate of $-7.84 \times 2000 = -15{,}680$ ft-lb/sec. The total power is therefore $27{,}500 - 15{,}680 = 11{,}820$ ft-lb/sec, and the acceleration is $a = 11{,}820 \div [(2000 \div 32.2)(150)] = 1.27$ ft/sec².

(Note that this problem could be solved readily by determining the propeller thrust T from the relation: $P = Tv$, and then finding the acceleration in the usual way by means of the equation $\Sigma F_x = ma_x$.)

NOTE ON ROLLING RESISTANCE. In Art. 70 rolling resistance was defined from the viewpoint of statics; we now discuss it briefly from the viewpoint of dynamics, with especial reference to the work required to overcome it.

If, by means of a force applied at its mass-center, a wheel is made to roll with uniform velocity on a yielding and nonelastic surface like a dirt road, the reaction has a component that opposes the motion, and this component is called the rolling resistance. Figure 281, which is a vertical section through the wheel and the rut it makes in the road, represents the assumed conditions. The forces on the wheel are the propelling force F, the combined weight of wheel and load W, and the road reaction R, which must be concurrent with F and W since the three forces are in equilibrium. The horizontal component of R is the rolling resistance, and it is obviously equal to F. Now R is made up of elementary forces dR, distributed over that part of the wheel in contact with the road. Since the wheel is rotating about the instantaneous center O and pressing back the dirt in front of it, any such point as P is moving in a direction normal to the radius OP, and the force dR at P opposes that motion. Therefore each elementary force dR is doing negative work at any instant, and so the total reaction R does negative work. Since the wheel neither gains nor loses kinetic energy, the negative work done by R is of the same magnitude as the positive work done by F, and so is equal to the rolling resistance times the distance the center of the wheel moves.

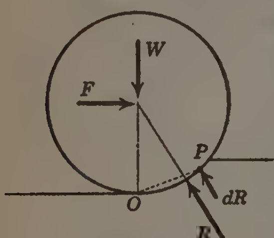

FIG. 281.

If the road is yielding but perfectly elastic, there is contact at the rear as well as at the front of the wheel, and for every rearward-acting dR in front of O there is a forward-acting dR behind O; therefore there is no rolling resistance, and no force F is required to keep the wheel rolling at constant velocity once it is in motion. Under these circumstances no net work is being done on the wheel; the negative work done by each rearward dR is canceled by the positive work done by the corresponding forward dR. A similar analysis shows that, when an elastic wheel (e.g., ideal pneumatic tire) rolls on either a rigid or perfectly elastic pavement, there is no rolling resistance. For an actual tire and pavement there is a small resistance, because, owing to imperfect elasticity, the forward dR's are slightly less than the rearward dR's.

If the wheel and road are both rigid (which is the assumption made in Ex. 1 of Art. 122), then the only point of contact is O; R acts vertically, and there is no rolling resistance. If the wheel is at rest and the contact surfaces are smooth, then on the application of a propelling force F the wheel slides without rolling, because there is no force that has a moment about its mass-center. If the contact surfaces are rough, the wheel rolls when F is applied, because the friction induced at O has a moment about its mass-center (see the

example referred to above). This friction is a resistance in the sense that it acts oppositely to the direction of motion and reduces the acceleration of the mass-center below what F alone would produce, but it is not called rolling resistance and is not present when the wheel rolls with uniform speed. So long as F acts, the friction will act and produce an angular acceleration, and the center of the wheel will have an acceleration due to F and the friction jointly. If F ceases to act, the friction also ceases to act and the wheel continues to roll with uniform speed under the action of W and the vertical reaction only. The friction force does no work at any time, since O, its point of application, is at no time moving.

EXAMPLE 6. A certain pair of car wheels and their axle weigh 2000 lb. Their diameter is 33 in., and the radius of gyration of wheels and axle is 9 in. They are rolled along a level track until their speed is 60 rev/min and are then left under the influence of the rolling resistance of the track, coming to rest after rolling a distance of 1000 ft. (Data not from an actual experiment.) It is required to determine the average rolling resistance, air resistance being assumed negligible.

Solution. When released, the wheels have an angular velocity of 1 rev/sec = 6.28 rad/sec, and the linear velocity of their centers = $33\pi \div 12 = 8.64$ ft/sec. They have plane motion; hence their kinetic energy is

$$E = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 = \frac{1}{2}\,\frac{2000}{32.2} \times 8.64^2 + \frac{1}{2}\,\frac{2000}{32.2}\left(\tfrac{9}{12}\right)^2 \times 6.28^2 = 3010 \text{ ft-lb}$$

The increment in kinetic energy is -3010 ft-lb, and this is equal to the work done by the rolling resistance (air resistance neglected). Hence, the rolling resistance is equal to $3010 \div 1000 = 3$ lb.

155. Conservation of Energy. The law of conservation of energy states that the total amount of energy in the universe is constant. But, although energy is indestructible, it can be transformed from one kind to another. Thus if a body is dropped from a height in a vacuum its potential energy is progressively converted into kinetic energy as it gains speed in falling. If a bullet is fired into a box of sand it is stopped after penetrating a few inches, and its kinetic energy is converted into thermal energy, as attested by the fact that the sand near the place of impact will be found hot to the touch. Such a conversion or transfer does not represent a real loss of energy, but it does generally represent a change to a less available form, and so it is not uncommon to speak of energy as being "lost" or "dissipated" in heat. Energy expended in overcoming friction, or in agitating a viscous fluid, or in producing plastic deformation (as when a block of lead is flattened by hammering) is thus dissipated in heat.

CONSERVATIVE FORCES. A system of bodies the external and internal forces on which depend only upon the configuration of the system, and not upon its motion, is called a *conservative system*, and the forces are called *conservative forces*. A conservative force is always the same for the same configuration; thus the tension in a perfectly elastic spring is always the same for a given amount of stretch, regardless of whether the spring is shortening or elongating.

A nonconservative force depends upon the motion of the body on which it acts, and is not always the same for the same configuration; thus friction always acts to oppose slipping, and reverses direction when the direction of slipping is reversed. The force of gravity, electromagnetic attraction or repulsion, and the stress in an elastic body are examples of conservative forces. Friction, the stress in a plastic body, and the resistance of a viscous fluid are examples of nonconservative forces.

In any change in the configuration of a conservative system the work done by the forces acting depends upon the relative displacement of their points of application, and not upon the actual paths traversed by those points. Thus the work done in separating the earth and a body (by raising the body) depends only upon the increase in distance between the centers of the body and the earth; the work done in slowly stretching a spring depends only upon the amount by which the ends of the spring approach or recede from each other. In any change from one configuration to another of a conservative system the work done is equal and opposite in sign to the work done in returning from the second to the first configuration. All positive work done in any such change represents a conversion of potential into kinetic energy, and all negative work a conversion of kinetic into potential energy. For a conservative system, then, the law of conservation of energy may be stated thus: *the sum of the potential and kinetic energies is constant.* This is true also for a system some of the forces on which are nonconservative provided that these nonconservative forces have fixed points of application and so do no work during any change in configuration of the system.

This relationship can be used to advantage in the solution of certain types of problems, as illustrated in the following example.

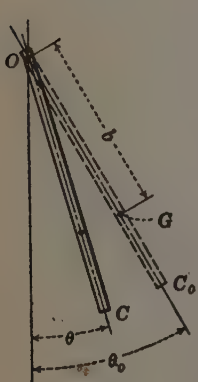

FIG. 282.

EXAMPLE 1. It is required to derive a formula for the period T of a compound pendulum (Art. 116), using the principle of conservation of energy.

Solution. Let OC (Fig. 282) represent the pendulum; O is the point of suspension and OC_0 represents the highest position of the pendulum. The distance from O to the center of gravity of the pendulum G is b. The forces on the pendulum as it swings (disregarding air resistance and friction at the support) are its weight W and the reaction R of the support. W is a conservative force; R is not, but since its point of application does not move it does no work, and so the sum of the potential and kinetic energies of the pendulum is constant.

Taking as datum the lowest level of G, the pendulum has when at the highest position OC_0 potential energy equal to $W(b - b\cos\theta_0)$ and kinetic energy equal to zero, and when at an intermediate position OC it has potential energy equal to $W(b - b\cos\theta)$ and kinetic energy equal to $\frac{1}{2}I\omega^2 = \frac{1}{2}I(d\theta/dt)^2$. Equating the total energies for the two positions gives $Wb(1 - \cos\theta_0) = Wb(1 - \cos\theta) + \frac{1}{2}I(d\theta/dt)^2$, or $Wb(\cos\theta - \cos\theta_0) = \frac{1}{2}I(d\theta/dt)^2$. Now for small values of θ, $\cos\theta$ is given with sufficient accuracy by the first two terms of the cosine series — that is, $\cos\theta = 1 - \frac{1}{2}\theta^2$.

Making this substitution the above equation becomes

$$Wb(\theta_0^2 - \theta^2) = I\left(\frac{d\theta}{dt}\right)^2 \quad \text{or} \quad 1 - \left(\frac{\theta}{\theta_0}\right)^2 = \frac{I}{Wb}\left[\frac{d(\theta/\theta_0)}{dt}\right]^2$$

For convenience, x is substituted for θ/θ_0, giving

$$1 - x^2 = \frac{I}{Wb}\left(\frac{dx}{dt}\right)^2 \quad \text{whence} \quad \frac{dx}{dt}\sqrt{\frac{I}{Wb}} = \sqrt{1 - x^2} \quad \text{or } dt = \sqrt{\frac{I}{Wb}}\frac{dx}{\sqrt{1 - x^2}}$$

Integrating gives $t = \sqrt{I/Wb}$ arc sin $x + C$. Let t be dated from the instant the pendulum is in the lowest position; then, when $t = 0$, $x = 0$, and so $C = 0$. When the pendulum has reached its highest position, $t = \frac{1}{4}T$ (one-fourth of a period), $x = 1$, and arc sin $x = \frac{1}{2}\pi$. Therefore

$$\tfrac{1}{4}T = \tfrac{1}{2}\pi\sqrt{\frac{I}{Wb}} \qquad \text{whence } T = 2\pi\sqrt{\frac{I}{Wb}} = 2\pi\sqrt{\frac{k^2}{gb}}$$

You should compare this solution with the one carried out in Art. 116 where T was found by using the equation of rotation.

POTENTIAL ENERGY AND EQUILIBRIUM. In Art. 10 it is stated that a body at rest is in equilibrium. For some purposes it is essential to distinguish three kinds or types of equilibrium, the distinction being made on the basis of what the body would do if displaced slightly from the position of equilibrium. If on being thus displaced the body would return to its original position, it is said to be in *stable equilibrium;* if it would move so as to increase the displacement, it is said to be in *unstable equilibrium;* if it would remain in the new position, it is said to be in *neutral* or *indifferent equilibrium.* The three cases are illustrated in Fig. 283, which represents a uniform bar in a vertical plane supported by a frictionless pin on which it can turn freely. In (a) the center of gravity is below the pin and the bar is in stable equilibrium; if turned through a small angle and released it would return to the vertical position. In (b) the center of gravity is above the pin and the bar is in unstable equilibrium; if turned slightly and released it would swing on over. In (c) the center of gravity is at the pin and the bar is in neutral equilibrium; it would remain in any position to which it was turned. Another example is afforded by the bar-spring combination shown in Fig. 284. When the springs are stretched as in (a) the system is in stable equilibrium; when the springs are compressed as in (b) it is in unstable equilibrium.

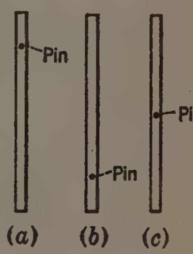

FIG. 283.

FIG. 284.

It is possible for a body or system of bodies to be in different kinds of equilibrium with respect to different kinds of displacement. For example, a sphere resting in a horizontal semicircular trough is in stable equilibrium with respect to transverse displacement and in neutral equilibrium with respect to

longitudinal displacement. It is important to note that in thus judging or ascertaining the kind of equilibrium that obtains in any case the supposed displacement must be of a nature that would be permitted by the supports or constraints; thus in Fig. 283 we considered only a slight rotation of the bar about the axis of the pin, and not a vertical or horizontal translation, which the supporting pin would not permit.

It is shown in Art. 146 that if a body acted on by balanced forces suffers an infinitesimal displacement the balanced forces collectively do no work, but that if the displacement is finite the forces may cease to be balanced, in which case they would do work during the displacement. If the originally balanced forces do negative work during the displacement they are resistances, opposing the displacement and tending to restore the body to its original position. If they do positive work they are efforts, favoring the displacement and tending to increase it. If they do no work they neither oppose nor favor the displacement. Therefore a body is in stable, unstable, or neutral equilibrium according to whether, in any small finite displacement, the original forces do negative, positive, or zero work.

Suppose that a conservative system (of bodies) is in equilibrium, and is supported or constrained so that certain displacements are possible. Suppose this system to be slightly displaced by a disturbing force, the displacement taking place with infinite slowness, so that no kinetic energy is acquired. The work then done by the disturbing force is equal to that done by the original forces, but opposite in sign, since the total work is zero. If the system was originally in stable equilibrium, the disturbing force does positive work, and, since no kinetic energy is imparted, the potential energy of the system must increase. But if the potential energy of the system is increased by any displacement, then the original configuration of the system must have been such as to make its potential energy a minimum — that is, less than for any other possible configuration. Similarly, if a system is in unstable equilibrium, its configuration must be one of maximum potential energy, and, to be in indifferent equilibrium, the system must have the same potential energy for all possible configurations. You should test this criterion for type of equilibrium by applying it to the bar of Fig. 283 and the system of Fig. 284. Another illustration is afforded by the spring shown in Fig. 285. In (*a*) the spring is compressed and inserted between two rigid surfaces whose distance apart is less than the normal length of the spring. The spring is here in unstable equilibrium; if bent the least bit it will "kick out" and assume the position shown in (*b*). In this configuration it is much less severely stressed and has much less potential energy (strain energy, as potential energy due to internal stress is often called) than in the first. Furthermore, of all possible forms of curvature, the form assumed will be that corresponding to minimum strain energy. Thus the spring would have more strain energy, and be unstable, if bent in the form of an S.

The fact that minimum potential energy is a criterion for stable equilibrium

is helpful in solving many problems, but most problems in which it is of particular advantage are of a type beyond the scope of this book. The following example serves to illustrate its use, although it could be solved equally well by the equations of statics.

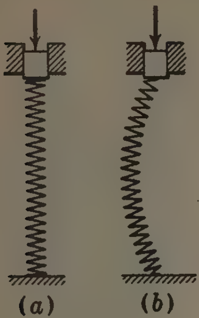

FIG. 285.

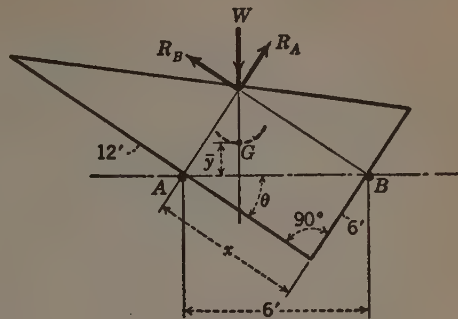

FIG. 286.

EXAMPLE 2. Figure 286 represents a uniform homogeneous triangular plate of weight W which is to be supported on edge by the two smooth pegs A and B, these pegs being on the same level and 6 ft apart. It is required to determine the position of stable equilibrium for the plate.

Solution. The level of the pegs is taken as datum; $\bar{y}$, the elevation above datum of the center of gravity G, is expressed in terms of the distance x, and the value of x to make $\bar{y}$ minimum is then found. Since the potential energy is equal to $W\bar{y}$, this value of x gives minimum potential energy and corresponds to the position of stable equilibrium.

It is apparent from the figure that $\bar{y} = 2 \cos \theta - (x - 4) \sin \theta$, that $\cos \theta = x/6$, and that $\sin \theta = \sqrt{6^2 - x^2}/6$. Therefore $\bar{y} = \frac{1}{3}x - \frac{1}{6}x\sqrt{6^2 - x^2} + \frac{2}{3}\sqrt{6^2 - x^2}$, and when $d\bar{y}/dx$ is set equal to zero the simplified expression takes the form $\frac{1}{9}x^4 - \frac{4}{9}x^3 - 3\frac{4}{9}x^2 + 8x + 32 = 0$. Solving for x gives $x = 4.95$. Inspection shows that for $x = 4.95$ the value of $\bar{y}$ is minimum, not maximum.

The correctness of this solution can be tested by drawing the plate to scale in the position defined by $x = 4.95$ and ascertaining whether normals through A and B, representing the lines of action of the reactions R_A and R_B, intersect on the line of action of W, as they must if the three forces on the plate are in equilibrium. This construction is indicated in the figure.

The plate would be in neutral equilibrium if supported on either of its longer edges with G between A and B, but for the dimensions given the position shown is the only one of stable equilibrium. If the pins were put closer together, it would appear that the plate might be stable with one of the other corners down. See if you can devise a criterion for telling whether, for a given span AB, either such position would be stable.

Problems of this type, in which it is required to determine the position assumed by a body supported in a certain way, usually involve considerable tedious algebraic work if a mathematical solution is to be made. It is often easier to solve graphically. Thus this example could be solved as follows: Cut from cardboard a triangle of the specified proportions and mark its center of gravity. To the same scale lay off points A and B on a line drawn on a sheet of paper. Then move the triangle to various positions, keeping the lower edges on points A and B, and for each position punch through G with a pin, thus marking out on the paper the path followed by G as the triangle slides around on the pins. It is then easy to locate the lowest point of this path and ascertain the corresponding distance x. The path of G for the triangle described is indicated in Fig. 286.

CHAPTER XIV

MOMENTUM AND IMPULSE

156. Linear Momentum of a Particle. By **linear momentum** of a moving particle is meant the product of its mass and its velocity; it is a vector quantity, and its direction is the same as that of the velocity. Unlike a velocity and like a force, the linear momentum of a particle is a localized vector quantity; it lies in a line that passes through the particle. For brevity, we shall speak of the linear momentum of a particle as a momentum, and of several such quantities as momentums.

We denote a momentum by U and its magnitude by u, following the same convention as in denoting velocity and acceleration. The magnitude can be expressed in any units of mass and velocity, as slug-feet per second (slug-ft/sec); no single word has been generally adopted for any unit of momentum.

Being a vector quantity, a momentum can be resolved into components, and several momentums can be compounded to obtain their resultant, just as is done with forces. The components most often useful are those parallel to rectangular axes x, y, z. The component of a momentum U parallel to any axis x inclined at an angle θ to U is

$$u_x = u \cos \theta = mv \cos \theta = mv_x$$

and so the momentum of a particle parallel to any line is the product of the mass of the particle and the component of its velocity parallel to that line.

157. Linear Momentum of a Body. The component, parallel to any axis x, of the linear momentum of a body is the sum of the x components of the momentums of all the particles of the body. It will now be shown that for any body, rigid or nonrigid, having any kind of motion, this is equal to the product of the mass of the body and the x component of the velocity of its mass-center, or

$$u_x = m\bar{v}_x$$

Let u_x', u_x'', etc., be the x components of the momentums of the particles, dm', dm'', etc., be their masses, and v_x', v_x'' the x components of their velocities. Then by definition

$$u_x = u_x' + u_x'' + \cdots = dm' \cdot v_x' + dm'' \cdot v_x'' + \cdots$$

Now, if at any instant x', x'', etc., = the x coordinates of the moving particles and $\bar{x}$ the x coordinate of the mass-center, then

$$dm' \cdot x' + dm'' \cdot x'' + \cdots = m\bar{x} \quad \text{(Art. 87)}$$

Differentiating with respect to t gives

$$dm' \cdot \frac{dx'}{dt} + dm'' \cdot \frac{dx''}{dt} + \cdots = m \frac{d\bar{x}}{dt}$$

or

$$dm' \cdot v_x' + dm'' \cdot v_x'' + \cdots = m\bar{v}_x$$

as stated.*

The total or resultant linear momentum of a body can be found from the components of its momentum parallel to three rectangular axes x, y, z by the relationship

$$u = (u_x^2 + u_y^2 + u_z^2)^{\frac{1}{2}} = [(m\bar{v}_x)^2 + (m\bar{v}_y)^2 + (m\bar{v}_z)^2]^{\frac{1}{2}} = m\bar{v}$$

Therefore the total linear momentum of a body is equal to the product of the mass of the body and the velocity of its mass-center. Like the linear momentum of a particle, it is a localized vector quantity; its direction is the same as that of $\bar{v}$, but in general it does not lie in a line passing through the mass-center. Its position, when required, can be found as explained in Art. 164.

EXAMPLE 1. A vertical column 30 ft long, uniform in section and weighing 40 tons, falls over, rotating about an axis through its base. When it reaches a position inclined at 30° to the vertical it has an angular velocity of 0.66 rad/sec. What is the horizontal component of its linear momentum at that instant?

Solution. The mass-center (halfway between the ends) has a velocity $v = 15 \times 0.66 = 9.9$ ft/sec, directed down at an angle of 30° to the horizontal. The x component of this velocity is $v_x = 9.9 \times 0.866 = 8.58$, and the x component of the momentum is $u_x = 80000 \div 32.2) \times 8.58 = 21300$ slug-ft/sec.

NONRIGID BODIES. The relationship $u_x = m\bar{v}_x$ provides perhaps the easiest method for determining any desired component of the momentum of a rigid body, but for a nonrigid body or system of bodies it is usually easiest to consider only those parts which move so as to have momentum in the direction in question, and to sum up the momentum components of those parts.

EXAMPLE 2. A 10-in. horizontal water main runs north, then east for 50 ft, then south. Water runs through it north-east-south with a uniform velocity of 8 ft/sec. It is required to determine how much eastward momentum the flowing water has at any instant.

Solution. Obviously only the water in the east-west section of the pipe need be considered. The mass of this water is $50 \times \pi \times (5 \div 12)^2 \times 62.4 \div 32.2 = 52.9$ slugs, and its momentum is $52.9 \times 8 = 423$ slug-ft/sec.

158. Rate of Change of Linear Momentum. Since $u_x = m\bar{v}_x$, differentiation with respect to t gives $du_x/dt = m(d\bar{v}_x/dt) = m\bar{a}_x$; that is, the rate of change of the component of the linear momentum of a body parallel to any line is equal to the product of the mass of the body and the acceleration, parallel to that line, of its mass-center.

* This equation was also obtained in Art. 106 as a step in the proof of the principle of motion of the mass-center.

This relationship might seem to imply that to find the rate of change of momentum of a body it is necessary to determine the acceleration of its mass-center, but this is not always so. Often, especially for a nonrigid body or system, the rate of change of momentum can be found by considering only those parts of the body or system whose momentum, in the direction in question, changes. This momentum can be expressed as a function of time, and its rate of change can be determined by ordinary differentiation. If the rate of change is constant, it can often be found most easily by calculating the change in momentum that takes place in each unit of time. Both methods are illustrated in the examples of the following article.

159. Force-Momentum Relationship. In the preceding article it was shown that, for any body, $du_x/dt = m\bar{a}_x$. But, according to the principle of motion of the mass-center (Art. 106), $\Sigma F_x = m\bar{a}_x$. Therefore $\Sigma F_x = du_x/dt$; that is, *the component parallel to any line of the external system of forces acting on any body is equal to the rate at which the component of the linear momentum of the body parallel to that line is changing.*

It follows from the statement just made that, if the system of external forces acting on a body has no component in any direction, or if there are no external forces, the linear momentum of the body remains constant, and, no matter how parts of the body may move, the mass-center either moves with constant speed along a straight line or remains at rest. This is known as the **principle of conservation of linear momentum.** This principle is implied by the principle of motion of the mass-center, which shows that, if the external force system that acts on a body has no component in any direction, the mass-center of the body has no acceleration. Indeed, the relationship between force and linear momentum stated above is essentially an alternative form of the principle of motion of the mass-center. But for some problems, especially those involving the motion of nonrigid bodies or the flow of liquids or gases, the force-momentum relationship affords the more convenient means of solution.

EXAMPLE 1. It is required to determine the propelling force on the car of Ex. 2, Art. 107, by means of the force-momentum relationship. The car weighs 1000 lb.

Solution. At any instant the total horizontal momentum of the system is equal to the product of the mass of the car and contained gravel and the velocity, or $u = [(1000 + 500t) \div 32.2] \times 15$ slug-ft/sec, where t is the time in seconds after dumping commences. Therefore $P = du/dt = (500 \div 32.2) \times 15 = 232.5$ lb. (On comparison, you will see that this solution is simpler than that made in Art. 107. Note that the weight of the car makes no difference.)

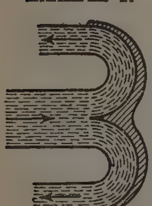

FIG. 287.

EXAMPLE 2. Figure 287 represents a cross section of a stationary Pelton water-wheel bucket against which a jet of water impinges. Because of the curved form of the bucket the jet is split, turned back, and made to flow in the opposite direction with undiminished speed.* Show that if the flow of all the

* This assumption is here made in order to simplify the problem. Actually the direction of the jet is not reversed; it is deflected only about 170° instead of 180°, so as not to strike the following bucket, and some velocity is necessarily lost because of friction.

water is thus reversed the jet exerts a steady force on the bucket equal to $0.02114d^2v^2$ lb, where v is the velocity of flow in feet per second and d is the diameter of the jet in inches.

Solution. The forward-moving water has momentum to the right (as viewed in the figure), and the rearward-moving water has momentum to the left. In each second all the water that strikes the bucket loses its momentum toward the right, and this loss represents an increment of momentum toward the left equal to the mass of water striking per second times its velocity, or $\Delta u = [\frac{1}{4}\pi(d \div 12)^2(v)(62.4 \div 32.2)] \times v = 0.01057d^2v^2$ slug-ft/sec toward the left. But this water also gains an equal amount of momentum toward the left, and this gain constitutes another increment of momentum toward the left equal to $0.01057d^2v^2$. The total increment of momentum each second is therefore $0.02114d^2v^2$, and, since the condition is one of steady flow, the (constant) rate of change of linear momentum is $0.02114d^2v^2$ slug-ft/sec² toward the left. The force exerted by the bucket on the jet is equal to this and in the same direction; the force exerted by the jet on the bucket is therefore $0.02114d^2v^2$ lb toward the right.

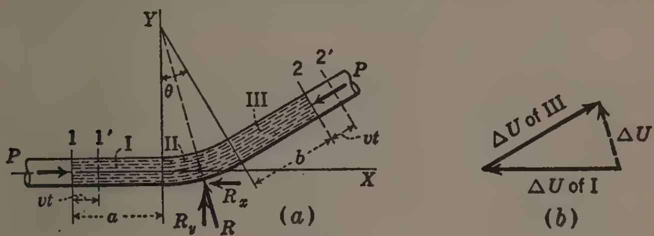

FIG. 288.

EXAMPLE 3. Figure 288*a* represents a top view of a horizontal water pipe of uniform cross-sectional area A, which is bent through an angle θ. Liquid of density (mass per unit volume) ρ flows through the pipe from left to right at a uniform speed v. It may be assumed that the speed of the water is the same at all points of any cross section in the straight portions of the pipe, and that there is no frictional resistance, so that the total pressures P, P on sections 1 and 2 are equal. It is required to determine the resultant force exerted by the water on the pipe at the bend.

Solution 1. Consider the column of water between sections 1 and 2 as made up of three parts: that in the straight part of the pipe I, that in the curved part II, and that in the straight part III. Disregarding the weight of the water, the only forces acting on the water column between sections 1 and 2 are the equal pressures P, P and the resultant force from the pipe R, with components R_x, R_y parallel to the x and y axes shown. In order to find R_x and R_y we express the x and y components of the momentum of the water column as functions of time, and then differentiate to get the rates at which they change.

At time t after the water column is in the position shown, end 1 will have moved forward a distance vt to 1′ and end 2 will have moved an equal distance to 2′; the length of I will therefore have become $a - vt$, and the length of III will have become $b + vt$. The x momentum of the column is then

$$u_x = (u_x \text{ of I}) + (u_x \text{ of II}) + (u_x \text{ of III}) = (a - vt)A\rho v + (u_x \text{ of II}) + (b + vt)A\rho v \cos\theta$$

and the y momentum of the column is

$$u_y = (u_y \text{ of I}) + (u_y \text{ of II}) + (u_y \text{ of III}) = 0 + (u_y \text{ of II}) + (b + vt)A\rho v \sin\theta$$

Now v is constant, and also u_x and u_y of II are constant, since part II of the pipe is always full of water flowing at the same rate. Therefore, differentiating u_x and u_y with respect to t and

setting the rates of change of momentum thus obtained equal to the summation of forces in the corresponding direction, we have

$$\Sigma F_x = P - P\cos\theta - R_x = -v^2A\rho + v^2A\rho\cos\theta$$

whence

$$R_x = (P + v^2A\rho)(1 - \cos\theta)$$

$$\Sigma F_y = R_y - P\sin\theta = v^2A\rho\sin\theta$$

whence

$$R_y = (P + v^2A\rho)\sin\theta$$

From these components R is readily found to be equal to $2(P + v^2A\rho)\sin\frac{1}{2}\theta$ and to act at an angle $\frac{1}{2}\theta$ with the y axis; that is, it bisects the angle subtended by the bend in the pipe. Obviously that part of R due to the flow of the water is $2v^2A\rho\sin\frac{1}{2}\theta$, and that part due to the pressures (P, P) is $2P\sin\frac{1}{2}\theta$. The force exerted by the water on the pipe is equal and opposite to R.

Solution 2. As before, we consider the water in the pipe as made up of parts I, II, and III. Each second the mass of part I decreases an amount $A\rho v$; hence there is a loss in the momentum of this part equal to $A\rho v^2$, and this loss constitutes an increment of momentum ΔU_{I} directed opposite to the flow. Also each second the mass of part III increases an amount $A\rho v$; hence there is a gain in the momentum of this part equal to $A\rho v^2$, and this gain constitutes an increment of momentum ΔU_{III} directed like the flow. There is no change in the volume of part II, and hence no increment of momentum for that part. The total increment of momentum ΔU is the vector sum of ΔU of I and ΔU of III, as shown in the vector diagram, Fig. 288*b*. Obviously $\Delta U = 2A\rho v^2\sin\frac{1}{2}\theta$, and is so directed as to bisect the angle subtended by the curved portion of the pipe. Since the condition is one of steady flow, the (constant) rate of change of momentum is $2A\rho v^2\sin\frac{1}{2}\theta$, directed like ΔU, and so the resultant force on the water column is equal to $2A\rho v^2\sin\frac{1}{2}\theta$. This is the resultant of P, P and R, that is, $2A\rho v^2\sin\frac{1}{2}\theta = R - 2P\sin\frac{1}{2}\theta$, whence $R = 2(P + v^2A\rho)\sin\frac{1}{2}\theta$.

(Note that this problem was solved without considering the acceleration of the mass-center, or of any part, of the water, and that the radius or form of curvature at the bend makes no difference whatever as far as the resultant force required to deflect the flow of water is concerned. But this resultant is made up of the pressures on all the elementary segments of the water column, and the distribution and intensity of these pressures do depend on the form of the bend. In order to compare methods, you should solve for the deflecting force $2A\rho v^2\sin\frac{1}{2}\theta$ by considering the body of water II and using the equation of motion of the mass-center, and also by considering the elementary segments of the water column.)

EXAMPLE 4. A rifle fires a bullet weighing 220 grains with a muzzle velocity of 2200 ft/sec. The rifle weighs 8 lb, and the powder charge weighs 30 grains. It is required to determine the velocity of free recoil for this rifle, that is, the velocity with which the rifle would recoil if no resistance were offered to the motion.

Solution. Consider the rifle, bullet, and powder gas as a body. During the time it takes for the bullet to reach the muzzle the only external forces that act on this body are its own weight, the supporting force that balances the weight, and the pressure of the air. The first two are mutually in equilibrium; the resistance of the air to recoil and the increased pressure at the muzzle due to the discharge may be considered negligible. Therefore the external force system on the body has no component in any direction, and so there can be no change in the linear momentum of the body; hence the linear momentum of the body remains zero. This momentum is the sum of the forward momentum of the bullet u_b, the forward momentum of the powder gas u_p, and the rearward momentum of the rifle u_r. Noting that the masses of the bullet, gun, and powder gas respectively are 0.000976, 0.248, and 0.000133 slug, and taking the forward direction as positive, we have: $u_b = 0.000976 \times 2200 = 2.15$ slug-ft/sec, and $u_r =$

$-0.248v$, where v is the velocity of recoil. To find u_p, we assume that the mass-center of the powder gas is at all times halfway between breech block and bullet; its velocity would then be $\frac{1}{2}(2200 - v)$, and its momentum $0.000133 \times \frac{1}{2}(2200 - v) = 0.146 - 0.0000665v$. Setting the total momentum equal to zero, we have

$$2.15 + (0.146 - 0.0000665v) - 0.248v = 0$$

Solving for v, we find $v = 9.25$ ft/sec.

(Because of the small mass of the powder the momentum of the rifle and the momentum of the bullet are very nearly of the same magnitude. Calculate the kinetic energy of each, and explain why the energy of the bullet is very much greater than that of the rifle.)

Rockets and jet propulsion. Interesting illustrations of the relationship between force and linear momentum are afforded by the rocket and by the jet-propelled plane. As far as method of propulsion is concerned, the two differ principally in this respect: the rocket carries all its fuel in the form of an explosive, whereas the jet engine carries fuel but depends upon air for combustion and may also take up and discharge air in addition to that used for combustion. In either device the propulsive force depends upon the mass of jet (gasified fuel and air) discharged per unit time and the relative velocity with which it is discharged, that is, upon the momentum increment added to the jet per unit time. The efficiency, which may be defined as the ratio of the useful work per unit time done in maintaining the kinetic energy of the airplane or rocket to the total power developed by the fuel in expanding, depends upon the ratio between the velocity of the plane or rocket and the relative velocity of the jet. (By " jet " we mean the entire quantity of moving gas and air, inside and outside the engine; the mass of the jet is being constantly augmented as fuel is gasified and still air is taken up and added to the jet. By relative velocity of the jet we mean the velocity, relative to the engine, with which the jet is moving at the escape orifice.)

We now derive expressions for the propulsive force F and the efficiency η of a jet-propulsion engine. The expressions can readily be modified, as will be shown, to apply either to a rocket or to an engine in which the mass of fuel consumed is negligible compared with the mass of air " passed through."

Let v = constant velocity of the engine or plane, m_f = mass of gasified fuel discharged per unit time, m_a = mass of air discharged per unit time, $m = m_f + m_a$ = mass of jet discharged per unit time, and u = velocity of the jet relative to the engine. Then the absolute velocity of the jet is $u - v$ rearward, and the increment of momentum added to the jet per unit time is $m_f(u - v) - (-m_fv) + m_a(u - v) = m_fu + m_a(u - v)$, rearward. The rearward force exerted by the engine on the jet, and therefore the forward force F exerted by the jet on the engine, is of this magnitude, or

$$F = m_fu + m_a(u - v) \tag{1}$$

Now the rate at which work is done on the jet is equal to the increment in its kinetic energy per unit time. The rate at which work is done on the jet is

$P - Fv$, where P represents the power developed by the fuel gas in expanding and $-Fv$ represents the negative work done in unit time by the rearward pressure of the engine on the jet. The increment in the kinetic energy of the jet per unit time is $\frac{1}{2}m_f(u - v)^2 - \frac{1}{2}m_f v^2 + \frac{1}{2}m_a(u - v)^2$. Setting $P - Fv = \frac{1}{2}m_f(u - v)^2 - \frac{1}{2}m_f v^2 + \frac{1}{2}m_a(u - v)^2$, substituting for F its value from Eq. 1, and solving for P, gives

$$P = \tfrac{1}{2}mu^2 - \tfrac{1}{2}m_a v^2 \tag{2}$$

The useful work done per unit time in propelling the plane is Fv, but there is lost, in the discharged fuel, kinetic energy in the amount $\frac{1}{2}m_f v^2$ per unit time. Therefore the net rate at which useful work is done is $Fv - \frac{1}{2}m_f v^2$, and so

$$\eta = \frac{Fv - \frac{1}{2}m_f v^2}{P} = \frac{[m_f u + m_a(u - v)]\,v - \frac{1}{2}m_f v^2}{\frac{1}{2}mu^2 - \frac{1}{2}m_a v^2} \tag{3}$$

For the rocket, $m_a = 0$ and $m_f = m$, and so Eq. 1 reduces to $F = mu$ and Eq. 3 to $\eta = (2uv - v^2)/u^2$. From this last equation it is obvious that η has its maximum value, unity, when $u = v$. This is consistent with the fact that when $u = v$ the absolute velocity of the jet is zero, and no kinetic energy is lost.

For a jet propulsion engine of the type in which the mass of discharged fuel is zero or negligibly small, $m_f = 0$ and $m_a = m$, and so Eq. 1 reduces to $F = m(u - v)$ and Eq. 3 to

$$\eta = \frac{2}{1 + u/v}$$

Here also the maximum efficiency is attained when $u = v$ (no kinetic energy lost), and it is apparent that η diminishes as the velocity ratio u/v increases.

It should be noted that in the above discussion the efficiency is not defined as the ratio of useful work to the total potential energy of the fuel, but as the ratio of useful work to actual mechanical work done by the fuel. The loss of thermal energy in the discharged gas and air is not taken into consideration.

EXAMPLE 5. A rocket weighing 150 lb is traveling 1260 ft/sec and discharging, at the rate of 24 lb/sec, a jet which has a relative velocity of 6700 ft/sec. It is required to determine the propulsive force and the efficiency.

Solution. Here $v = 1260$, $u = 6700$, and $m = 24 \div 32.2 = 0.745$ slug/sec. Using Eqs. 1 and 3 as modified for the rocket, we have

$$F = 0.745 \times 6700 = 5000 \text{ lb}$$

$$\eta = [(2 \times 6700 \times 1260) - 1260^2] \div 6700^2 = 0.34$$

(If, as is assumed, the rocket has no acceleration, the propulsive force of 5000 lb is just sufficient to overcome air resistance at a velocity of 1260 ft/sec, plus the component of the weight of the rocket tangent to the trajectory.)

EXAMPLE 6. The engine of a jet-propelled plane burns fuel at the rate of 0.8 lb/sec and takes up and discharges air at the rate of 50 lb/sec when flying 400 mi/hr. The jet velocity is 2000 ft/sec. It is required to determine the propulsive force, the efficiency, and the horsepower developed by the engine.

Solution. Here m_f may be regarded as negligible in comparison with m_a. Therefore $m = 50 \div 32.2 = 1.55$ slugs/sec, $u = 2000$, and $v = 400 \times 5280 \div 3600 = 587$ ft/sec. Using Eqs. 1 and 3 as modified for the engine with $m_f = 0$, we have

$$F = 1.55(2000 - 587) = 2190 \text{ lb}$$

$$\eta = 2 \div [1 + (2000 \div 587)] = 0.453$$

The useful power developed is $2190 \times 587 \div 550 = 2340$ hp; the total power developed by the engine is the useful power divided by the efficiency, or $2340 \div 0.453 = 5160$ hp.

160. Angular Momentum of a Particle. The angular momentum (also called moment of momentum) of a particle about a line is analogous to the moment of a force about a line (see Art. 18) and is computed in the same way. That is, it is computed by resolving the momentum into three rectangular components, one of which is parallel to the line, and summing the moments of these components.

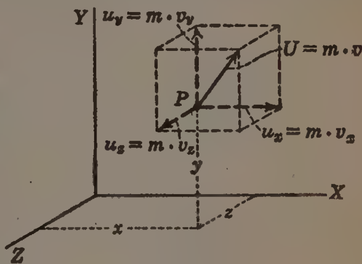

FIG. 289.

For example, let P (Fig. 289) be a particle having momentum $U = mv$ as shown, and suppose that it is desired to compute its angular momentum about any line such as OX. The momentum U is resolved into the three rectangular components

$$u_x = mv_x \qquad u_y = mv_y \qquad u_z = mv_z$$

The sum of the moments of these three components about OX, or about either of the other axes, is readily found. Denoting angular momentum by h, with subscript to indicate the axis about which it is taken, and following the same rule for sign as in computing moments of forces (Art. 18), it is evident that

$$h_x = mv_zy - mv_yz \qquad h_y = mv_xz - mv_zx \qquad h_z = mv_yx - mv_xy$$

where x, y, and z are the coordinates of the particle with respect to the axes shown.

Angular momentum can be expressed in any units of momentum and distance, as (slug-ft/sec) $\times$ ft or slug-ft²/sec.

161. Angular Momentum of a Body. The angular momentum (or moment of momentum) of a body about a line is the sum of the angular momentums of all the particles of the body about that line. If P of Fig. 289 is thought of as an elementary particle, of mass dm, of a body not shown, then for the body

$$h_x = \int (dmv_zy - dmv_yz) \qquad h_y = \int (dmv_xz - dmv_zx)$$

$$h_z = \int (dmv_yx - dmv_xy)$$

In general, these expressions are not easily evaluated; they are of use mainly in proving certain relationships discussed in later articles. But for certain motions and axes the formulas reduce to simpler terms; for such cases h can readily be found, either from the formulas or by simply summing the angular momentums of the constituent particles. The procedure will be illustrated for three important motions of a rigid body.

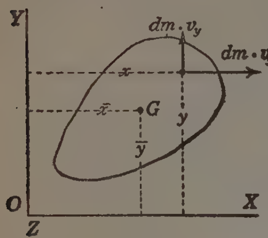

FIG. 290.

1. ANGULAR MOMENTUM OF A BODY IN TRANSLATION ABOUT AN AXIS NORMAL TO THE DIRECTION OF MOTION. Let Fig. 290 represent a body having motion of translation with velocity to the right equal to v, at the given instant, and suppose that it is desired to determine the angular momentum of the body about the z axis, normal to the paper. If axes OX and OY are taken as shown (OX parallel to the velocity and OY vertical), then the angular momentum about OZ is

$$h_z = \int(-dmv_xy + dmv_yx) = -v_x\int dmy + v_y\int dmx = -mv_x\bar{y} + mv_y\bar{x}$$

where m is the mass of the body and $\bar{x}$ and $\bar{y}$ are the coordinates of the mass-center G. Therefore the angular momentum of a body in translation about an axis normal to the direction of motion can be computed as though the entire mass of the body were concentrated at the mass-center.

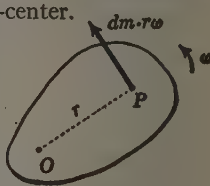

FIG. 291.

2. ANGULAR MOMENTUM OF A ROTATING BODY ABOUT THE AXIS OF ROTATION. Let Fig. 291 represent the body, which rotates in the plane of the paper about an axis through O. P is any elementary particle of the body, of mass dm, and distant r from the axis of rotation. The momentum of P has the same direction as its velocity and is equal to $dmv = dmr\omega$; it is indicated by the vector in the figure. The angular momentum of P about the axis of rotation is the product of this momentum and the arm r, or $dmr^2\omega$, and the angular momentum of the body is

$$h_0 = \int dmr^2\omega = \omega\int dmr^2 = I_0\omega$$

That is, the angular momentum of a rotating body about the axis of rotation is equal to the product of the angular velocity of the body and its moment of inertia about the axis of rotation.

3. ANGULAR MOMENTUM OF A BODY IN PLANE MOTION ABOUT AN AXIS NORMAL TO THE PLANE OF MOTION. Let Fig. 292 represent a body having plane motion in the plane of the paper, and let the z axis normal to the paper be the axis in question. G is the mass-center of the body; $\bar{v}_x$ and $\bar{v}_y$ represent the velocity

$\bar{V}$ of G; ω is the angular velocity of the body. Axes OX and OY are fixed; GX_1 and GY_1 are parallel to OX and OY but move with G. If G is chosen as base point, any particle P has a velocity that is the vector sum of $\bar{V}$ (or $\bar{v}_x$ and $\bar{v}_y$) and the relative velocity $r\omega$ (perpendicular to the radius r). The particle has corresponding momentum components $dm \cdot \bar{v}_x$, $dm \cdot \bar{v}_y$, and $dm \cdot r\omega$, indicated by the vectors in the figure. The moments about OZ of these momentum components are computed separately. The moment of $dm\bar{v}_x$ is $-dm\bar{v}_x(\bar{y} + y_1)$; the moment of $dm\bar{v}_y$ is $+dm\bar{v}_y(\bar{x} + x_1)$; the moment of $dmr\omega$ is $dmr\omega \sin\phi\ (\bar{y} + y_1) + dmr\omega \cos\phi\ (\bar{x} + x_1)$. The angular momentum of the particle is the sum of these moments or

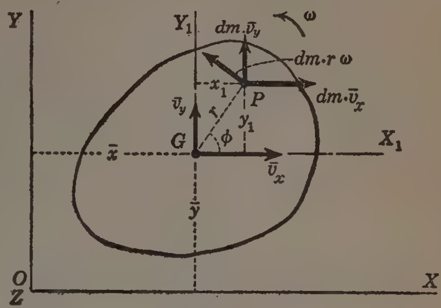

FIG. 292

$$-dm\bar{v}_x(\bar{y} + y_1) + dm\bar{v}_y(\bar{x} + x_1) + dmr\omega \sin\phi\ (\bar{y} + y_1) + dmr\omega \cos\phi\ (\bar{x} + x_1)$$
$$= -dm\bar{v}_x(\bar{y} + y_1) + dm\bar{v}_y(\bar{x} + x_1) + dm\omega(\bar{y}y_1 + y_1^2) + dm\omega(\bar{x}x_1 + x_1^2)$$

And for the body the angular momentum is

$$\begin{aligned} h_z &= \int(-dm\bar{v}_x\bar{y} - dm\bar{v}_x y_1 + dm\bar{v}_y\bar{x} + dm\bar{v}_y x_1 + dm\omega\bar{y}y_1 \\ &\qquad + dm\omega y_1^2 + dm\omega\bar{x}x_1 + dm\omega x_1^2) \\ &= -\bar{v}_x\bar{y}\int dm - \bar{v}_x\int dmy_1 + \bar{v}_y\bar{x}\int dm + \bar{v}_y\int dmx_1 \\ &\qquad + \omega\bar{y}\int dmy_1 + \omega\bar{x}\int dmx_1 + \omega\int dm(x_1^2 + y_1^2) \\ &= -m\bar{v}_x\bar{y} + m\bar{v}_y\bar{x} + \omega\bar{I} \end{aligned}$$

This is equal to the angular momentum the body would have if it moved with motion of translation and with the velocity of the mass-center, plus the product of the angular velocity and the moment of inertia of the body about an axis through its mass-center and parallel to the given axis. Obviously, if the z axis passes through the mass-center, $h_z = \bar{I}\omega$, and is independent of the motion of the mass-center.

Note that in each of the expressions for h under paragraphs 1, 2, and 3 the signs depend upon the arbitrarily assumed position of the axis and direction of the motion.

162. Rate of Change of Angular Momentum. As shown in Art. 161, the angular momentum of a body about any axis X is

$$h_x = \int(dmv_zy - dmv_yz)$$

Differentiation gives

$$\frac{dh_x}{dt} = \int \left[dm \left(\frac{dv_z}{dt} y + v_z \frac{dy}{dt} \right) - dm \left(\frac{dv_y}{dt} z + v_y \frac{dz}{dt} \right) \right]$$

and, since $dy/dt = v_y$, $dz/dt = v_z$, $dv_z/dt = a_z$, and $dv_y/dt = a_y$,

$$\frac{dh_x}{dt} = \int (dma_z y - dma_y z)$$

Similarly,

$$\frac{dh_y}{dt} = \int (dma_x z - dma_z x) \qquad \text{and} \qquad \frac{dh_z}{dt} = \int (dma_y x - dma_x y)$$

These expressions are, in general, not easily evaluated; their use, like that of the formulas for the components of h, is mainly in proving further relationships which are discussed in the next article.

For a rigid body that has motion of translation, motion of rotation, or plane motion, the rate of change of angular momentum about an axis normal to the plane of the motion can be readily found by differentiating the corresponding simplified expression for h given in Art. 161. Thus for translation

$$h_z = -mv_x\bar{y} + mv_y\bar{x} \qquad \text{hence} \qquad \frac{dh_z}{dt} = -ma_x\bar{y} + ma_y\bar{x}$$

for rotation

$$h_0 = I_0\omega \qquad \text{hence} \qquad \frac{dh_0}{dt} = I_0\frac{d\omega}{dt} = I_0\alpha$$

and for plane motion

$$h_z = -m\bar{v}_x\bar{y} + m\bar{v}_y\bar{x} + \bar{I}\omega \qquad \text{hence} \qquad \frac{dh_z}{dt} = -m\bar{a}_x\bar{y} + m\bar{a}_y\bar{x} + \bar{I}\alpha$$

For three-dimensional motion of a rigid body, the rate of change of angular momentum can be found by the methods explained in Chapter XV. For nonrigid bodies or systems the rate of change of angular momentum can usually best be found by considering only those parts whose angular momentums change; and for steady states of motion it can be found by calculating the increment of angular momentum per unit time without determining the actual amount of angular momentum present in the system at all. The several methods are illustrated in the examples of the following article.

163. Force-Angular Momentum Relationship. *The moment about any line of the external system of forces acting on any body is equal to the rate at which the angular momentum of the body about that line is changing,* or

$$\Sigma M_x = \frac{dh_x}{dt} \tag{1}$$

where x denotes any axis.

Proof. Let P, Fig. 293, be any elementary particle of the body; a_x, a_y, and a_z the axial components of the acceleration of P, and dm its mass. The resultant F of all forces acting on P is equal to ma, and the axial components of F are $F_x = dma_x$, $F_y = dma_y$, $F_z = dma_z$, as marked in the figure. The moment about OX of all forces acting on P is therefore $dma_zy - dma_yz$, and so

$$\begin{matrix}\text{Moment of all forces acting}\\ \text{on all particles of the body}\end{matrix} = \int (dma_zy - dma_yz)$$

Now the left-hand member of this equation includes the moments of all forces, external and internal, that act on the body. Since the internal forces occur in pairs of equal, opposite, and collinear forces, their moments cancel, and the value of the left-hand member is the same as that of the moment of the external force system, that is, ΣM_x. It has already been shown (Art. 162) that $dh_x/dt = \int(dma_zy - dma_yz)$.

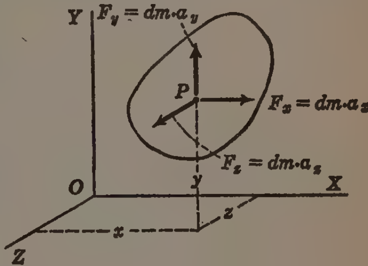

FIG. 293.

A simple illustration of this relationship is afforded by a rigid body having motion of rotation about, say, the x axis. Here, as shown in Art. 161, $h_x = I_x\omega$; therefore $dh_x/dt = I_x(d\omega/dt) = I_x\alpha$, and so for this particular motion the equation $\Sigma M_x = dh_x/dt$ reduces to the familiar equation of rotation (Art. 113).

It follows from Eq. 1 that, if the moment, about any line, of the external system of forces acting on any body equals zero, the angular momentum of the body about that line remains constant. This is known as the **principle of conservation of angular momentum.**

APPLICATIONS. The force-angular momentum relationship is especially useful in the solution of problems that involve the motion of nonrigid bodies or systems and in problems that involve three-dimensional motion of rigid bodies (Chapter XV). The following examples illustrate its use in solving the first type of problem.

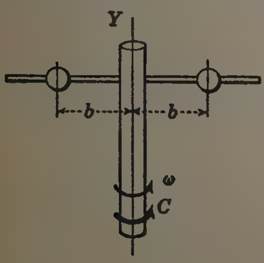

FIG. 294.

EXAMPLE 1. Figure 294 represents a vertical shaft to which is fixed a horizontal bar along which two small pierced spheres can be made to slide. The whole system is made to rotate with constant angular velocity ω about the vertical axis Y of the shaft by a driving couple C, while the spheres are made to slide outward with a uniform radial speed v. This radial motion of the spheres is controlled by a mechanism of negligible mass (not shown) attached to and regarded as part of the shaft-bar combination. The weight of each sphere is W; they may be regarded as particles, and when their outward motion starts they are at a distance b from the axis of the shaft as shown. It is required to derive an expression for the driving couple C

in terms of t, the elapsed time after outward motion starts, and to evaluate C for $W = 5$ lb, $v = 10$ ft/sec, $\omega = 100$ rev/min, $b = 1$ ft, and $t = 2$ sec.

Solution. At time t the distance r from the axis Y to either sphere is $(b + vt)$. The component of the velocity of each sphere normal to the arm is $(b + vt)\omega$; the corresponding component of the linear momentum of each sphere is $m(b + vt)\omega$, and the angular momentum of each sphere about the axis Y is

$$m(b + vt)\omega(b + vt) = m\omega(b^2 + 2bvt + v^2t^2)$$

The total angular momentum of the system is twice this plus the constant angular momentum of the shaft-bar combination, which latter is $I_y\omega$, I_y being the moment of inertia of the shaft-bar combination. Therefore, for the entire system, the angular momentum is

$$h_y = I_y\omega + 2m\omega(b^2 + 2bvt + v^2t^2)$$

and

$$C = \frac{dh_y}{dt} = 2m\omega(2bv + 2v^2t)$$

For $b = 1$, $m = 5 \div 32.2$, $v = 10$, $\omega = 100$ rev/min $= 10.5$ rad/sec, $t = 2$, the value of C is found to be 1370 ft-lb.

(You should determine what difference, if any, it would make if the spheres were so large that they could not be regarded as particles. You should also solve this problem by the application of d'Alembert's principle, and compare with the above solution.)

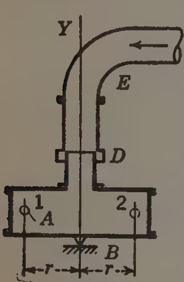

FIG. 295.

EXAMPLE 2. Figure 295 represents, in principle, an hydraulic motor sometimes called a "Barker's mill." Essentially, the motor consists of a horizontal cylinder A, mounted on a vertical pivot B, and an inlet D connected by a water-tight sleeve joint to a feed pipe E. On opposite sides of the cylinder and near its ends there are orifices or nozzles 1 (in front as viewed) and 2 (behind) through which the water flows horizontally, causing the motor to turn clockwise about axis Y as viewed from above. On the assumption of a condition of steady flow, it is required to determine the torque or turning couple C exerted by the water on the motor.

Solution. The water has no angular momentum about the axis of rotation when it enters the motor at D but does have when it escapes at the orifices; therefore the mill imparts angular momentum to the water, and the amount imparted per unit time equals the rate of change of angular momentum of the water, which will now be determined.

The mass of water which escapes per unit time is m. Its absolute velocity when it escapes is the vector sum of its velocity relative to the orifice and the absolute velocity of the orifice; the former is equal to v, the latter is equal to $r\omega$ and has the opposite direction, and so the absolute velocity of the water is equal to $v - r\omega$ and is directed perpendicularly to r. The linear momentum of this water is $m(v - r\omega)$, and its angular momentum about the axis of rotation is $m(v - r\omega)r$. This is the increment in angular momentum of the water per unit time, or rate of change of angular momentum, and so is equal to the torque exerted by the mill on the water; the couple C exerted on the mill by the water is equal and opposite.

The mass of water m escaping per unit time can be controlled by the pressure in the feed pipe. The angular velocity ω depends on the resisting torque of the machinery driven by the mill, which is equal and opposite to C, since the mill itself suffers no change in angular momentum. The useful power developed is $C\omega$, or $m(v - r\omega)r\omega$.

EXAMPLE 3. A small horizontal platform or wheel is so mounted on frictionless bearings that it can rotate about a vertical axis through its center. A man stands erect on this platform, his arms hanging straight down at his sides and his center of gravity directly over the center of the platform. The platform and man are made to rotate with an angular velocity ω, and, while so rotating, the man raises his arms so that they are outstretched horizontally. This changes his moment of inertia about the axis of rotation from 0.88 slug-ft² to 1.30 slug-ft². With what angular velocity ω' will he rotate in the new position?

Solution. If air resistance is neglected, the external forces on the platform-man system have no moment about the axis of rotation; therefore the angular momentum about that axis remains constant or $1.30\omega' = 0.88\omega$, and so $\omega' = 0.677\omega$.

(Compare the kinetic energy of the rotating system before and after the man changes position, and explain, on the basis of the principle of work and kinetic energy, how it is possible for the change in kinetic energy to occur.)

EXAMPLE 4. Figure 296 represents a ballistic pendulum, a device sometimes used for determining the velocity of a bullet. It consists of a block A of material into which the bullet will penetrate without either rebounding or going clear through, and a slender rod B, the combination forming a compound pendulum which can swing about an axis of suspension O at the upper end of the rod. The bullet is fired horizontally into the block, causing the pendulum to swing. By means of the principle of conservation of angular momentum and the principle of work and kinetic energy, it is possible to calculate, from the angle θ through which the impact moves the pendulum, the striking velocity of the bullet. Given the mass of the bullet $= m_b$, the mass of the pendulum $= m_p$, the radius of gyration of the pendulum $= k$ with respect to the axis O, the distance from O to the center of gravity of the pendulum $= d$, and the vertical distance from O to the point of impact $= h$, it is required to determine v, the striking velocity of the bullet, in terms of θ, the maximum angular displacement of the pendulum.

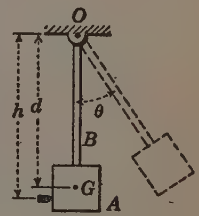

FIG. 296.

Solution. We take as the body under consideration the pendulum-bullet combination. It may be assumed that during the extremely short period of impact (time required for the bullet to attain its maximum penetration) the pendulum suffers very little displacement, and that during that period the only external forces acting on the body (its weight and the axle reaction) have at no time any appreciable moment about the axis O. Therefore the angular momentum of the system about O is the same before and immediately after impact. Before impact the angular momentum of the system is that of the bullet only, and is equal to $m_b vh$. After impact the body has motion of rotation about O, and its angular momentum is then $I_O\omega = (m_pk^2 + m_bh^2)\omega$. Therefore

$$m_bvh = (m_pk^2 + m_bh^2)\omega \quad \text{or} \quad v = \frac{(m_pk^2 + m_bh^2)\omega}{m_bh}$$

Now the kinetic energy of the system after impact is $\frac{1}{2}I_O\omega^2 = \frac{1}{2}(m_pk^2 + m_bh^2)\omega^2$. This energy is lost during the displacement θ, and, by the principle of work and kinetic energy, this loss is equal to the negative work done in bringing the body to rest. The only force that does work on the body during the swing (if air resistance and axle friction are disregarded) is gravity; this does negative work on each body equal to its weight times the vertical rise of its center of gravity, and equating the total work to the increment of kinetic energy we have

$$-32.2m_pd(1 - \cos\theta) - 32.2m_bh(1 - \cos\theta) = -\tfrac{1}{2}(m_pk^2 + m_bh^2)\omega^2$$

whence

$$\omega = \left[\frac{64.4(1 - \cos\theta)(m_pd + m_bh)}{m_pk^2 + m_bh^2}\right]^{\frac{1}{2}}$$

Substituting this value of ω in the above expression for v gives

$$v = \left[\frac{64.4(1 - \cos\theta)(m_p d + m_b h)(m_p k^2 + m_b h^2)}{m_b^2 h^2}\right]^{\frac{1}{2}}$$

Since $m_b h$ is small compared with $m_p d$ and $m_b h^2$ is small compared with $m_p k^2$, the terms in the numerator in which m_b appears may be dropped without appreciable error, and the expression for v then reduces to

$$v = \frac{m_p k}{m_b h}[64.4d(1 - \cos\theta)]^{\frac{1}{2}}$$

(How do you think the kinetic energy of the system after impact compares with the kinetic energy before impact? How do you account for the difference, if any? See, in this connection, Art. 169. This device should be so designed and used that the bullet strikes along a horizontal line through the center of percussion of the pendulum. Why?)

164. Resultant Momentum of a Body. How to determine the magnitude and direction of the linear momentum of a body is explained in Art. 157, and how to determine the angular momentum of a body about a given line is explained in Art. 161. We now introduce the concept of *resultant* or *total momentum* of a body, which may be defined as the resultant of the momentums of all the constituent particles of the body.

Since a system of forces and a system of momentums are precisely alike in the sense that forces and momentums are both localized vector quantities, the resultants of the two kinds of systems can be found in the same way and expressed in the same form, though of course in different units. Therefore we choose a form of resultant already developed for forces. It is shown in Art. 32 that any system of forces can be compounded into a single force R and a couple C. The line of action of R may be taken through any chosen point O of the body; the couple C is not localized but is generally represented by a vector lying in a line through O (see Art. 32 for discussion). Now in the same way a system of momentums can be compounded into a single linear momentum U and a *momentum couple* H. This momentum couple is in every respect analogous to a force couple; it is a vector quantity, and therefore it can be represented by a vector and can be resolved into components.

The position of U may be taken in a line passing through any chosen point O (Fig. 297); H is not localized but may conveniently be represented by a vector lying in a line through O. Obviously U does not depend on the position of O, but H does. As shown in Art. 157, the magnitude of the linear momentum is $u = (u_x^2 + u_y^2 + u_z^2)^{\frac{1}{2}} = m\bar{v}$, and U has the same direction as $\bar{V}$, with direction-cosines u_x/u, u_y/u, u_z/u. The momentum couple H has the same moment about any line through O as do the momentums of the particles, and so its x, y, and z components are respectively equal to h_x, h_y, and h_z. Also $h = (h_x^2 + h_y^2 + h_z^2)^{\frac{1}{2}}$, and the direction-cosines of the vector representing H are h_x/h, h_y/h, h_z/h. Of all lines through O, the one about which the angular

momentum of the body is greatest is the one coinciding with the vector, drawn through O, that represents H.

Since, as pointed out in Art. 161, h_x, h_y, and h_z are as a rule not easily evaluated, the resultant momentum of a body is usually hard to determine. For certain simple kinds of motions and bodies, however, U and H can be readily found. Thus, *if O is taken coincident with the mass-center of the body*, it can be shown that: (i) for a body having motion of translation, $U = mv$ directed like V, $H = 0$; (ii) for a body rotating about an axis of symmetry, $U = 0, H = \bar{I}\omega$, where $\bar{I}$ is the moment of inertia about the axis of rotation, and the vector for H is parallel to that axis; (iii) for a body having a plane of symmetry and having plane motion in that plane, $U = m\bar{v}$ directed like $\bar{V}$, $H = \bar{I}\omega$, where $\bar{I}$ is the moment of inertia about an axis through the mass-center and perpendicular to the plane of motion, and the vector for H is parallel to that axis. You should draw a figure to illustrate each of these cases and prove that the resultant momentum is as stated.

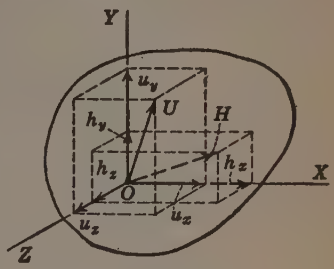

FIG. 297.

It should be noted that, for the conditions specified in (i), (ii), and (iii), H is independent of $\bar{V}$. It is shown in Chapter XV that for any motion of any body H is thus independent of $\bar{V}$ if the point O through which U is taken to act is coincident with the mass-center.

The concept of resultant momentum is useful in the discussion of spherical motion and general motion, subjects dealt with in Chapter XV. It also provides a basis for a statement of what may be called the principle of conservation of resultant momentum, namely, if the system of external forces acting on any body is in equilibrium, or if there are no external forces, then the resultant momentum of the body remains constant. This follows from the principles of conservation of linear momentum and conservation of angular momentum.

EXAMPLE. Suppose that a man stands on a turntable as described in Ex. 3 of Art. 163, the man and turntable being stationary, and someone hands the man a rapidly rotating flywheel mounted in a frame, so that it can be held without interfering with its rotation. The axis of the flywheel is vertical, and the rotation is clockwise as viewed from above. If the flywheel is handed to the man with negligible velocity of translation, the resultant momentum of the man-turntable-flywheel system, both before and after the transfer is made, is a momentum couple $H = I\omega$, where I is the moment of inertia and ω the angular velocity of the flywheel. Now no external force acting on the man-turntable-flywheel has a moment about the vertical central axis; hence the angular momentum about that axis h_y must always equal H. Suppose now that the man inverts the flywheel, so that as viewed from above it rotates counterclockwise. If the man and platform were to remain stationary, the resultant momentum of the system would become $-H$ and this would represent an increment in angular momentum about the y axis $\Delta h_y = -2H$. But this is impossible; there can be no increment in h_y, and so the man and turntable must take on a clockwise rotation at such angular velocity as to make h_y

for the system exactly the same as before the flywheel was inverted. (We are not yet ready to discuss the circumstances while the wheel is being inverted, or when it is in some intermediate position. We shall return to that in Chapter XV.)

Again, suppose that, instead of inverting the flywheel, the man holds it in the original upright position and stops the spinning by braking it with one hand against the rim. The forces between hand and rim are internal forces; they cannot affect the angular momentum of the system, and so again the man and turntable must take on a clockwise rotation at such angular velocity as to make h_y for the system equal to its original value $I\omega$. This angular velocity will obviously be less than when the flywheel is inverted.

(Show that there is no loss of kinetic energy when the wheel is inverted, and derive an expression for the loss of kinetic energy when the wheel is braked.)

165. Linear Impulse of a Force. If the magnitude and direction of a force are constant for any interval of time, the product of the magnitude of the force and the interval is called the **impulse** of the force for that interval; it is a localized vector quantity, directed like the force and having the same line of action. If the force varies in magnitude only, the impulse for any interval is the sum of the elementary impulses $F\,dt$ for all the elementary periods of time that make up the interval. If the force varies in direction, or in both direction and magnitude, the impulse for any interval is the resultant or vector sum of all the elementary impulses $F\,dt$.

We denote impulse by L; its magnitude by l. It can be expressed in any units of force and time, as pound-seconds; no single word has been generally adopted for any unit of impulse.

COMPONENT OF IMPULSE. It is evident that the component parallel to any axis X of the elementary impulse of a force F is equal to the elementary impulse of the x component of the force; that is $(F\,dt)_x = F_x\,dt$. Hence, a like relation holds for finite impulses, or

$$l_x = \int (F\,dt)_x = \int F_x\,dt$$

For any system of forces, F', F'', etc.

$$l_x = \int (F'\,dt)_x + \int (F''\,dt)_x + \cdots = \int F_x'\,dt + \int F_x''\,dt + \cdots = \int (\Sigma F_x)\,dt$$

Any of these sums we shall call the x component of the impulse of the system of forces. Obviously the integration indicated can be performed graphically if a graph or table showing the (ΣF_x)-t relationship is available. By finding the components of the impulse parallel to each of three rectangular axes the magnitude and direction of the total or resultant impulse can be found. To find the position of this resultant impulse, the concept of angular impulse or moment of impulse, which is discussed in Art. 167, must be utilized.

166. Impulse-Momentum Relationship. *For any interval of time the component parallel to any line X of the impulse of the external forces acting on a body*

is equal to the increment of the x component of the momentum of the body, or

$$I_x = \Delta u_x$$

For, according to Art. 159, $\Sigma F_x = du_x/dt$; hence $(\Sigma F_x)\,dt = d(u_x)$ and $\int(\Sigma F_x)\,dt = I_x = \Delta u_x$.

The relationship stated and proved above is useful in solving problems somewhat like those that were solved in Art. 154 by the principle of work and kinetic energy. That principle, however, is especially adapted to finding the velocity change that occurs during a certain displacement, whereas the impulse-momentum relationship is especially adapted to finding the velocity change that occurs during a certain time interval.

EXAMPLE 1. A block weighing 100 lb rests on a horizontal floor; the coefficient of kinetic friction between block and floor is 0.3. A horizontal force of 60 lb acts on the block for 5 sec. It is required to determine the velocity with which the block is moving at the end of that time.

Solution. The only horizontal forces on the block are the 60-lb force, which acts in the direction of the motion, and the frictional force, which is 30 lb and which acts opposite to the direction of the motion. The impulse in the direction of motion is therefore $I = (60)(5) - (30)(5) = 150$ lb-sec. Since the block is initially at rest, the increment in its momentum is equal to its final momentum $(100 \div 32.2)v$ slug-ft/sec; equating this to 150 and solving for v gives $v = 48.3$ ft/sec.

(You should also solve this example by means of the equation $\Sigma F_x = ma_x$ and compare the two solutions. Note that if the statement of the problem had been that the 60-lb force acted while the block moved a certain distance the principle of work and kinetic energy could have been used to advantage.)

EXAMPLE 2. A block weighing 50 lb rests on a smooth horizontal floor (Fig. 298). A force F, always equal to 30 lb, is applied to the block by means of a cord which, originally vertical, is rotated downward with an angular velocity of 0.5 rad/sec until it reaches the horizontal position. It is required to determine the velocity of the block at the instant the cord becomes horizontal.

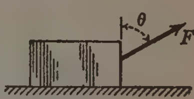

FIG. 298.

Solution. Dating time from the instant the cord starts to swing down, we have: $\Delta t = \Delta\theta/\omega = \frac{1}{2}\pi \div 0.5 = \pi$ sec, and $\Sigma F_x = 30 \sin \theta = 30 \sin (0.5t)$. Therefore $I_x = \int_0^\pi 30 \sin (0.5t)\,dt = 60$ lb-sec. This is equal to $\Delta u_x = (50 \div 32.2)v$; hence $v = 38.64$ ft/sec. (Since at all times $\Sigma F_y = 0$, $I_y = 0$ and the block acquires no vertical momentum.)

EXAMPLE 3. A block weighing 100 lb rests on a horizontal floor; the coefficient of friction (static and kinetic) between floor and block is 0.3. A horizontal force F which is initially zero and which increases uniformly with time at the rate of 10 lb/sec acts on the block for 8 sec. It is required to determine the velocity of the block at the end of that time.

Solution. The limiting friction under the block is $0.3 \times 100 = 30$ lb, and until F attains that value (which it does in 3 sec) the friction will at all times be equal and opposite to F; hence for the first 3 sec their impulses will be equal and opposite and I_x will be zero. Dating time from the end of the third second, therefore, we have $F = 30 + 10t$, friction $= 30$ in the opposite direction, and for the remaining 5 sec

$$I_x = \int_0^5 (30 + 10t)\,dt - \int_0^5 30\,dt = 125 \text{ lb-sec}$$

This is equal to $\Delta u_x = (100 \div 32.2)v$, whence $v = 40.2$ ft/sec. As in Ex. 2, $I_y = 0$, since

$\Sigma F_y = 0$ at all times. (This example is similar to Ex. 5 of Art. 91. Compare the two solutions.)

167. Angular Impulse. If the line of action of a force is fixed in position, the **angular impulse** of that force for any interval of time about any line is the moment, about that line, of the impulse of the force for the interval. The moment of an impulse about a line is computed just like the moment of a force, that is, by resolving the impulse into three rectangular components, one being parallel to the line, and then summing the moments of these three components.

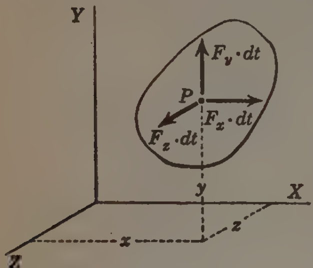

FIG. 299.

Thus let F be a force acting at a point P of a body, Fig. 299. The x, y, z components of the elementary impulse $F\,dt$ are as indicated. The angular impulse of the force for the elementary time dt about the x axis, say, is $(F_z\,dt)y - (F_y\,dt)z = (F_zy - F_yz)\,dt$, and the angular impulse of the force for any finite interval of time, which we denote by J_x, is

$$J_x = \int (F_zy - F_yz)\,dt = \int M_x\,dt$$

where M_x denotes the (varying) moment of the force about the x axis. By angular impulse about a line of a system of forces is meant the sum of their angular impulses about that line. Thus if M_x', M_x'', etc., denote the individual moments of the several forces about the x axis, the angular impulse of the system is

$$J_x = \int M_x'\,dt + \int M_x''\,dt + \cdots = \int (\Sigma M_x)\,dt$$

where ΣM_x denotes the (varying) moment of the system about the line.

Angular impulse can be expressed in any units of impulse and distance, as pound-seconds $\times$ feet.

168. Angular Impulse-Momentum Relationship. *For any time interval the angular impulse, about any line, of the external system of forces acting on a body equals the increment in the angular momentum of the body about that line.* This follows from Art. 163, for it was there shown that $\Sigma M_x = dh_x/dt$, and hence $\Sigma M_x\,dt = dh_x$ and $\int(\Sigma M_x)\,dt = J_x = \Delta h_x$.

This relationship is useful in solving problems that involve finding the change in angular velocity of a body that occurs in a certain time interval.

EXAMPLE 1. The pulley-drum combination shown in Fig. 300 weighs 200 lb and has a radius of gyration of 1.6 ft, an inner radius of 1.5 ft, and an outer radius of 2 ft. The suspended blocks A and B weigh 100 and 80 lb, respectively; the rope may be regarded as flexible and weightless and the bearings as frictionless. It is required to determine the angular velocity of the pulley 4 sec after the system is released from rest.

Solution. We use the equation $J_x = \Delta h_x$, applying it to the entire pulley-block system with reference to the axis of the pulley. The only external forces that have a moment about the axis are the weights of the blocks; their combined moment is constant and is $\Sigma M_x = (100 \times 2) - (80 \times 1.5) = 80$ ft-lb, and so the angular impulse for the 4-sec interval is $J_x = \int_0^4 80\,dt = 320$ lb-sec-ft. For the pulley the angular momentum is $I_x\omega = (200 \div 32.2)\cdot(1.6^2)\omega = 15.9\omega$; for A it is $(100 \div 32.2)(2\omega)(2) = 12.4\omega$, and for B it is $(80 \div 32.2)(1.5\omega)\cdot(1.5) = 5.58\omega$. All these angular momentums are counterclockwise; therefore their sum equals Δh_x; and so, equating J_x and Δh_x, we have $320 = 33.88\omega$, whence $\omega = 9.45$ rad/sec.

(Note that this solution does not involve finding the angular acceleration α, but that α can be determined very readily since all the forces and moments, and therefore α also, are constant, and so $\alpha = \Delta\omega/\Delta t = 2.36$ rad/sec². You should compare the above solution of this problem with the solution that would be carried out using the equations $\Sigma F_x = ma_x$ and $\Sigma M = I\alpha$.)

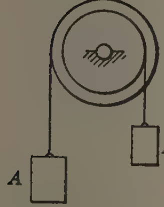

FIG. 300.

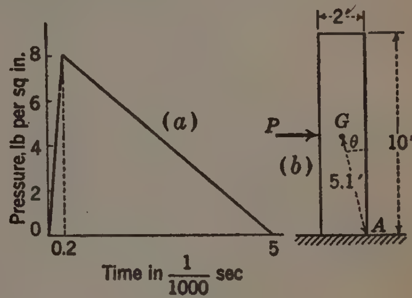

FIG. 301.

EXAMPLE 2. Figure 301*a* is a simplified copy of a graph showing how the pressure wave from the explosion of a 500-lb bomb at a distance of 40 ft varies with time. (The short period of high positive pressure is followed by a longer period of small negative pressure which is not represented on the graph.) In order to protect airplanes on the ground from bomb splinters and blast effect, a wall of concrete 10 ft high and 2 ft thick (Fig. 301*b*) weighing 150 lb/ft³ is built in segments or panels each 10 ft long; these panels simply rest on level ground in a vertical position. It is required to ascertain whether the pressure from an explosion represented by the graph would overturn the wall. It may be assumed that the wall rests in a slight depression and so does not slide on the ground, and that the pressure is uniform over the entire face of the panel.

Solution. Obviously any motion of the wall that occurs is a rotation about the edge A. As long as the moment of the pressure about A is less than the moment of gravity, the wall will be in equilibrium under the pressure, the ground reaction, and its own weight, and during that time the angular impulse about A will be zero. Throughout the interval during which the moment of the pressure exceeds the moment of the weight, there will be a net angular impulse about A. We determine this net angular impulse and from it determine the angular momentum acquired by the wall, and then, by the principle of work and energy, ascertain whether the corresponding kinetic energy is sufficient to carry the wall beyond the angle of equilibrium.

The moment of the weight about A is $100 \times 2 \times 150 \times 1 = 30{,}000$ ft-lb. To overcome this moment the total pressure P must equal $30{,}000 \div 5 = 6000$ lb, and the unit pressure p must equal $6000 \div 14{,}400 = 0.417$ lb/in.² From the graph we find that p has this value when $t = 0.00001$ sec and again when $t = 0.00475$ sec. For this interval the linear impulse of the pressure on each square inch of wall surface is equal to the corresponding area under the graph, which is practically (within the limits of accuracy of the data) equal to the whole area,

or 0.02 lb-sec. For the total pressure P the linear impulse is $14{,}400 \times 0.02 = 288$ lb-sec, and the angular impulse about A is therefore $288 \times 5 = 1440$ lb-sec-ft clockwise. For the interval $t = 0.00001$ to $t = 0.00475$, the angular impulse of the weight is $30{,}000 \times 0.00475 \times 1 = 142$ lb-sec-ft counterclockwise. The total angular impulse is therefore $J_A = 1440 - 142 = 1298$ lb-sec-ft. This is equal to the acquired increment of angular momentum $I_A\omega$, and, since

$$I_A = \frac{1}{12}\,\frac{30{,}000}{32.2}\,(2^2 + 10^2) + \frac{30{,}000}{32.2}\,(5.1^2) = 32{,}280 \text{ slug-ft}^2$$

$\Delta h_A = 32{,}280\omega$. Equating J_A to Δh_A and solving for ω gives $\omega = 0.0403$ rad/sec. The kinetic energy of the rotating wall is $\frac{1}{2}I_A\omega^2 = 26.2$ ft-lb. This is the kinetic energy at the instant $t = 0.00475$ sec, and the wall will then have rotated through a small angle θ'. To find θ' we should have to express the variable angular velocity as a function of t and evaluate $\int \omega\, dt$, either mathematically or graphically. But this is not necessary, because it is obvious that, since the maximum value of ω is 0.0403 rad/sec, θ' must be less than $0.00475 \times 0.0403 = 0.000191$ rad $= 0.011°$, and so we can without appreciable error assume $\theta' = 0$ in comparison with $\theta = \tan^{-1}\frac{1}{5}$, the angle through which the wall must tilt to pass the position of equilibrium (mass-center G directly over A). We assume then that the wall, still vertical, is rotating clockwise with $\omega = 0.0403$ rad/sec. During the subsequent displacement gravity does negative work on the wall equal to the weight times the rise of the center of gravity, and for the very short period $t = 0.00475$ to $t = 0.005$ the diminishing blast pressure does positive work which will be disregarded as negligibly small (show that this positive work is less than 0.3 ft-lb). If the wall rotated to the position of equilibrium, it would have to turn through the angle $\theta = \tan^{-1}\frac{1}{5} = 11.3°$; the center of gravity would rise a distance $5.1 \times (1 - \cos 11.3) = 0.10$ ft, and gravity would do negative work equal to $30{,}000 \times 0.10 = 3000$ ft-lb. But the kinetic energy is only 26.2 ft-lb, and so the wall does not rotate far enough to reach the position described and so does not overturn.

EXAMPLE 3. A solid homogeneous cylinder 4 ft in diameter weighing 400 lb is made to roll up a 30° incline by a force P of 300 lb applied as shown in Fig. 302*a*. The plane is rough enough to prevent slipping, and there is no rolling resistance. It is required to determine the velocity of the cylinder after P has acted for 10 sec, the cylinder being initially at rest.

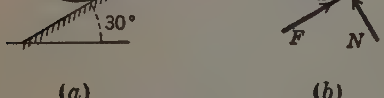

FIG. 302.

Solution. The fbd for the cylinder is shown in Fig. 302*b*; the weight and the reaction of the plane are each represented by components normal and parallel to the plane. The fixed line O, the original line of contact between cylinder and plane, is chosen as the line with respect to which J and Δh are computed. Since the normal reaction of the plane N is equal and opposite to the normal component of the weight, and the friction F always acts through O, the only forces producing angular impulse about O are P and the component of the weight parallel to the plane, and so $J_0 = 300(4)(10) - 200(2)(10) = 8000$ ft-lb-sec. The original $h_0 = 0$; the final $h_0 = \Delta h_0 = (m\bar{v} \times 2) + \bar{I}\omega = 2(400 \div 32.2)\bar{v} + \frac{1}{2}(400 \div 32.2)(2^2)(\bar{v}/2) = 37.26\bar{v}$. Equating J_0 and Δh_0 and solving for $\bar{v}$ gives $\bar{v} = 214.4$ ft/sec.

169. Collision; Impact Forces. When two moving bodies collide, each exerts on the other an impact force, usually of brief duration but great magnitude, and as a result the motions of the bodies are altered. Impact forces depend on the relative motion and positions of the bodies at the instant of

impact, on their masses and shapes, and on the materials of which they are composed. In order to calculate either these forces or their effects, information is necessary about certain physical properties of the materials which can be determined only by experiment. When these experimental data are known, something concerning the impact forces and the behavior of the bodies subsequent to collision can be determined analytically by means of the principle of impulse and momentum.

Before showing how this is done we define the several types of collision and explain some simplifying assumptions that may be made.

Definitions. If the mass-centers of the two bodies are moving along the same straight line immediately before impact, the collision is called *direct;* otherwise it is called *oblique.* If the impact forces which the two bodies exert upon each other act along the line joining their mass-centers, the collision is called *central;* otherwise it is called *eccentric.*

Assumptions. In most collisions the impact forces are so great compared with any other forces acting on the bodies, such as their weights, friction, or air resistance, that such other forces may be disregarded, and it may be assumed that during the impact the two bodies jointly are not acted on by any external forces whatever. It follows that the total momentum of the system comprising the two bodies is the same immediately before and immediately after the collision.

The total kinetic energy of the system is not the same before and after impact, for some energy is always converted into heat. If the bodies are almost perfectly elastic* there is very little energy loss; if they are inelastic there will be a considerable loss of kinetic energy. But the proportional loss of kinetic energy does not depend solely upon the material of which the bodies are composed; it depends also, as will be shown presently, upon their relative masses and velocities. However, it has been learned experimentally that, in direct central collision of spheres, the velocity of separation (rate of increase of distance between the bodies after impact) is always less than the velocity of approach (rate of decrease of distance between the bodies before impact), and that the ratio of the former to the latter, called **coefficient of restitution** and here denoted by e, does depend almost solely on the material.† Approximate values of e for some materials are: glass $\frac{15}{16}$; ivory $\frac{8}{9}$; steel and cork $\frac{5}{9}$; wood $\frac{1}{2}$; clay and putty 0.

* An elastic body is one which, when moderately stretched, compressed, or otherwise deformed by applied forces will spring back or recover its original shape and dimensions when those forces are removed, and in thus recovering can do work equal to that done in deforming it. An inelastic or plastic body is one which remains deformed after the loading is removed; it recovers partially or not at all, and returns only part or none of the work done in deforming it, the rest of that work being converted into heat. Steel, glass, and ivory are examples of almost perfectly elastic material; lead and putty are examples of almost wholly plastic material.

† The collision may be regarded as comprising two phases, the first lasting from the instant of contact until the mass-centers are closest together (at which instant their relative velocity

We shall assume that e is constant for any given material or combination of materials, and, on the basis of this assumption and the assumption of constant momentum, we discuss two simple examples of impact.

DIRECT CENTRAL COLLISION. We assume the bodies to have motion of translation before collision; since the impact force on each acts through the mass-center, each body will continue to have motion of translation after collision. Let A and B denote the two bodies, m_1 and m_2 their masses, v_1 and v_2 their velocities just before impact, v_1' and v_2' their velocities just after impact. We regard these velocities as having sign, velocity in one direction along the line of motion, say to the right, being positive, and in the opposite direction negative. We further assume A to be on the negative side, that is to the left, of B. Then during approach v_1 must be algebraically greater than v_2, and the velocity of approach is $v_1 - v_2$; during separation v_2' must be algebraically greater than v_1', and the velocity of separation is $v_2' - v_1'$. Therefore $e = (v_2' - v_1')/(v_1 - v_2)$ or

$$v_2' - v_1' = e(v_1 - v_2) \tag{1}$$

Now, whatever the direction of the velocities (whether positive or negative), the momentum of the system just before impact is $m_1v_1 + m_2v_2$, and just after impact it is $m_1v_1' + m_2v_2'$, and since there is no change in total momentum

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \tag{2}$$

Equations 1 and 2 solved simultaneously for the final velocities give

$$\left.\begin{aligned} v_1' &= v_1 - (1 + e)\frac{m_2}{m_1 + m_2}(v_1 - v_2) \\ v_2' &= v_2 - (1 + e)\frac{m_1}{m_1 + m_2}(v_2 - v_1) \end{aligned}\right\} \tag{3}$$

If one of the colliding bodies, say B, is fixed, then $v_2 = 0$ and m_2 is the mass of B and its supports, infinitely great. The above equation then gives $v_1' = -ev_1$. This would be the velocity of rebound of a ball thrown against a wall or dropped upon a pavement.

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is zero) and the second lasting from then until the bodies break contact. The first phase may be called the blow; the second phase, the rebound. During the blow the bodies are being compressed and their relative velocity (velocity of approach) is being reduced to zero; during the rebound the bodies are recovering their original form and their relative velocity (velocity of separation) is being built up. The velocity of separation would be just as great as the velocity of approach if the impulse given each body during the rebound were as great as that given it during the blow, and this would be the case if the colliding bodies were perfectly elastic. Because of the imperfect elasticity of all actual bodies, however, the pressure at a given stage of the rebound is somewhat less than that at a corresponding stage of the blow, and therefore the impulse received by each body is less during the rebound than during the blow. For spheres, the ratio is practically constant for a given material, and this ratio is also the ratio between the velocities of separation and approach.

OBLIQUE CENTRAL COLLISION. Suppose the bodies A and B to be moving with motion of translation along the paths 1 and 2, with velocities V_1, V_2 when they collide as shown in Fig. 303. As in central impact, the impact force on each body acts through its mass-center and so does not cause turning. But

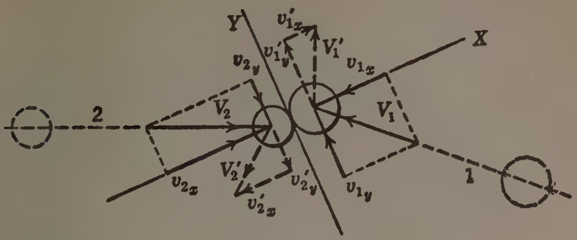

FIG. 303.

now these impact forces act, not parallel to the paths of the bodies, but along the x axis joining the mass-centers of the bodies. They therefore produce no change in the momentum of either body perpendicular to the x axis (along the y axis shown), and so there is no change in the y component of the velocity of either body, or

$$v_{1y}' = v_{1y} \qquad \text{and} \qquad v_{2y}' = v_{2y}$$

With respect to the x components of momentum exactly the same analysis applies as in direct collision, and so the x components of V_1' and V_2' are given by

$$\begin{aligned} v_{1x}' &= v_{1x} - (1 + e)\frac{m_2}{m_1 + m_2}(v_{1x} - v_{2x}) \\ v_{2x}' &= v_{2x} - (1 + e)\frac{m_1}{m_1 + m_2}(v_{2x} - v_{1x}) \end{aligned} \tag{4}$$

The final velocities V_1' and V_2' are readily determined from their x and y components.

For the oblique collision represented in Fig. 303, possible paths and velocities are indicated. The x component of the velocities after impact might be quite different from those shown, and hence the paths might be different, but the y components of the velocities after impact must be exactly the same as before impact.

In the above discussion of direct and oblique impact it has been assumed that the bodies have motion of translation. If either or each of the bodies is spinning at the instant of collision, the formulas derived for velocities will still apply to the velocities of the mass-centers of the bodies provided that the collision is central. But for spinning bodies, unless their surfaces are perfectly smooth, the impact will in general not be central. In any eccentric

impact, whether the eccentricity is due to spin or to the shape of the bodies, it is necessary to take angular momentum as well as linear momentum into account. Further discussion of eccentric impact is beyond the scope of this book.

LOSS OF KINETIC ENERGY IN IMPACT. From the equations for velocities derived above it is easy to obtain an expression for the loss of kinetic energy in impact. Thus for central oblique impact

$$\text{Loss} = [\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2] - [\tfrac{1}{2}m_1(v_1')^2 + \tfrac{1}{2}m_2(v_2')^2]$$

Now $v_1^2 = v_{1x}^2 + v_{1y}^2$, $v_2^2 = v_{2x}^2 + v_{2y}^2$, $(v_1')^2 = (v_{1x}')^2 + (v_{1y}')^2$, and $(v_2')^2 = (v_{2x}')^2 + (v_{2y}')^2$. Hence

$$\text{Loss} = \tfrac{1}{2}m_1\,[v_{1x}^2 - (v_{1x}')^2] + \tfrac{1}{2}m_2\,[v_{2x}^2 - (v_{2x}')^2]$$

Substituting for v_1' and v_2' their values from Eq. 4 and simplifying we get

$$\text{Loss} = \tfrac{1}{2}(1 - e^2)\,\frac{m_1m_2}{m_1 + m_2}\,(v_{1x} - v_{2x})^2$$

For perfectly elastic bodies $e = 1$ and the loss $= 0$. For imperfectly elastic bodies $1 - e^2$ is always a positive quantity since $e < 1$, and $v_{1x} - v_{2x}$ cannot be zero, else the bodies would never collide. Therefore the expression for loss of kinetic energy is always a positive quantity for imperfectly elastic bodies.

It is of practical interest to ascertain the factors that determine the proportional loss of kinetic energy for one important case — that of direct central impact when one body is at rest, as when a pile driver strikes a pile or a projectile strikes a stationary target. Let m_1 be the mass of the moving body and m_2 that of the stationary body; then, since, for direct collision, $v_{1x} = v_1$ and $v_{2x} = v_2$, we have for the original kinetic energy $\frac{1}{2}m_1v_1^2$, for the loss of energy $\frac{1}{2}(1 - e^2)\,\dfrac{m_1m_2}{m_1 + m_2}\,(v_1)^2$. Dividing the loss by the original energy we have

$$\text{Proportional loss of kinetic energy} = \frac{1 - e^2}{1 + m_1/m_2}$$

Obviously the proportional loss is less when the mass of the moving body is large compared with that of the stationary body. This is consistent with everyday experience and instinctive perception. For example, it is much easier to drive a post into the ground with a heavy sledge than with a light one, even if the light one is swung with such high velocity as to have just as much kinetic energy. Again, a .22 rifle bullet fired into a block of wood suspended by a string (ballistic pendulum) will hardly move it; nearly all the kinetic energy is lost. But if the bullet is suspended, and the block of wood is thrown against it with the same kinetic energy, there will be very little energy loss.

IMPACT FORCES. In the above discussion the effect of impact upon the motions of the colliding bodies was determined without determining the impact forces themselves; when the two bodies are considered jointly the impact forces

are internal forces and so can be disregarded. If, instead of considering the bodies jointly, we consider either one separately, the impulse $\int F\,dt$ of the impact force on that body is equal to the increment of momentum for that body, and, since the impact forces are equal and opposite at all instants, their impulses are also equal and opposite and give equal and opposite increments of momentum to the bodies. This is really another way of showing that the total momentum of the system is unchanged during the collision.

From the equations derived above for V_1' and V_2' it is easy to compute the increment of momentum for either body, and thus to determine the impulse of either impact force. But to calculate the maximum magnitude of that force, its duration, and the manner in which it varies, is very difficult. It is a problem in elasticity as well as dynamics, and it has been solved for a few special cases only, as impact between spheres and between prismatic bars of elastic materials. Discussion of these solutions is beyond the scope of this book, but it is possible to arrive at some conclusions about impact forces by relatively simple analysis.

Suppose we consider the familiar example of direct central collision between a moving body and a fixed body, as when a bullet strikes and is stopped by a fixed target, and limit the discussion to the first phase of the impact, that is, the blow. Now from observation we know that the duration of a blow is usually extremely short, and shorter for hard bodies than for soft; yet the impulse of the impact force, being just as great as the momentum given up by the moving body, may be very large. Therefore the *time-average* force of impact, which is defined as the impulse divided by the duration of the blow, is usually very large, and much larger for hard bodies than for soft. Again, we know from observation that in a blow of this kind the moving body is stopped within a distance which is usually very short, and shorter for hard bodies than for soft. If no work is done on the moving body by internal forces — that is, if the moving body is not deformed — the negative work done on it by the resistance of the fixed body is equal to the loss of kinetic energy, which may be large. Therefore the *space-average* force of impact, which is defined as the work done divided by the distance through which the resistance acts, is usually very large, and much larger for hard bodies than for soft. In general the time-average force and the space-average force will not be equal, but in the collision of hard bodies both are usually very large, and the maximum magnitude of an impact force may be enormous even for a blow in which very little of either momentum or kinetic energy is involved.

An understanding of even so elementary a discussion of impact forces as the above should enable one to avoid many common errors and misconceptions. Usually when the term "force of a blow" or "power of a blow" is used the meaning is vague. In referring to a moving body striking and being stopped by a stationary body, one can definitely and correctly speak of (i) the momentum of the blow, meaning the momentum of the moving body, or of (ii) the energy of a blow, meaning the kinetic energy of the moving body, or of (iii) the impulse

of the blow, meaning the impulse of the impact force exerted on the stationary body. In speaking of the force of the blow it is necessary, in order to be definite, to specify whether the maximum or average force is meant and, if average, whether the time-average or space-average. Correctly used, "power of a blow" can mean only the time rate at which the impact force exerted on the stationary body does work (if the stationary body is immovable and rigid this rate is zero), and here again it must be specified whether the maximum rate or time-average rate is meant. It is important to understand that the maximum force of a blow, or either one of the average values of an impact force, depends largely on the material of the colliding bodies. A child can produce a greater impact force by tapping on an anvil with a tack hammer than the strongest boxer can produce by striking the body of an opponent with his gloved fist, in spite of the fact that the tack hammer has very little of either momentum or kinetic energy.

A very common misconception among big-game hunters and even among many military men is that a large-caliber rifle or even pistol bullet has great "stopping" or "knocking down" effect merely by virtue of the mechanical effect of its impact, quite apart from any physiological effect. Thus, in Theodore Roosevelt's *African Game Trails*, this statement is made concerning a rhinoceros the author shot: "I fired for the chest, and the heavy Holland bullet knocked it clean off its feet," and again, with reference to a charging lion which was shot with a lighter rifle, "The blow brought him up all standing, and he fell forward on his head." It used to be a common saying among older army men that the .45 caliber pistol was adopted because it was capable of "stopping" or "knocking down" a man "by sheer impact." Now the forward momentum of any bullet can be no greater than the rearward momentum of the recoiling gun from which it is fired (actually it is less by the amount of momentum of the powder gas), and so the bullet can exert on the object it strikes no greater impulse than the gun exerts on the man who shoots it. Since this impulse is not great enough to upset the man, it is hardly reasonable to suppose that it would knock a rhinoceros weighing two or three tons "clean off its feet." For Mr. Roosevelt's bullet to have actually stopped a charging lion by impact, it would have been necessary for the bullet to have at least as much momentum as the lion, whereas it actually had less than one-hundredth as much. The momentum of a .45 caliber pistol bullet is less than that of a baseball thrown by a good pitcher, though the bullet has far more kinetic energy. Misstatements such as those quoted spring from misinterpretation of observed happenings; an animal stops charging or falls down suddenly because it is hurt, and reflex muscular action may make either act so sudden that it seems to be the result of the impact alone.

CHAPTER XV

THREE-DIMENSIONAL (OR SOLID) MOTION OF A RIGID BODY

170. Introductory. In this chapter, our principal purpose is to show how to determine the forces acting on a rigid body having a given three-dimensional motion. For that purpose we employ d'Alembert's principle. Generally the resultant effective system in any particular case is regarded as consisting of a force R, acting at a chosen point, and a couple C. As shown in Arts. 132 and 163, when the origin of coordinates is at the chosen point of application of R,

$$R_x = m\bar{a}_x \qquad R_y = m\bar{a}_y \qquad \text{and} \qquad R_z = m\bar{a}_z \tag{1}$$

and

$$C_x = \frac{dh_x}{dt} \qquad C_y = \frac{dh_y}{dt} \qquad C_z = \frac{dh_z}{dt} \tag{2}$$

The magnitude and direction of R do not depend in any way on the position of the point of application or on the direction of chosen coordinate axes; the couple C does depend on the position of the point. In general it is advantageous to choose the center of gravity of the moving body under discussion as the point of application of R and to use the notation $\bar{R}$ and $\bar{C}$. However, in the special motion called spherical (defined in Art. 171), it is generally advantageous to choose the fixed point O of the moving body as the point of application of R and to use the notation $\mathring{R}$ and $\mathring{C}$.

It is necessary to discuss next some preliminaries, namely: kinematics of solid motion and methods for calculating the rates in Eq. 2.

A. Spherical Motion

171. Definitions and Illustrations. A motion such that one point of the moving (rigid) body, or of an imagined rigid extension of it, is stationary or fixed is called *spherical*. For example, consider a cane supplied with a sharp spike in its lower end. If the spike is pressed lightly into a floor and the upper end moved about in any way, the motion of the cane is spherical. For another example, consider the gyroscope represented in Fig. 333*b*; axes A, B, and C intersect in a single point O. Suppose that the wheel is spinning and gimbal G_1 is turning about axis B and/or gimbal G_2 is turning about axis C. The wheel has a single "fixed point" (imaginary) at O; the motion of the wheel is spherical.

The name spherical motion derives from the fact that in such a motion any

point P of the body moves on the surface of an imaginary sphere. The center of the sphere is at the fixed point O and the radius of the sphere is OP.

ILLUSTRATION 1. Figure 304 represents a rotor consisting of a wheel and axle. It is supported in bearings in the frame F and is prevented from sliding lengthwise in the bearings by suitable stops; the frame is supported in fixed bearings in G; the axes S and P intersect. When the frame is rotated, the wheel rolls on a conical floor and travels around axis P.

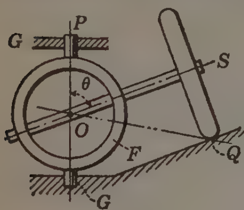

FIG. 304.

The intersection O is a single fixed point of the rotor, and so the motion of the rotor is spherical. The point Q of the wheel in contact with the floor has no velocity, nor has any point of the rotor, in the line OQ; the line OQ is the instantaneous axis of the motion at the instant when the wheel is where shown. All such points Q form a circle on the wheel; the lines joining O with all points Q form a cone, fixed in the rotor. Let Q' be a point of the floor, coinciding with Q. All such points as Q' form a circle on the floor, and lines joining O with all points Q' form another cone, fixed in space.

This motion of the rotor can be looked upon as a combination of two component rotations, namely, one about the moving axis OS and the other about the fixed axis OP. The first rotation is relative to the frame or to the plane SOP and is called *spin* of the rotor. The second rotation is relative to the supports G and is called *precession* of the rotor; it is just like the rotation of the axis of spin OS about the axis of precession OP. Since the bearings in G are fixed with respect to the earth, the precessional motion is relative to the earth.

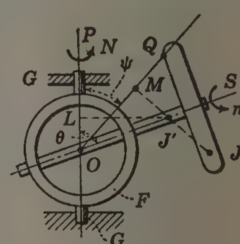

FIG. 305.

ILLUSTRATION 2. The device represented in Fig. 305 is like the one of Fig. 304 except that here there is no floor under the wheel; the spin and precession are independent. Suppose that the rotor is spinning and precessing in the directions indicated by the curved arrows marked n and N, and let these letters respectively denote the angular velocities of spin and precession at a given instant.

The instantaneous axis is determined, as in the preceding illustration, by the fixed point O and any other point of the rotor whose velocity is zero. Here the second point is not apparent; it must be searched for. We note first that the velocity due to n only and the velocity due to N only of any point of the rotor in the plane of the axes S and P are normal to that plane; and that these velocities are oppositely directed if the point is in the angle SOP or in its opposite. If Q is a point for which the two component velocities are not only opposite but also equal, then the resultant velocity is zero and Q is the point sought. To locate OQ, let θ and ψ denote angles as indicated, and v_1 and v_2

the velocities of Q due to n and N, respectively. Then

$$v_1 = OQ \sin(\theta - \psi) \cdot n \quad \text{and} \quad v_2 = OQ \sin \psi \cdot N$$

Equating these expressions and solving for ψ you find

$$\tan \psi = \frac{n \sin \theta}{N + n \cos \theta} \tag{1}$$

The *state of the motion* of the body (rotor) at the given instant is one of rotation about OQ. Just as in a rotation about a fixed axis, so here the velocities of various points of the body at this instant are proportional to their distances from the instantaneous axis, and the velocity of any point of the body at the given instant is perpendicular to the plane of the point and the axis. By angular velocity of the body at a given instant is meant the rate at which any line fixed in the body and then perpendicular to the axis is turning. Let JM be such a line but chosen for simplicity so that M is on the axis at the instant. The rate of turning is v/r, where v denotes velocity of J and r the length JM.

A formula for the angular velocity ω can be arrived at readily by taking J at J', on the axis of spin (see Fig. 305); the velocity of such a point is due to precession only. Thus for J'

$$v = (OJ') \sin \theta \cdot N \qquad r = (OJ') \sin(\theta - \psi)$$

hence

$$\omega = \frac{(OJ') \sin \theta \cdot N}{(OJ') \sin(\theta - \psi)} = \frac{\sin \theta}{\sin(\theta - \psi)} N \tag{2}$$

172. Composition of Velocities of Spin and Precession. Let OS and OP of Fig. 306 correspond precisely to OS and OP of Fig. 305, and let vectors Oa and Ob represent the angular velocities of spin and precession, n and N of Art. 171. Then the vector Oc, which is $Oa \leftrightarrow Ob$, represents the angular velocity Ω of the spherical motion of the rotor. For

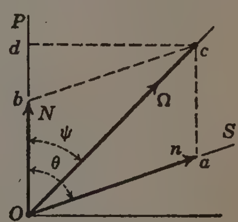

FIG. 306.

$$\tan bOc = \frac{cd}{Ob + bd} = \frac{n \sin \theta}{N + n \cos \theta}$$

which agrees with Eq. 1 of Art. 171 and therefore shows that angle bOc is equal to ψ of Fig. 305; hence the line Oc is the instantaneous axis. Also $Oc \sin(\theta - \psi) = Ob \sin \theta$, or

$$Oc = \frac{\sin \theta}{\sin(\theta - \psi)} Ob$$

which agrees with Eq. 2 of Art. 171 and therefore shows that the length Oc represents ω, the magnitude of the angular velocity of the spherical motion. Moreover, the arrowhead on Oc indicates the sense of Ω correctly. That is, the

parallelogram construction described yields the instantaneous axis and the magnitude and sense of the angular velocity of a spherical motion due to spin and precession.

Note that it was not assumed in the foregoing that the velocities compounded are constant, only that both pertain to the same instant; we are concerned with a state of velocities.

Obviously, this parallelogram construction can be extended to the case where the frame F turns in bearings in another frame F' which can turn in bearings in a fixed support, provided that the axes of the three sets of bearings intersect at a single point. The result of the first construction is the angular velocity of the rotor relative to F', and the result of the second construction is the absolute angular velocity of the rotor.

173. Spherical Motion as Rotation about a Moving Axis. Any spherical motion may be looked upon as a continuous rotation about a continuous succession of lines or instantaneous axes, as proved below. Each axis may be looked upon as a line fixed in the body or fixed in space. As in Art. 171, the successions of lines in the body and in space form surfaces, called *body cone* and *space cone*, respectively. The motion may be looked upon also as a continuous rotation about a moving line which coincides at each instant with one of the fixed lines just mentioned. The instantaneous axis is said to move about in the body and in space.

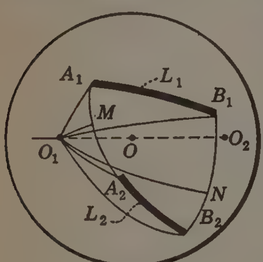

FIG. 307.

Our proof is much like that for an analogous proposition relating to plane motion (see Art. 123). The moving body employed for proof here is a strip-like lamina AB (Fig. 307) fitted to the surface of a sphere on which the lamina can be moved about. Obviously the motion is spherical; the fixed point is at the center of the sphere.

We show first that any displacement of the lamina could be accomplished by a rotation about a line through the fixed point. The two heavy lines represent the lamina L in its initial and final positions, L_1 and L_2. Other curves represent arcs of great circles on the sphere; M and N are midpoints of the arcs A_1A_2 and B_1B_2; MO_1 and NO_1 are normals to the arcs respectively; they determine O_1 and, if extended, the diametrically opposite point O_2. Now A can be displaced from A_1 to A_2 by a rotation about O_1O_2 through an angle $A_1O_1A_2$, and B can be displaced from B_1 to B_2 by a rotation about O_1O_2 through the angle $B_1O_1B_2$. Since these angles are equal, a single rotation about O_1O_2 would displace A and B, and hence the lamina, from the initial to the final position.

Now imagine any motion of the lamina on the sphere which would displace the lamina from the initial to the final position. Let t be the time for the displacement, and consider the displacement to be made up of many small displacements corresponding to intervals $(\Delta t)_1$, $(\Delta t)_2$, $(\Delta t)_3$, etc., which comprise

the whole interval. Each small displacement could be accomplished by a rotation about a line through the fixed point. Imagine all such rotations to be made in continuous succession and each in its proper interval Δt. Obviously, such a succession of rotations would approximately duplicate the imagined motion. "In the limit," the succession *would* duplicate that motion. Thus any spherical motion may be looked upon as a continuous rotation about a continuous succession of axes, all passing through the fixed point.

Visualization of angular velocity in spherical motion. In general, to an observer, a body having spherical motion would appear to be turning about in some more or less complicated manner. Its apparent rate of turn would depend on the observer's point of view, somewhat as in the simpler case of a spoked wheel rotating about a fixed axis, say horizontal. If viewed obliquely, not along the axis, some of the spokes appear to be turning more rapidly than others; and if viewed edgewise, from a point in the plane of the wheel, the spokes do not seem to be turning at all, since they appear at all times vertical. But by looking along the axis, so that the spokes turn in a plane normal to the line of vision, one gets a true perception of the angular velocity.

To achieve a corresponding visualization of the angular velocity in spherical motion one may resort to some such concept or device as this: Let Fig. 308 represent a body of any form having a spherical motion with O as fixed point, and imagine the body to be transparent and to have imbedded in it two rods, 1 and 2, mutually perpendicular and intersecting at O. Assume that at some instant rod 1 coincides with the instantaneous axis OQ which in turn coincides with the fixed line OQ'. The rate of turn of rod 2 at that instant is the angular velocity of the body then; and an observer stationed at any point on OQ or OQ' would perceive this velocity correctly.

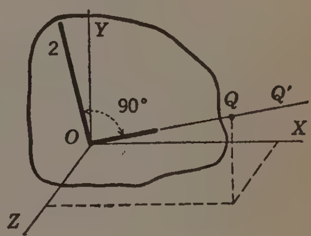

FIG. 308.

174. Rectangular Components of Angular Velocity. Let α, β, and γ (Fig. 309) be the direction-angles of the instantaneous axis OQ, of a moving body not shown, with respect to a set of axes O-XYZ fixed in space. Then the x, y, and z components of Ω are respectively

$$\omega_x = \omega \cos \alpha \qquad \omega_y = \omega \cos \beta \qquad \omega_z = \omega \cos \gamma$$

We now show that the visualized angular velocity about the x axis, say, is equal to $\omega \cos \alpha$. Suppose that the rod device of the preceding article is at some instant where indicated in the figure, rod 1 on OX and rod 2 on OY. The lower end of 2 is fixed; the other end P appears to be moving across the line of vision of an observer stationed anywhere on OX. The rate of turn of rod 2 as observed by him at that instant is the angular velocity of the body about OX. Now to calculate this rate: Since the velocity V of P is perpendicular to the plane

POQ, which is vertical, V is horizontal, and so $v_y = 0$; hence, *the angular velocity about* OX is $v_z \div OP$. Now $v = (PM)\omega$, where M is the projection of P on OQ; hence

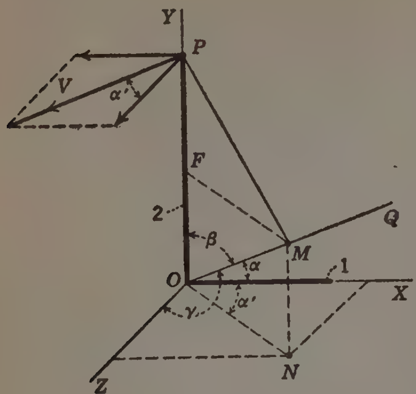

FIG. 309.

$$\frac{v_z}{OP} = \frac{(PM)\omega \cos \alpha'}{OP} = \omega \sin \beta \cos \alpha'$$

Since $\sin \beta = FM \div OM$ also, and $\cos \alpha' = OX \div ON$, the angular velocity about OX is

$$\omega \frac{FM}{OM}\frac{OX}{ON} = \omega \frac{OX}{OM} = \omega \cos \alpha$$

Hence, one may regard ω_x as standing not only for the magnitude of the x component of the angular velocity but also for the magnitude of the angular velocity about the x axis.

EXAMPLE. O-XYZ, Fig. 310a, is a fixed frame of axes; O is also the fixed point of a rotor spinning about OS and precessing about OP, which coincides with OY. O-ABC is a moving frame; OB coincides with OY, and OA is the projection of OS on the xy plane. (This frame rotates with the plane SOP.) O-123 is another moving frame; $O1$ coincides with OS, and $O2$ is in the plane SOP. (This frame also rotates with the plane SOP.) The angle ϕ locates the plane SOP with respect to the fixed frame, and θ locates OS in SOP. The rotor is supposed to be homogeneous and symmetrical with respect to OS, which is therefore a principal axis of the rotor at or for O (see Appendix B); moreover, its moments of inertia about all perpendiculars to OS at O are equal; $O1$, $O2$, and $O3$ are principal axes of the rotor at O.

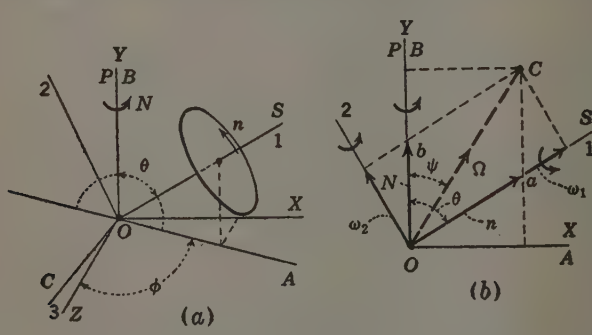

FIG. 310.

Given the angular velocities of spin and precession, n and N, respectively, with directions as indicated, and the angle θ. Required the components about the principal axes of the angular velocity Ω of the rotor.

Solution. Figure 310b represents various lines in the plane SOP of Fig. 310a in their true relations. Let Oa and Ob represent n and N, respectively; then the diagonal Oc of the par-

allelogram $Oacb$ represents the angular velocity Ω. The projections of Oc on the principal axes respectively represent ω_1, ω_2, and ω_3. You see readily from the figure that

$$\omega_1 = n + N\cos\theta \qquad \omega_2 = N\sin\theta \qquad \omega_3 = 0$$

175. Angular Momentums about Principal Axes. In a spherical motion the angular momentums of the moving body about its principal axes through the fixed point are given by the following simple formulas (proved below)

$$h_1 = I_1\omega_1 \qquad h_2 = I_2\omega_2 \qquad h_3 = I_3\omega_3 \tag{1}$$

where the h's denote angular momentum, the I's moment of inertia, and the ω's angular velocity, all about the principal axes, respectively. The h and ω in any one of these equations have the same sign or sense.

Proof. Preliminary to the main proof, we develop some necessary expressions for the axial components of the velocity of any particle P of the moving body; see Fig. 311, where O is the fixed point of the moving body not shown; $x, y,$ and z are fixed axes. Imagine the indicated parallelepiped to be a portion of the moving body. Let V denote the velocity of P at the instant when the moving parallelepiped is where shown, and V_1, V_2, and V_3 the then velocities of the corners P_1, P_2, and P_3, respectively. Also let L, L_1, L_2, and L_3 denote the "position vectors" OP, OP_1, OP_2, and OP_3, all constant in magnitude during the entire motion. At each instant $L = L_1 \nrightarrow L_2 \nrightarrow L_3$; hence

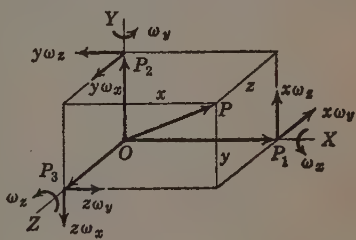

FIG. 311.

$$\frac{dL}{dt} = \frac{dL_1}{dt} \nrightarrow \frac{dL_2}{dt} \nrightarrow \frac{dL_3}{dt}$$

or

$$V = V_1 \nrightarrow V_2 \nrightarrow V_3$$

Hence also

$$V_x = (V_1)_x \nrightarrow (V_2)_x \nrightarrow (V_3)_x$$

The corners P_1, P_2, and P_3, respectively, have no x, y, z component velocities; each corner has, in general, two components as indicated by vectors. Hence

$$v_x = +z\omega_y - y\omega_z \qquad v_y = +x\omega_z - z\omega_x \qquad v_z = +y\omega_x - x\omega_y$$

For the main proof, see Fig. 312, where O is the fixed point of the moving body, not shown. P is any particle of the body, and O-123 is the position of the set of principal axes (of the body) at or for O, at a particular or given instant. (The principal axes are fixed in the body; they move with it.) O-XYZ is a set of axes fixed in space and chosen in advance so that the principal axes coincide

with them at the given instant. From Art. 161 or from Fig. 312, you see that

$$h_x = \int dm \cdot (v_z y - v_y z)$$

Note that the velocities and coordinates in the above integrals are relative to the fixed axes, that is absolute. Substituting for v_y and v_z here their values above gives

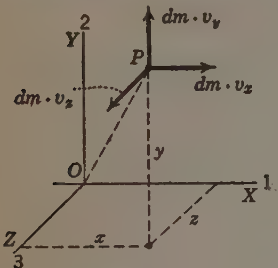

FIG. 312.

$$h_x = \int dm[(y\omega_x - x\omega_y)y - (x\omega_z - z\omega_x)z]$$

$$= \omega_x \int dm(y^2 + z^2) - \omega_y \int dm \cdot xy - \omega_z \int dm \cdot zx$$

Now the first integral in the preceding line is I_x; the other two are K_{xy} and K_{zx}, respectively. Since the x, y, and z axes coincide with the principal axes, K_{xy} and $K_{zx} = 0$, and so $h_x = I_x\omega_x$. And, since h_x, I_x, and ω_x are respectively equal to h_1, I_1, and ω_1, we may write $h_1 = I_1\omega_1$, which is preferable to $h_x = I_x\omega_x$, since the former notation serves to remind one that all symbols refer to a principal axis.

176. Vector Representation of Angular Momentum; Maximum Angular Momentum. In some discussions that follow, angular momentums will be represented by vectors laid off on the lines respectively to which they pertain. If h_x, say, is positive, the arrow on the vector points in the positive x direction. H_x, H_1, etc., will be used to denote angular momentums as vector quantities, about axes or lines x, 1, etc.; and, as heretofore, h_x, h_1, etc., will denote the magnitudes of H_x, H_1, etc.

Obviously, the angular momentums of the moving body about various lines through the fixed point are, at any particular instant, unequal. There is one such line for which the angular momentum is maximum; this momentum will be denoted by H_m, and the magnitude of H_m by h_m.

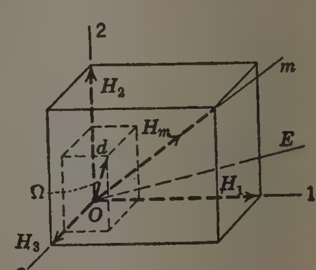

FIG. 313.

Angular momentum of a body, or the moment of the system of momentums of all particles comprising the body, is strictly analogous to moment of a system of forces. Hence you may find H_m in a given problem in the manner explained in Art. 32 for finding the maximum moment of a system of forces. You have only to sum the vectors H_1, H_2, and H_3; the sum is H_m; see Fig. 313 for this summation. The line Om about which H is maximum does not in general coincide with the instantaneous axis Od. It can be shown that they coincide only if $I_1 = I_2 = I_3$.

The angular momentum H_e about any line OE is obviously represented by the projection of H_m on OE; also by the sum of the projections on OE of any three rectangular components of H_m.

EXAMPLE. To illustrate composition and resolution of angular momentums, we use the rotor of Fig. 310, reproduced with some omissions and additions as Fig. 314. Since $h_1 = I_1\omega_1$, $h_2 = I_2\omega_2$, and $h_3 = I_3\omega_3$, you see from the example of Art. 174 that

$$h_1 = I_1(n + N\cos\theta) \qquad h_2 = I_2N\sin\theta$$
$$h_3 = 0 \tag{1}$$

See the figure for H_1 and H_2; notice that the pointing of these vectors agrees with the senses of ω_1 and ω_2. Vector OT represents H_m.

Let it be required to find h_a, h_b, and h_c. They may be found from H_m, but more simply from H_1, H_2, and H_3. Thus you see that

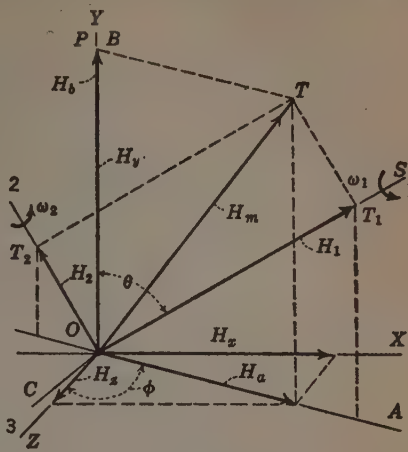

FIG. 314.

$$\left.\begin{aligned} h_a &= h_1\sin\theta - h_2\cos\theta = I_1n\sin\theta + \tfrac{1}{2}(I_1 - I_2)N\sin 2\theta \\ h_b &= h_1\cos\theta + h_2\sin\theta = I_1n\cos\theta + (I_1\cos^2\theta + I_2\sin^2\theta)N \\ h_c &= 0 \end{aligned}\right\} \tag{2}$$

Components h_x, h_y, and h_z may be found readily from H_a, H_b, and H_c.

$$h_x = h_a\sin\phi \qquad h_y = h_b \qquad h_z = h_a\cos\phi \tag{3}$$

177. Rate of Change of Angular Momentum. Two methods for calculating the rates dh_x/dt, dh_y/dt, and dh_z/dt, which are values of C_x, C_y, and C_z, respectively (Art. 170), are explained in this article. (Note that the axes x, y, and z contain the fixed point of the moving body, and are fixed in direction.) The first method employs differentiation of scalar quantities (ordinary differentiation); the second method employs differentiation of vector quantities.

SCALAR DIFFERENTIATION METHOD. One first sets up general expressions for h_x, h_y, and h_z which hold for the entire motion, not merely for a particular instant or position of the moving body, and then differentiates these expressions with respect to time and arrives at general expressions for the desired rates. From these expressions one can calculate the rates for any given instant or position. In a given numerical example, dh_x/dt, say, has sign and can be represented by a vector on the x axis or on a line parallel thereto. If the rate is positive (negative) the vector points in the positive (negative) x direction. See Fig. 315, where OA represents a positive dh_x/dt. And, since $dh_x/dt = C_x$, the vector represents the couple C_x, the sense of which is indicated by the curved arrow also.

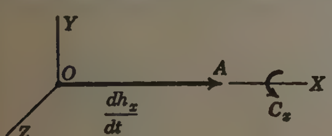

FIG. 315.

Vector differentiation method. Since angular momentum is a vector quantity, its rate of change also is a vector quantity; and this rate is represented by the "velocity" of the tip of the vector drawn from a fixed point to represent the (varying) angular momentum (see Art. 98). Thus, in Fig. 316, vector OT represents H_m; and the velocity of T, the tip of this vector, represents dH_m/dt. Also if T_x, T_y, T_z are tips of vectors representing components of H_m about fixed axes O-XYZ, the velocities of these tips are the rates of change of h_x, h_y and h_z, respectively. But the velocity of T_x is the x component of the velocity of T, or $dh_x/dt = (dH_m/dt)_x$. And so the problem of determining the rate of change of angular momentum about any fixed axis becomes the problem of determining the component, along that axis, of the velocity of the vector tip T. We now describe several ways of doing this. In each way dh_x/dt, say, is found from one or more vectors that give not only the magnitude but also the sense of this rate and therefore of C_x.

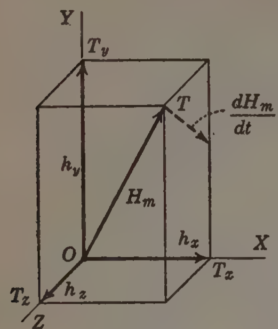

FIG. 316.

1. Solution from the motion of the vector for H_m. If the spherical motion is such that the H_m vector moves in some simple and readily discernible way, the component of the velocity of the tip T along any axis can be found directly. This is especially easy when the vector changes in direction only, as in Ex. 1 below.

2. Solution from the motion of the vectors of components of H_m. It is not practical to describe this solution prior to its development, which we now give. The frame O-XYZ (Fig. 317) represents, as before, fixed axes intersecting at the fixed point O of the body (not shown) having spherical motion, and OT is the vector for H_m. O-ABC is a set of rectangular axes having any arbitrary spherical motion, with fixed point O, and T_a, T_b, T_c are tips of vectors representing the components of H_m about these moving axes. Then at all times $H_m = H_a \mathrel{+\!\!\!\!\rightarrow} H_b \mathrel{+\!\!\!\!\rightarrow} H_c$, and

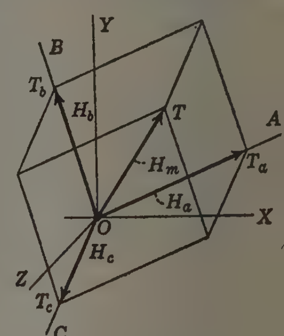

FIG. 317.

$$\frac{dH_m}{dt} = \frac{dH_a}{dt} \mathrel{+\!\!\!\!\rightarrow} \frac{dH_b}{dt} \mathrel{+\!\!\!\!\rightarrow} \frac{dH_c}{dt}$$

$$\left(\frac{dH_m}{dt}\right)_x \quad \text{or} \quad \frac{dh_x}{dt} = \left(\frac{dH_a}{dt}\right)_x + \left(\frac{dH_b}{dt}\right)_x + \left(\frac{dH_c}{dt}\right)_x$$

That is, the x component of the velocity of T (which represents dh_x/dt) is equa

to the algebraic sum of the x components of the velocities of T_a, T_b, and T_c. This sum can be found readily if the two sets of axes (O-XYZ and O-ABC) are so chosen that they coincide at the instant for which the rates dh_x/dt, dh_y/dt, and dh_z/dt are to be determined, as will now be shown. Figure 318 represents the two sets of axes at coincidence; frame O-ABC is moving with an angular velocity whose components are ω_x', ω_y', and ω_z'. Now the velocity of T_a is the vector sum of its velocity relative to the line OA and the velocity of the coincident point on OA (Art. 126). The velocity of T_a relative to OA is the rate of change of length of vector H_a, or dh_a/dt. The coincident point on OA has a y velocity $h_a\omega_z'$ due to the rotation ω_z', and a z velocity $h_a\omega_y'$ due to the rotation ω_y'; T_a has therefore a velocity made up of the three components indicated. Similarly T_b and T_c have velocities made up of the components indicated. The component of the velocity of T along any one of the fixed axes is now readily found by summing the component velocities of T_a, T_b, and T_c along that axis; thus

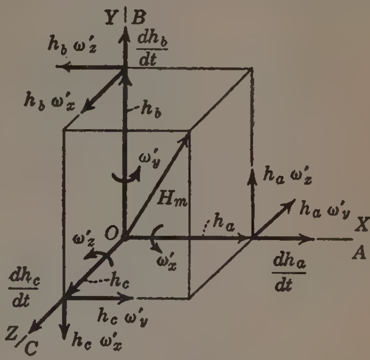

FIG. 318.

$$\frac{dh_x}{dt} = \frac{dh_a}{dt} - h_b\omega_z' + h_c\omega_y'$$

The axes O-ABC are arbitrarily chosen. If they are taken as fixed in the body, and moving with it, then $\omega_x' = \omega_x$, $\omega_y' = \omega_y$, $\omega_z' = \omega_z$. Note that the figure was drawn for positive ω's and h's; also that the two coordinate frames coincide precisely; that is, the positive directions along any two coinciding axes are the same.

EXAMPLE 1. Suppose that the rotor described in the example of Art. 174 is moving with constant velocities of spin and precession (n and N) and constant angle θ, between axes of spin and precession. It is required to find the rates dh_x/dt, dh_y/dt, and dh_z/dt for the instant when the axis of spin is passing through the xy plane ($\phi = 90°$).

1. *Solution by scalar differentiation.* From Eq. 3, Art. 176,

$$h_x = h_a \sin\phi \qquad h_y = h_b \qquad \text{and} \qquad h_z = h_a \cos\phi$$

Since n, N, and θ are constant, h_a and h_b are constant (see Eq. 2 of Art. 176); and since $d\phi/dt = N$

$$\frac{dh_x}{dt} = h_aN\cos\phi \qquad \frac{dh_y}{dt} = 0 \qquad \frac{dh_z}{dt} = -h_aN\sin\phi$$

When the axis of spin is passing through the xy plane ($\phi = 90°$),

$$\frac{dh_x}{dt} = 0 \qquad \frac{dh_y}{dt} = 0 \qquad \frac{dh_z}{dt} = -h_aN \qquad (1)$$

Substituting for h_a its value from Eq. 2, Art. 176,

$$\frac{dh_z}{dt} = -I_1nN \sin\theta - \tfrac{1}{2}(I_1 - I_2)N^2 \sin 2\theta$$

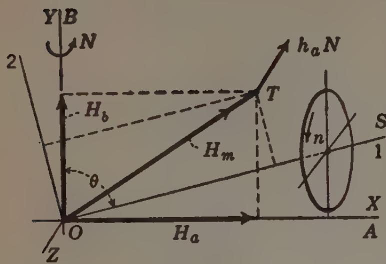

FIG. 319.

These are the required results.

If the axes of spin and precession are at right angles, $\theta = 90°$ and $dh_z/dt = -I_1nN$.

2. *Solution by vector differentiation of H_m.* Figure 319 represents the rotor, various vectors, etc., for the instant when the axis of spin is passing through the xy plane. During the entire motion, the vector H_m is rotating about the axis of precession; and, since h_m is constant, T is describing a horizontal circle, radius $= h_a$. The velocity of T is equal to h_aN, and is directed, at the given instant, as indicated (parallel to the z axis). The x, y, and z components of this velocity are 0, 0, and $-h_aN$. Hence, as in solution 1,

$$\frac{dh_x}{dt} = 0 \qquad \frac{dh_y}{dt} = 0 \qquad \frac{dh_z}{dt} = -h_aN \tag{2}$$

EXAMPLE 2. It is required to solve Ex. 1 supposing that N is not constant, the precession being hastened or retarded. Two solutions by differentiation of components of H_m follow.

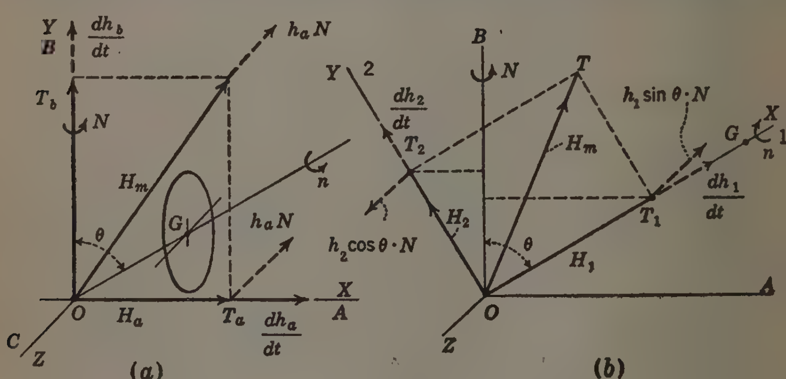

FIG. 320.

1. With x, y, and z axes as in Fig. 320a: The velocity of T_a has a transvese component h_aN and a radial component (along OA) equal to dh_a/dt. The velocity of T_b has a radial component only (along OB) equal to dh_b/dt. From these three represented components you see that

$$\frac{dh_x}{dt} = \frac{dh_a}{dt} \qquad \frac{dh_y}{dt} = \frac{dh_b}{dt} \qquad \frac{dh_z}{dt} = -h_aN$$

From the example in Art. 176, $h_a = h_1 \sin\theta - h_2 \cos\theta$; also

$$h_1 = I_1(n + N\cos\theta) \qquad \text{and} \qquad h_2 = I_2N \sin\theta$$

The last two equations show that h_1 and h_2 vary with N, and so

$$\frac{dh_a}{dt} = \frac{dh_1}{dt}\sin\theta - \frac{dh_2}{dt}\cos\theta = \left(I_1\frac{dN}{dt}\cos\theta\right)\sin\theta - \left(I_2\frac{dN}{dt}\sin\theta\right)\cos\theta$$

hence,

$$\frac{dh_x}{dt} = \tfrac{1}{2}(I_1 - I_2)\sin 2\theta\,\frac{dN}{dt} \tag{1}$$

Similarly, you find

$$\frac{dh_y}{dt} = (I_1\cos^2\theta + I_2\sin^2\theta)\frac{dN}{dt} \tag{2}$$

and

$$\frac{dh_z}{dt} = -I_1nN\sin\theta - \tfrac{1}{2}(I_1 - I_2)N^2\sin 2\theta \tag{3}$$

2. With x, y, and z axes as in Fig. 320b: The velocity of T_1 has a transverse component $(h_1 \sin\theta)N$ parallel to OZ, and a radial component dh_1/dt along $O1$. The velocity of T_2 has a transverse component $(h_2\cos\theta)N$ parallel to OZ, and a radial component dh_2/dt along $O2$. From these four represented components, you see that

$$\frac{dh_x}{dt} = \frac{dh_1}{dt} = I_1\frac{dN}{dt}\cos\theta \qquad \frac{dh_y}{dt} = \frac{dh_2}{dt} = I_2\frac{dN}{dt}\sin\theta \qquad \frac{dh_z}{dt} = -h_1\sin\theta\cdot N + h_2\cos\theta\cdot N \tag{4, 5, 6}$$

Equations 4, 5, and 6 respectively can be transformed into Eqs. 1, 2, and 3.

178. Inertia Systems (R'-C') Represented. In the preceding article it is shown how to calculate the rate of change of the angular momentum of a body, in spherical motion, about given or chosen x, y, and z axes. In Art. 170 it was shown that these rates respectively give the couples C_x, C_y, and C_z of the effective system. Hence the rates with signs changed give the couples C_x', C_y', and C_z' of the reversed effective or inertia system. We now show how to represent these inertia couples in a few examples that follow. In the figures thus formed we represent also the inertia forces R_x', R_y', and R_z', and thus the entire inertia system is represented.

EXAMPLE 1. It is required to represent the inertia system for the rotor of Ex. 1, Art. 177, with these data: OG (Fig. 321) = 1 ft, $\theta = 60°$, $m = 1.6$ slugs; $I_1 = 3$ and $I_2 = 2$ slug-ft²; $n = 4$ and $N = 2$ rad/sec.

From the example referred to above,

$$\frac{dh_x}{dt} = 0 \qquad \frac{dh_y}{dt} = 0 \qquad \frac{dh_z}{dt} = -h_aN$$

From the figure, it is plain that

$$\omega_1 = n + N\cos\theta = 5$$
$$\omega_2 = N\sin\theta = 1.732 \text{ rad/sec}$$

hence

$$h_1 = I_1\omega_1 = 15 \qquad h_2 = I_2\omega_2 = 3.464$$
$$h_a = h_1\sin\theta - h_2\cos\theta = 11.25$$

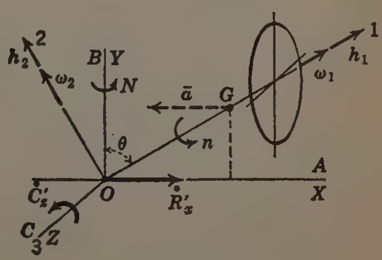

FIG. 321.

Finally,

$$\frac{dh_z}{dt} = -(11.25 \times 2) = -22.5 \text{ ft-lb} \qquad \mathring{C}_x' = 0 \qquad \mathring{C}_y' = 0 \qquad \mathring{C}_z' = +22.5 \text{ ft-lb}$$

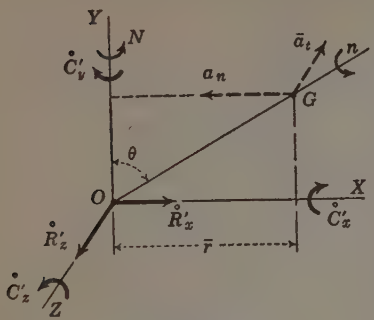

FIG. 322.

The inertia couple has, you see, only one axial component. The acceleration $\bar{a}$ of the center of gravity is directed as indicated; hence $\mathring{R}_x' = m\bar{a}_x = 1.6(OG \times \sin\theta)N^2 = 5.55$ lb. $\mathring{R}_y'$ and $\mathring{R}_z' = 0$. So the inertia system as represented consists of a single force and a single couple.

EXAMPLE 2. It is required to represent the inertia system for the rotor of Ex. 2, Art. 177, with data as above, except that the velocity of precession when it equals 2 is increasing at a rate of 10 rad/sec/sec. See Fig. 322, which corresponds to Fig. 320*a*.

For the given data, Eqs. 1, 2, and 3 of the example just referred to give

$$\frac{dh_x}{dt} = +4.33 \qquad \frac{dh_y}{dt} = +22.5 \qquad \frac{dh_z}{dt} = -22.5$$

Hence the inertia couples are

$$\mathring{C}_x' = -4.33 \qquad \mathring{C}_y' = -22.5 \qquad \mathring{C}_z' = +22.5 \text{ ft-lb}$$

The acceleration of the center of gravity has two components

$$\bar{a}_n = \bar{r}N^2 = 3.46 \qquad \text{and} \qquad \bar{a}_t = \bar{r}\frac{dN}{dt} = 8.67 \text{ ft/sec}^2$$

$$\mathring{R}_x' = 1.6 \times 3.46 = 5.55 \qquad \mathring{R}_y' = 0 \qquad \mathring{R}_z' = 1.6 \times 8.67 = 13.9 \text{ lb}$$

Thus the inertia system, as represented, consists of two forces and three couples.

179. Determination of Reactions on a Rotor with Spherical Motion. For a given state of motion, proceed as follows: (i) determine the inertia system; (ii) make a fbd of the rotor showing the d'Alembert system (inertia and external systems); (iii) solve that system for the desired unknowns.

EXAMPLE 1. The wheel and axle (Fig. 323*a*) are rigidly fastened together; the axle extends through a suitable head on the vertical shaft which can turn freely in the pedestal support; a nut on the axle prevents motion of the axle in the direction *OG*. The wheel is made to roll in a circular path by means of a horizontal push perpendicular to the axle at *A*. The wheel weighs 200 lb; its diameter is 6 ft; its radius of gyration with respect to the axis of the axle is 2 ft and with respect to a diameter 1.5 ft. The coefficient of rolling resistance (Art. 70) is 0.1 ft. The rolling is uniform and such that the axle makes 30 rev/min about *OY*. It is required to find the forces acting on the rotor (wheel and axle) at *O*, *A*, and *B*.

Solution. For simplicity we disregard the mass of the axle so that the center of gravity of the rotor is at *G*, the center of the wheel. We disregard also all friction *on the shaft* and its inertia; then the shaft exerts only a single force *P*, say, on the axle. The indicated *x*, *y*, and *z* axes are fixed in space. We consider the forces on the rotor at the instant when the principal axes *O*1, *O*2, and *O*3, respectively, coincide with the *x*, *y*, and *z* axes.

Suppose that the precession is as indicated by the arrow *N*; then the rolling is as indicated

by the arrow n, and $n = (5 \div 3)N$; and since

$$N = 30 \text{ rev/min} = 3.14 \text{ rad/sec} \qquad n = 50 \text{ rev/min} = 5.24 \text{ rad/sec}$$

$$\bar{a}_n = \bar{r}N^2 = 5 \times 3.14^2 = 49.3 \text{ ft/sec}^2 \quad \text{and} \quad \bar{a}_t = 0 \quad \text{(see Fig. 323b)}$$

Since $m = 200 \div 32.2 = 6.22$ slugs, $\mathring{R}_x = m\bar{a}_n = 6.22 \times 49.3 = 307$ lb, $\mathring{R}_y = m\bar{a}_y = 0$, $\mathring{R}_z = m\bar{a}_t = 0$ (see Fig. 323c).

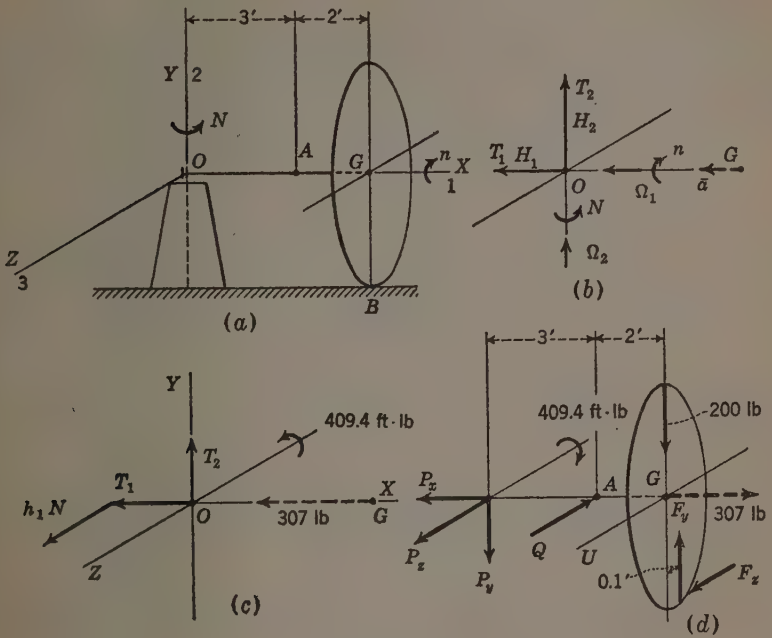

FIG. 323.

$I_1 = 6.22 \times 2^2 = 24.9$, and $I_2 = 6.22(1.5^2 + 5^2) = 169.5$ slug-ft²; $h_1 = I_1\omega_1 = 24.9 \times 5.24 = 130.4$, $h_2 = I_2\omega_2 = 169.5 \times 3.14 = 532.2$, $h_3 = I_3\omega_3 = 0$. The vectors H_1 and H_2 point as the vectors Ω_1 and Ω_2 do.

The velocity of T_1 is equal to h_1N, or 409.4, directed as indicated by the vector marked h_1N. The velocity of T_2 is zero. Hence

$$\frac{dh_x}{dt} = 0 \qquad \frac{dh_y}{dt} = 0 \qquad \frac{dh_z}{dt} = 409.4 \text{ ft-lb}$$

and these, respectively, are $\mathring{C}_x$, $\mathring{C}_y$, and $\mathring{C}_z$.

Figure 323*d* is a fbd for the rotor. The external system consists of the weight 200 lb, the force Q applied at A, the force P at O (represented by three components), and the reaction F of the roadway (represented by two components). The inertia system consists of the centrifugal force $\mathring{R}_x' = 307$ lb and the gyroscopic couple* $\mathring{C}_z' = -409.4$ ft-lb. The d'Alembert

* This is a term often used to refer to the reversed effective or inertia couple whose plane is parallel to the plane of the axes of spin and precession. Any force or component of force due wholly to a gyroscopic couple is said to be a gyroscopic effect.

or combined system has six unknowns; it is solvable. Equilibrium equations written in this order

$$\Sigma F_x = 0 \qquad \Sigma M_u = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_z = 0 \qquad \Sigma M_y = 0 \qquad \Sigma F_z = 0$$

give the following results directly:

$$P_x = 307 \qquad P_y = 82 \qquad F_y = 282 \qquad F_z = 9.4 \qquad Q = 15.7 \qquad P_z = 6.3 \text{ lb}$$

(P_y is due wholly to $\mathring{C}_z{}'$ and is a gyroscopic effect; F_y may be regarded as consisting of two components 200 lb and 82 lb; the 82-lb component is a gyroscopic effect.)

EXAMPLE 2. Suppose that the precession of Ex. 1 is not constant, and that when the velocity N reaches 30 rev/min the acceleration is 5 rev/min/sec. It is required to find the forces then acting on the rotor at A, O, and B.

Solution. Since N is changing, n, h_1, and h_2 are changing too. At the given instant

$$\frac{dN}{dt} = 0.524 \qquad \text{and} \qquad \frac{dn}{dt} = \frac{5}{3} \times 0.524 = 0.873 \text{ rad/sec}^2$$

$$\bar{a}_t = 5 \times 0.873 = 4.37 \qquad \text{and} \qquad \bar{a}_n = 5 \times 3.14^2 = 49.3 \text{ ft/sec}^2$$

$$\mathring{R}_x = m\bar{a}_n = 307 \qquad \mathring{R}_y = 0 \qquad \mathring{R}_z = m\bar{a}_t = 27.2 \text{ lb}$$

The radial velocities of T_1 and T_2, respectively, are

$$\frac{dh_1}{dt} = I_1\frac{d\omega_1}{dt} = 21.7 \qquad \text{and} \qquad \frac{dh_2}{dt} = I_2\frac{d\omega_2}{dt} = 88.8$$

The transverse velocity of T_1 is (as in Ex. 1) $h_1N = 409.4$; T_2 has no transverse velocity (see Fig. 324*a*). Hence

$$\frac{dh_x}{dt} = -21.7 \qquad \frac{dh_y}{dt} = +88.8 \qquad \frac{dh_z}{dt} = +409.4 \text{ ft-lb}$$

and these, respectively, are $\mathring{C}_x$, $\mathring{C}_y$, and $\mathring{C}_z$.

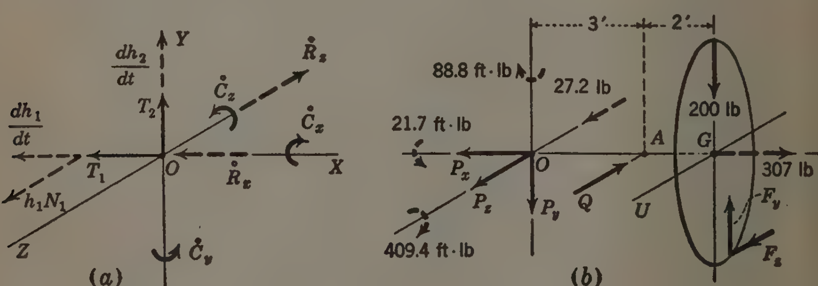

FIG. 324.

Figure 324*b* is a fbd, showing the rotor and the d'Alembert system. The external system consists of seven forces as in Ex. 1; the reversed effective system consists of two forces and three couples, indicated by dashed arrows. Equilibrium equations written in this order

$$\Sigma F_x = 0 \qquad \Sigma M_u = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_z = 0 \qquad \Sigma M_y = 0 \qquad \Sigma F_z = 0$$

give the following results directly:

$$P_x = 307 \qquad P_y = 82 \qquad F_y = 282 \qquad F_z = 16.6 \qquad Q = 57.3 \qquad P_z = 13.5 \text{ lb}$$

EXAMPLE 3. Figure 325 represents an old-time mill for grinding cement. The mill consists of (i) an upright frame or housing; (ii) a die fixed in the frame; (iii) a roll fastened to a shaft; (iv) a pulley with a hollow hub fastened to a shaft; and (v) a universal joint in the hollow hub connecting parts iii and iv. Part iv has three bearings, two for lateral and one (at the top) for vertical support. The weight of iii is taken by a spherical seat formed on the lower portion of the hub; this weight and that of iv and v are taken by the top bearing. When the mill is idle, the roll shaft hangs in a vertical position; when the pulley is rotated the roll shaft rotates in the vertical position with the pulley. When it is desired to start the mill for grinding, the roll is pulled outward and the power is turned on at the pulley; the roll and its shaft rotate, and the roll rolls on the die, a great pressure being developed between roll and die. Material to be ground is fed into the mill so that some is caught between the roll and the die and there pulverized.

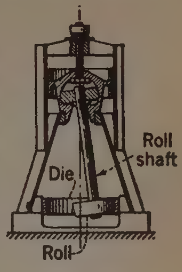

FIG. 325.

We now propose the problem of determining the pressure between the roll and the ring when the mill is run at a pulley speed of 165 rev/min. The die is 40 in. in diameter; the mean diameter of the roll is $\frac{1}{2}(24 + 22) = 23$ in., and its thickness is 8 in. The rotor (roll and its shaft) weighs 1480 lb; its center of gravity G (Fig. 326*a*, not to scale) is 55.6 in. from O, the " fixed point " of the rotor. The upper face of the roll is 66 in. from O, and the angle GOY' is 7°. The principal moments of inertia of the rotor at or for O are

$$I_1 = 13.9 \qquad I_2 = 1165 \text{ slug-ft}^2 = I_3$$

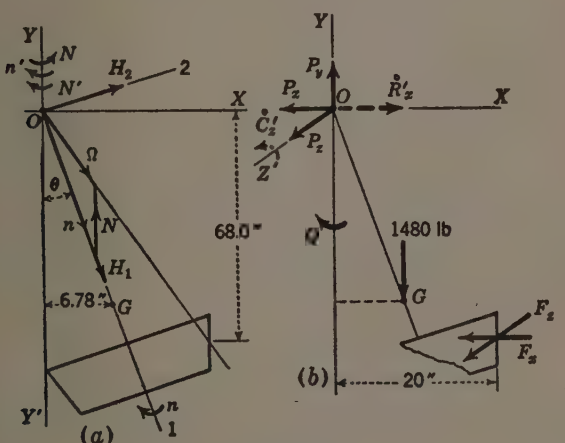

FIG. 326.

Solution. We neglect all friction but allow for rolling resistance, taking the coefficient as $1\frac{1}{4}$ in. (see Art. 70). Because the roll is conical and the die cylindrical, there is slipping between roll and die at the upper and lower points of their contact; but there is no slip at some intermediate point which we assume is the middle of the contact, where the roll diameter is 23 in. Let d denote this diameter, D the diameter of the die, n the velocity of spin or rolling (relative to the plane $Y'OG$), and N the velocity of that plane (velocity of precession); then $n = (D/d)N$. Suppose that the sense of the precession is as indicated by the curved arrow N; then the sense of the spin is as indicated by the curved arrow n. Let N' denote the (absolute)

velocity of the pulley and its shaft (165 rev/min), and n' their velocity relative to the plane $Y'OG$. Since the two shafts are connected as explained, n and n' are equal and have like senses, as indicated. Now N, n', and N' are velocities of rotation about the same line, and, since n' is greater than N, $N' = n' - N$, and the sense of N' is like that of n' as indicated. It follows that

$$N = \frac{d}{D-d}N' = 223.2 \text{ rev/min} = 23.37 \text{ rad/sec}$$

and $n = 388.2$ rev/min $= 40.64$ rad/sec. (A check on these calculations is afforded by a graphical construction for the angular velocity Ω of the rotor; $\Omega = n \leftrightarrow N$. Since this vector Ω is on the instantaneous axis of the spherical motion, its extension should pass through the contact point of no slip; it does so on a drawing to scale.)

The angular velocities and angular momentums of the rotor about the principal axes 01, 02, and 03 are

$$\omega_1 = 17.3 \qquad \omega_2 = 2.85 \qquad \text{and} \qquad \omega_3 = 0 \text{ rad/sec}$$

hence

$$h_1 = 242 \qquad h_2 = 3320 \qquad \text{and} \qquad h_3 = 0$$

The vectors H_1 and H_2 (not to scale) are rotating with the plane YOG; and since their lengths are constant their tips are describing horizontal circles centered on YY' with radii equal to $h_1 \sin\theta$ and $h_2 \cos\theta$. The velocities of the tips are in the negative z direction. Hence

$$\frac{dh_z}{dt} = -(h_1 \sin\theta)N - (h_2 \cos\theta)N = -77{,}500 \text{ ft-lb} = \mathring{C}_z$$

Since dh_x/dt and $dh_y/dt = 0$, $\mathring{C}_x = \mathring{C}_y = 0$. Since $\mathring{R} = m\bar{a} = (1480 \div 32.2)(6.78 \div 12)N^2 = 14{,}200$, $\mathring{R}_x = -14{,}200$ lb, $\mathring{R}_y = 0$, and $\mathring{R}_z = 0$.

The reversed effective system for the rotor is indicated in Fig. 326b; also the external system which consists of the following: the weight 1480 lb; the reaction F of the die, and the system S exerted by the pulley shaft on the roll shaft. F is represented by two components F_x and F_z; F_z is really $1\frac{1}{4}$ in. beyond the page. S is regarded as consisting of a force P at O and a couple Q. P is represented by three components; considering the forces acting on the pulley and its shaft, you see that the plane of Q is horizontal, and that Q is equal to the moment of belt pulls about the axis of the pulley shaft. The combined or d'Alembert system has six unknowns and is solvable. The answers are $F_x = 13{,}550$ lb, $F_z = 0$, $P_x = 650$, $P_y = 1480$, $P_z = 0$, $Q = 1410$ ft-lb. (The power required at the pulley is $1410 \times 23.37 = 32{,}950$ ft-lb/sec $= 59.9$ hp.)

B. Any Three-Dimensional, or Solid, Motion

180. Kinematics. In this section we make use of several sets of coordinate axes, some fixed and some moving with respect to the earth. For an example, see Fig. 327, O-XYZ is a set of fixed axes; G is the center of gravity of a moving body, not shown, and G-$X'Y'Z'$ is a set of axes always parallel to the first set. (The second set has, you see, a motion of translation.) The (varying) coordinates of G with respect to the first set are $\bar{x}$, $\bar{y}$, and $\bar{z}$.

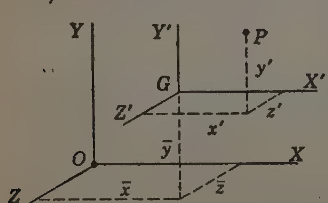

FIG. 327.

Let P be any point of the moving body.

Its (varying) coordinates with respect to the two sets are x, y, z and x', y', z'. Then

$$x = \bar{x} + x' \qquad y = \bar{y} + y' \qquad z = \bar{z} + z' \tag{1}$$

The velocities of P relative to the two sets of axes are not identical, for (see Art. 119)

$$v_x = \bar{v}_x + v_x' \qquad v_y = \bar{v}_y + v_y' \qquad v_z = \bar{v}_z + v_z' \tag{2}$$

and hence

$$V = \bar{V} \leftrightarrow V' \tag{3}$$

where V and V' denote the velocities of P relative to the fixed and moving sets, respectively, and $\bar{V}$ the velocity of G. Since the moving body is supposed to be rigid, P is at a constant distance from G. Hence, the motion of the body relative to the moving axes is spherical, and this relative motion has at each instant a definite instantaneous axis and a definite angular velocity which we denote by Ω'.

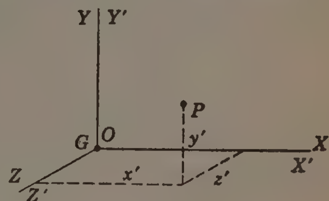

FIG. 328.

When the state of motion at a given instant is under consideration, it is advantageous generally to choose the fixed axes, in advance, so that the center of gravity G will have come into coincidence, at the given instant, with the fixed origin O. Then, at the given instant, $\bar{x}$, $\bar{y}$, and $\bar{z}$ are zero, and $x = x'$, $y = y'$, and $z = z'$ (Fig. 328); but Eqs. 2 and 3 remain unchanged.

181. Angular Momentum about Any Line through the Center of Gravity. We now show: (i) that the angular momentum of a moving body does not depend on the motion of the center of gravity; (ii) that the angular momentum may be calculated in the manner explained in part A of this chapter, only in place of ω in Art. 175 one must use ω', so that here

$$h_1 = I_1\omega_1' \qquad h_2 = I_2\omega_2' \qquad \text{and} \qquad h_3 = I_3\omega_3' \tag{1}$$

From these equations one can calculate the angular momentum about any line through the center of gravity, after the manner of Art. 175.

Proof. We use the two sets of axes of Fig. 328 explained in the preceding article. The angular momentums about the x and x' axes, say, are equal, of course, at any given instant. This common value is, according to Art. 175,

$$h_x = \int dm \cdot (v_z y - v_y z) \tag{2}$$

where all symbols in the parenthesis refer to the fixed axes. At the given instant, $y' = y$ and $z' = z$; also $v_y = \bar{v}_y + v_y'$ and $v_z = \bar{v}_z + v_z'$. Hence

$$h_x \quad \text{and} \quad h_x' = \int [(\bar{v}_z + v_z')y' - (\bar{v}_y + v_y')z']\, dm$$

Expanding the right-hand member to the sum of four integrals, you see that the preceding equation reduces to

$$h_x' = \int (v_z'y' - v_y'z')\, dm \tag{3}$$

where all primed symbols refer to the moving axes.

Observe that the last integral is independent of the motion of the center of gravity G. This fact holds for any line through G, since the direction of GX' was taken at random.

For illustration, we consider the flywheel of the engine of a power shovel in operation. Figure 329 is a plan view of the cab. A vertical through O is the axis about which the cab rotates or revolves; A is the flywheel of the engine, and G is the center of gravity of the wheel. G-$X'Y'Z'$ is a coordinate frame the axes of which have fixed directions; it does not revolve. Let n denote the velocity of spin of the wheel relative to the cab, and N the velocity of the swinging or revolving cab relative to the earth and to the frame. The angular velocity ω' of the wheel relative to the frame is the resultant of n and N; $\omega_1' = n$, $\omega_2' = N$, and $\omega_3' = 0$. Hence $h_1 = I_1n$, $h_2 = I_2N$, and $h_3 = 0$.

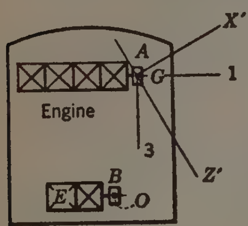

FIG. 329.

(Suppose that E is a second engine with a flywheel B just like A, its center of gravity being at O. The angular momentums of A and B about any two parallel lines through their respective centers of gravity are equal.)

182. Inertia Systems Represented. Whenever the axis of a spinning rotor changes its direction, that is precesses, the rotor develops a gyroscopic couple and perhaps other inertia couples.

EXAMPLE 1. Consider the flywheel of the preceding article. Suppose that $G1$, $G2$, $G3$ (Fig. 330) are the principal axes 1, 2, and 3 of Fig. 329, and suppose that the senses of n and N are as indicated. Then the vectors H_1 and H_2 are correctly pointed. If N is constant, H_2 is constant; if n is constant, H_1 changes in direction only and the rate of change of H_1 is h_1N directed as indicated. Hence the only inertia couple is $\bar{C}_z' = h_1N = I_1nN$, directed as indicated. The plane of this couple is parallel to the plane of axes 1 and 2; this is the gyroscopic couple. There is a centrifugal force $R' = m\bar{a} = m(OG)N^2$, directed from O to G.

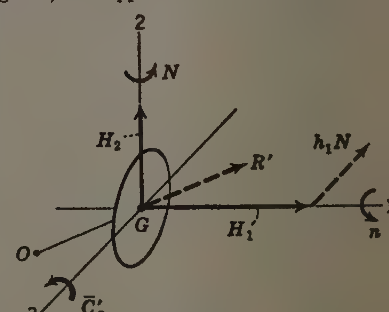

FIG. 330.

The rotation of the cab is to and fro, of course, so that there is a starting period and a stopping period. During each there is an inertia couple $\bar{C}_2' = I_2\,dN/dt$. In a starting period the sense of $\bar{C}_2'$ and that of N are unlike; during a stopping period they are alike.

EXAMPLE 2. Consider a pair of the wheels under a railway car rounding a curved track (Fig. 331). Suppose that the center of the curve is to your left and that the car is moving from you. Then N and n (velocities of precession and spin) are as indicated. ($N = v/r$, where v is velocity of the car and r is radius of curvature of the curve; $n = v \div \frac{1}{2}d$, where d is diameter of the wheels.) Angular momentums about principal axes at G are

$$h_1 = I_1 n \qquad h_2 = I_2 N \qquad h_3 = 0$$

For constant v and r, h_2 is constant. H_1 changes in direction only; its rate of change is $h_1 N$ directed as indicated. Hence, the only inertia couple is $\bar{C}_3' = h_1 N = I_1 n N$ directed as indicated. The plane of $\bar{C}_3'$ is parallel to the plane of the axes of spin and precession; it is the gyroscopic couple. This couple tends to rotate the pair of wheels upward about the outer rail. (There is a centrifugal force equal to $m\bar{a}_n = mrN^2$ directed from the center of the curve toward G.)

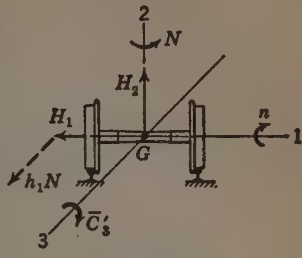

FIG. 331.

If the speed of the car were increasing, say, there would be two more inertia couples

$$\bar{C}_1' = -\bar{C}_1 = -\frac{dh_1}{dt} = -I_1\frac{dn}{dt} = -2\frac{I_1}{d}\frac{dv}{dt}$$

$$\bar{C}_2' = -\bar{C}_2 = -\frac{dh_2}{dt} = -I_2\frac{dN}{dt} = -\frac{I_2}{r}\frac{dv}{dt}$$

(Indicate the senses of these couples by curved arrows in the figure.) There are two inertia forces: a centrifugal force as above, and a force equal to $m\bar{a}_t = m\,dv/dt$ directed from G toward 3.

EXAMPLE 3. Consider an airplane coming out of a dive; see Fig. 332. We assume that the propeller has more than two blades so that it is kinetically symmetrical (Art. 6, App. B) and that the propeller spins clockwise as seen by the pilot. The precession is about the perpendicular to the paper at G and counterclockwise to you. This perpendicular is one of the principal axes at G; the other two are marked 1 and 3.

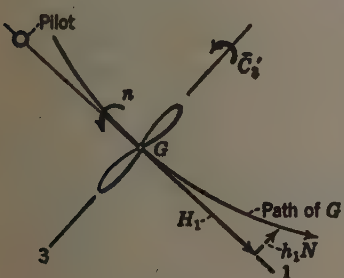

FIG. 332.

$$h_1 = I_1 n \qquad h_2 = I_2 N \qquad h_3 = 0$$

$N = v/r$, where v is velocity of the plane and r is radius of curvature of the path of G. We suppose that n, v, and r are constant. H_2 does not change; H_1 changes in direction only; the rate of change of H_1 is $h_1 N$ directed as indicated. Hence the only inertia couple is $\bar{C}_3' = I_1 n N$ with sense as indicated. The plane of this couple is parallel to that of the axes of spin and precession; this is the gyroscopic couple. This couple tends to turn the nose of the plane toward the pilot's right.

If the speed of the plane were increasing, say, there would be one more inertia couple

$$\bar{C}_2' = -\bar{C}_2 = -\frac{dh_2}{dt} = -I_2\frac{dN}{dt} = -\frac{I_2}{r}\frac{dv}{dt}$$

(How would this couple turn the nose of the airplane?)

183. Gyroscope. The gyroscope seems to have been devised originally, in 1832, as a scientific toy and to illustrate the composition of rotations and the dynamics of such motions. One was used in 1852 to make the rotation of the earth visible, whence the name of the instrument. In recent years, gyroscopes have been used in more practical devices, a few of which are described in Arts. 184–5.

Every gyroscope has a rotor in the form of a wheel and axle; other parts are the inner and outer gimbals G_1 and G_2 (Fig. 333) and a base or other support. There are three axes of rotation, A (the axis of spin), B, and C; A and B are

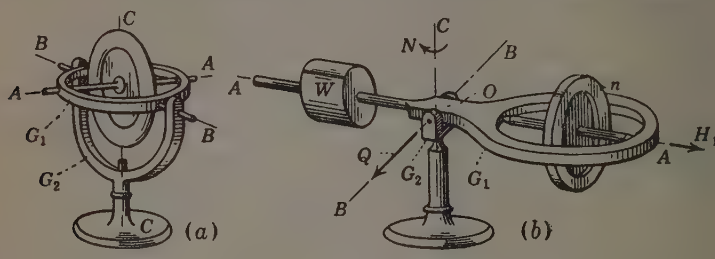

FIG. 333.

always perpendicular to each other, also B and C. By planes of the inner and outer gimbals respectively is meant the plane of A and B and that of B and C. Because the rotor can be rotated about any one of these axes, it is *universally mounted;* the gyroscope too is said to be universally mounted. The word "gyro" is used as a short substitute for gyroscope, and sometimes in place of "rotor."

TWO IMPORTANT PROPERTIES OF GYROSCOPES. 1. Consider a well-made gyroscope universally mounted; and suppose that its rotor is spinning with its axis at rest. If the base of the instrument were grasped and moved slowly about in any way, the direction of the axis of the rotor would be observed to remain practically unchanged. This permanence of direction of the axis is sometimes referred to as "a rigidity of the rotor or gyro." The explanation of the permanence lies in the fact that there is no reason why the direction of the axis should change, for the forces acting on the rotor are such that they have little or no combined moment about any line through the fixed point; hence they do not materially change the angular momentum of the rotor about any such line, and the angular momentum vector for the rotor does not change in any way. In any actual gyroscope, however, on account of the inertia of the gimbals and friction, the direction of the axis of spin can be changed more or less if the gimbals are initially in a suitable position. For example, suppose that the inner ring of the gyroscope in Fig. 333*a* has been put into the plane of the outer ring, the axis of spin being vertical then; and suppose the rotor to be spinning. A sudden rotation of the base about a perpendicular to the plane

of the rings, through 90° say, would leave the axis in a nearly vertical final position, but the axis would have changed its direction noticeably in the rotation.

2. Consider the motion of a gyroscope the inner ring of which is subjected to a force having a moment about the fixed point of the rotor. Refer to the gyroscope represented in Fig. 333*b* because it is especially suitable here. Suppose that the center of gravity of the part consisting of rotor, gimbals G_1 and G_2, and weight W is not at O, the intersection of axes A, B, and C; the part would rotate about axis B. But, if the rotor were spinning, the motion would be very different; soon after starting, the part would be rotating steadily about axis C. This steady rotation would not develop instantaneously, of course; during its build-up, G_1 would oscillate about axis B, but the oscillation would be damped by friction in the bearings at B and eventually extinguished.

Before describing the sense of this precession, we ask you to imagine W resolved into a vertical force at O and a couple Q. The moment of Q is just like the moment of W about O; Q is represented by a vector for a counterclockwise moment of W. The angular momentum H_1 of the rotor about the axis of spin is represented for a spin indicated by the curved arrow n. The precession for the assumed senses of spin and couple is clockwise viewed from above and is so indicated by the curved arrow N. The resultant H of H_1 and H_2 is shown horizontal in Fig. 334; this would be correct if the precession had been started with an initial velocity N of the proper value.

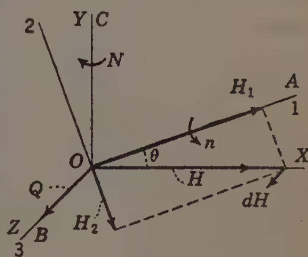

FIG. 334.

For high spin velocity, h_2 is small compared to h_1, θ is small, and h_1 and h are nearly equal. The rule for sense of precession is this: *The angular momentum vector H (or H_1) turns toward the couple vector Q.* Since Q recedes as fast as H (or H_1) advances, H (or H_1) is said to chase Q. (How would the precession appear if the sense of n were reversed? If the sense of Q were reversed? If the senses of both were reversed?)

In explanation of the precession above described, we consider that Q continually produces angular momentum about axis B. In an interval dt it produces an amount $dh = M\,dt$, where M stands for the moment of Q. Adding this increment to the angular momentum H gives a new angular momentum of a different direction. And so, in all successive intervals dt, the H vector and the axis of spin rotate through a small constant angle; they precess about the vertical with constant speed, and so as to turn the H vector toward the Q vector. Or, otherwise explained, since the rate of change of H must be just like Q, the velocity of the tip of the H vector has a direction just like the direction of the Q vector.

For a general discussion of the motion of the rotor, including its wobble or nutation, see any good treatise on the gyroscope.

184. Some Applications of Gyroscopes. The properties of gyroscopes described in the preceding articles are utilized in a variety of instruments and appliances. Nearly every airplane has one or more gyro instruments. One of the simplest is the gyrocompass. Its " card " or graduated circle, visible to the pilot, is steady even in rough weather, in marked contrast with the magnetic compass. Another simple instrument is the artificial horizon. The gyro always keeps a " horizon bar," visible to the pilot, in a horizontal position relative to the earth; the bar serves as a substitute for the natural horizon when the natural horizon is invisible. From the position with respect to the bar of a miniature airplane on the dial of the instrument, the pilot can tell whether his plane is heading up or down, rolling to right or left.

Gyroscopes serve also as the " brains " of automatic airplane pilots. They detect deviations from course, due to yaw, pitch, or roll, and signal to a " nerve center " which puts into operation a motor system that, like a human pilot, puts the control surfaces (rudder, elevator, ailerons) into corrective positions.

These gyroscopes are put into flying position by the pilot at the beginning of a flight or of a particular course, but friction and violent motions of the plane tend to displace them; for this reason most gyroscopes are provided with devices that automatically keep them in flying position.

THE MARINER'S GYROCOMPASS. Many persons have the impression that the instrument is based on this principle: if the axis of spin of a universally mounted gyroscope is set parallel to the earth's axis and the rotor is spinning rapidly, the axis of spin remains parallel to the earth's axis. This would be true for a perfect instrument, but no gyroscope has yet been made that would so maintain its axis of spin. This kind of instrument needs an auxiliary device that would continually counteract any tendency of the axis of spin to wander and deviate from parallelism. But even such a modified compass would have a serious practical defect, namely, the axis of spin would not be horizontal and so would not point to the geographic north unless the instrument were at the equator.

All mariner's gyrocompasses are contrived so that they point to the geographic north. They have special devices for precessing the axis of spin to the desired position or direction from any random initial setting and for keeping the axis in the desired position. To see what sort of corrective precessions are required, consider the circumstances of a spinning perfect gyroscope, on the earth, at four different times, 6 hours apart in one revolution of the earth. Let A, B, C, and D (Fig. 335a) be the positions, respectively. Suppose that the axis of spin was level and in the plane of the meridian in the initial position A; the axis was then pointing to the geographic north and to an imaginary star X. During the entire revolution the axis pointed to X. When the gyro was at C the axis was in the plane of the meridian at C but it was not, as you see,

level; it was tilted or elevated, 60° for the latitude indicated. When the instrument was at B, the axis of spin had an easterly deviation and an elevation of about 30°. When the instrument was at D, the axis had an equal elevation but a westerly deviation! Note that when the deviation was easterly (westerly) the elevation was increasing (decreasing).

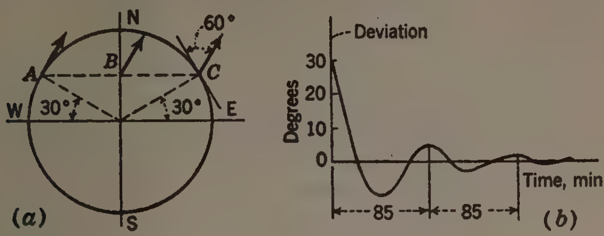

FIG. 335.

This changing elevation of the axis of spin is utilized by the mentioned auxiliary apparatus to develop a corrective force exerted on the inner gimbal, not a ring but a case completely enclosing the rotor. The force is applied near the bottom of the case and has a moment M_b about axis B (horizontal), and a moment M_c about axis C (vertical); see Fig. 333*a*. M_b precesses the axis of spin about axis C and so affects the deviation; M_c precesses the axis of spin about axis B and so affects the elevation. These precessions have a marked effect on the axis of spin. See Fig. 335*b*, made from a photographic record of a test or trial of a gyrocompass. It shows how the deviation changed from an initial easterly value of 30° (initial elevation probably zero) to a final value of zero. You see that the final value was reached in less than three oscillations. The period of oscillation was 85 minutes, and the successive amplitudes 30°, 10°, $3\frac{1}{3}$°, etc. (The above refers to one type of compass.)

185. Ship Stabilization. Various devices have been contrived for ameliorating the rolling of a ship in a rough sea. Figure 336 represents in part one type of such a stabilizer, essentially a huge gyroscope. The rotor is supported in bearings in the frame F which has gudgeons resting in bearings in pedestals securely fastened to the ship. The entire apparatus described partakes of any motion the ship may have; at the same time the rotor may be spinning about axis 1–1, and the frame and rotor may be rotating or precessing about axis 2–2. (We are not here using A and B to designate axes of the gyro, because the letters are needed below for another purpose.) The rotor is kept spinning by a motor the armature of which is mounted on the axle of the gyro, and the rotor can be made to precess as

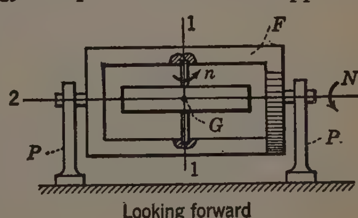

FIG. 336.

desired by an electric motor alongside the stabilizer, geared to a gear segment mounted on the frame.

We will call counterclockwise rolling viewed from the rear of the ship positive; and counterclockwise precession viewed from the right positive. The range of precession is limited to about 120°, 60° on either side of the upright position of the frame. When the roll is positive (negative) the gyro is made to precess in the positive (negative) direction; the gyro then opposes the roll (proved later).

Operation of the mentioned motor is not directed by a skillful person but by a tiny pilot gyroscope. The rolling ship makes it precess and thereby close or open a circuit which controls the motor and certain brakes so that the precessions are as required, that is, of the same sign as the sign of the rolling.

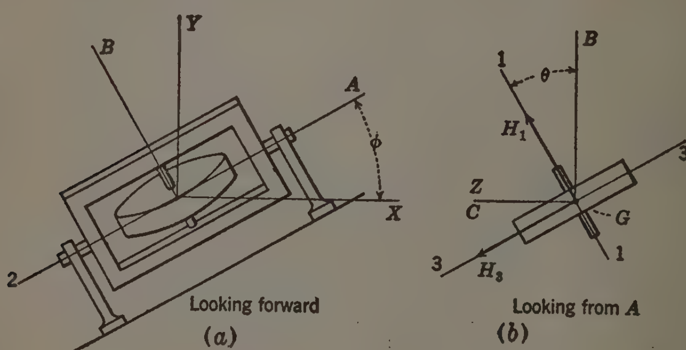

FIG. 337.

Derivation of a formula for the stabilizing couple of the gyro. Figure 337*a* represents the stabilizer as precessing and rolling. The origin of the x, y, and z axes is at the center of gravity G (not indicated) of the rotor; axis x is horizontal athwart the ship, axis y is vertical, and axis z (not shown) is horizontal fore and aft. G-ABC is another set of axes; A coincides with the axis of precession and hence is in the xy plane; B is in that plane, and hence C coincides with the z axis. G-ABC rolls with the ship. The angle ϕ defines the position of GA. The velocity of roll $\omega_r = d\phi/dt$, and the acceleration of roll $\alpha_r = d^2\phi/dt^2$. Figure 337*b* represents the rotor as seen from A. The angle θ is the inclination of the axis of spin to the vertical plane xy; the velocity of precession $N = d\theta/dt$. The axes of spin and precession are principal axes of the rotor at G and are numbered 1 and 2 respectively in Fig. 337*b* and Fig. 337*a*; the third principal axis is numbered 3 in Fig. 337*b*.

Since the stabilizing couple $\bar{C}_z' = -dh_z/dt$ we proceed to find an expression for h_z. Now the motion of the rotor relative to G-XYZ is spherical, and the angular velocity Ω' of that motion is the resultant of n, N, and ω_r. The com-

ponents of Ω' about axes 1, 2, and 3, respectively, are

$$\omega_1' = n + \frac{d\phi}{dt}\sin\theta \qquad \omega_2' = N \qquad \omega_3' = \frac{d\phi}{dt}\cos\theta$$

The angular momentums of the gyro about the same axes are

$$h_1 = I_1\omega_1' \qquad h_2 = I_2\omega_2' \qquad h_3 = I_3\omega_3'$$

The components of these h's about the z axis are, respectively, $h_1 \sin\theta$, 0, and $h_3 \cos\theta$; hence $h_z = h_1 \sin\theta + h_3\cos\theta$, or

$$h_z = I_1 n \sin\theta + (I_1 \sin^2\theta + I_3 \cos^2\theta)\frac{d\phi}{dt} \tag{1}$$

$$\frac{dh_z}{dt} = I_1 nN \cos\theta + (I_1 - I_3)\frac{d\phi}{dt} N \sin 2\theta + (I_1 \sin^2\theta + I_3\cos^2\theta)\frac{d^2\phi}{dt^2}$$
$$= (I_1 n \cos\theta + I'\omega_r \sin 2\theta)N + I_z\alpha_r \tag{2}$$

where I' is an abbreviation for $I_1 - I_3$, and I_z denotes the changing moment of inertia of the precessing gyro about OZ. Now the first term of the parenthesis of Eq. 3 is much larger than the second, because n is much larger than ω_r, I_1 is larger than I', and $\cos\theta$ is larger than $\sin 2\theta$ for the smaller values of θ (between 30° and −30°). And so, as a good approximation,

$$\bar{C}_z' = -(I_1 nN \cos\theta + I_z\alpha_r) \tag{3}$$

The second term of the right-hand member of this equation is negligible in comparison with the first except in the beginning and ending stages of a roll, to right or left, for then N is zero or small. Now these stages are relatively short in duration, and so in the major part of a roll

$$\bar{C}_z' = -I_1 nN \cos\theta \tag{4}$$

Since all factors on the right except N are always positive, the sign of $\bar{C}_z'$ and that of N are unlike; and, since, as already stated, the sign of N and that of the roll are alike, the sign of $\bar{C}_z'$ and that of the roll are unlike. That is, *the inertia couple always opposes the roll.*

The signs of the component couple $I_z\omega_r$ in the two stages are opposite; hence the angular impulses given and the works done by that couple in the two stages cancel or nearly so, and the angular impulse given and the work done by the couple $\bar{C}_z'$ in a roll may be calculated from the component $-I_1 nN\cos\theta$.

186. Kinetically Unsymmetrical Rotor. A two-blade airplane propeller is an example of this sort of rotor. See Fig. 338*a*, where G is the center of gravity of such a propeller; $G1$, the axis of the propeller (perpendicular to the paper) is one central principal axis, $G2$ and $G3$ are the others. Suppose that the plane (moving into the paper) is pulling out of a dive without bank. Given: the velocity n of spin of the propeller, clockwise as viewed in the figure; the

velocity N of pitching (upward) in the pull-out; and I_1, I_2, and I_3, the principal moments of inertia. Required: the components of the inertia couple $\bar{C}'$ about axes 1, 2, and 3.

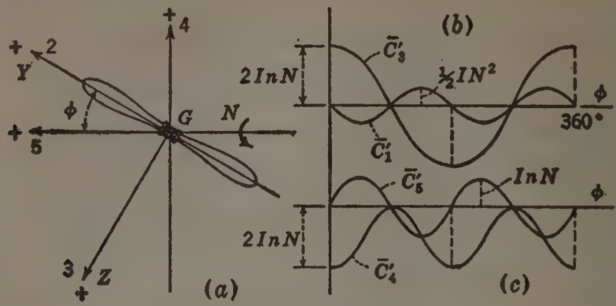

FIG. 338.

Solution. Imagine a coordinate frame with origin at G and fixed in direction. The motion of the propeller with respect to this frame is spherical; the angular velocity Ω' of this motion is the resultant of n and N. The components of Ω' parallel to the principal axes are

$$\omega_1' = -n \qquad \omega_2' = -N\cos\phi \qquad \omega_3' = -N\sin\phi$$
$$h_1 = -I_1 n \qquad h_2 = -I_2 N\cos\phi \qquad h_3 = -I_3 N\sin\phi$$

Let G-XYZ be a coordinate frame fixed as to direction and chosen so that it coincides with the frame G-123 at the instant when that frame is where shown. Then

$$\frac{dh_x}{dt} = \frac{dh_1}{dt} - h_2\omega_3' + h_3\omega_2' = -\tfrac{1}{2}(I_2 - I_3)N^2 \sin 2\phi$$

$$\frac{dh_y}{dt} = \frac{dh_2}{dt} - h_3\omega_1' + h_1\omega_3' = (I_1 + I_2 - I_3)nN\sin\phi$$

$$\frac{dh_z}{dt} = \frac{dh_3}{dt} - h_1\omega_2' + h_2\omega_1' = -(I_1 - I_2 + I_3)nN\cos\phi$$

The right-hand members are expressions for $\bar{C}_x$, $\bar{C}_y$, and $\bar{C}_z$.

For a conventional propeller, I_2 is very small compared to I_1 and I_3, and I_1 and I_3 are nearly equal. Therefore we assume, as a close approximation, that $I_2 = 0$ and $I_1 = I_3$; and we let I denote the common value of I_1 and I_3. Then the inertia couples are

$$\bar{C}_x' = -\tfrac{1}{2}IN^2\sin 2\phi \qquad \bar{C}_y' = 0 \qquad \bar{C}_z' = +2InN\cos\phi$$

These respectively are also values of $\bar{C}_1'$, $\bar{C}_2'$, and $\bar{C}_3'$.

The fluctuations of the first and third couples in one revolution of the pro-

peller are represented in Fig. 338*b*. $\bar{C}_1'$ is a torque load on the engine, a bending load on the blades of the propeller, and a rolling load on the airplane. $\bar{C}_3'$ is a bending load on the blades and a yawing and pitching load on the airplane. The yawing and pitching components, about axes 4 and 5, of $\bar{C}_3'$ are

$$\bar{C}_4' = -\bar{C}_3' \cos\phi = -2InN \cos^2\phi$$
$$\bar{C}_5' = +\bar{C}_3' \sin\phi = +InN \sin 2\phi$$

These are represented in Fig. 338*c*. The yawing couple is always negative, tending to turn the nose of the airplane toward the pilot's right; its average value is InN.

APPENDIX A

SYSTEMS AND DIMENSIONS OF UNITS

A1. Systems of Units. By a system of units is meant a group of units that have been chosen as suitable for use in a particular subject or field. Four systems are tabulated on pages 302 and 303; each is based on three independent units, called *fundamental units* or *fundamentals*. Other units in each system are based on one or more of its fundamentals and therefore are *derived units*. The fundamentals of the two systems on page 302 are units of

length, *mass*, and *time*

The fundamentals of the systems on page 303 are units of

length, *force*, and *time*

The centimeter-gram-second (CGS) system was formulated and adopted by the British Association in 1873 as generally suitable for physical science. It is in wide use. The meter-kilogram-second (MKS) system was adopted by the International Electrotechnical Commission in 1935 principally because the units derived from the meter and kilogram are — so it is said — of a more suitable size than those derived from the centimeter and gram for commercial and scientific purposes.

The British and Continental systems have no official sanction, not even universally accepted names. We have adopted the name " British system " because at least two of the fundamental units, the pound force and the foot, and the name* " slug " originated in Britain. The system is used in English-speaking countries. The name " Continental system " seems appropriate because the system is used on the Continent (of Europe); also in South America.

Nearly all of each table is self-explanatory; the last column of each is discussed in the following article.

To beginners in dynamics we recommend adherence to the units of a formal system. We prefer the British system because one of the fundamentals of that system is a unit of force, in keeping with the importance of force in dynamics for engineers, and because the unit is a familiar one. In statics, the kinds of quantities dealt with are comparatively few and familiar (forces, moments of

* Given by Professor A. M. Worthington " to the British Engineer's Unit of Mass, i.e., to the mass in which an acceleration of one foot-per-sec-per-sec is produced by a force of one pound." Quotation from the preface to his *Dynamics of Rotation*, sixth edition, 1906.

force and couples, lengths, areas, and volumes), and misuse of units is not likely to occur.

Adherence to one of the formal systems is not imperative. Departures are sometimes convenient, but, once fundamental units are chosen for the solution of a problem, they should be used throughout, that is consistently. Thus, suppose that it is desired to calculate the period of a pendulum by the formula

$$T = 2\pi\sqrt{k^2/bg} \qquad \text{(see Art. 116)}$$

The second for period T, and the inch for lengths k and b, seem suitable; if they are chosen, then g should be expressed in inches per second per second.

Formulas requiring the use of mixed units are contrived for greater convenience sometimes; they have been called "bastard formulas." The following formula for the lift on a wing of a flying airplane is an example:

$$L\ (\text{lb}) = 0.00256C_L(S\ \text{ft}^2)(V\ \text{mi/hr})^2$$

L denotes lift, S wing area, V speed of airplane relative to the air, and C_L lift coefficient, a number dependent on type of wing and angle of attack. You see that two different units of length are specified. No other units than those specified may be used in this formula as it stands.

A2. Dimensions of Derived Units. A statement of the way in which a derived unit depends on the fundamental units involved in it is called a statement of the dimensions of the unit. For example,

$$\frac{\text{one square yard}}{\text{one square foot}} = \frac{(\text{one yard, or three feet})^2}{(\text{one foot})^2} = 9$$

Thus, units of area depend only on the units of length used, and units of area are proportional to the square of the units of length on which they are based. This relation is expressed in the form of a *dimensional equation* or *formula*, thus

$$(\text{unit area}) = (\text{unit length})^2, \text{ or } A = L^2$$

Briefly a unit area is said to be "two dimensions in length"; similarly, a unit volume is said to be "three dimensions in length."

The dimensional formulas for derived units that have composite names can be got readily from these names respectively. Thus from the names cm/sec, ft/min, mi/hr you see at once that the dimensional formula for a unit of velocity is L/T or LT^{-1}; similarly you see that the dimensional formula for unit acceleration is LT^{-2}, and that the dimensional formula for unit angular velocity is T^{-1}.

Units of force in the systems of page 302 are derived units. Their dimensional formulas can be found from $F = ma$, which shows that these units vary directly as the units of mass and acceleration; hence the dimensional formula for units of force is

$$(M)(LT^{-2}) \text{ or } LMT^{-2}$$

Units of mass in the systems of page 303 are derived units. Their dimensional formulas can be found from $m = F/a$, which shows that these units vary directly as the units of force and inversely as the units of acceleration; hence the dimensional formula for unit masses is

$$(F) \div (A) = \frac{F}{LT^{-2}} \quad \text{or} \quad L^{-1}FT^2$$

Two Systems of Units Based on Units of Length, Mass, and Time

Name of Quantity	CGS	MKS	Dimensional Formula
	Name of Unit		
Length	centimeter (cm)	meter (m)	L
Mass	gram (gr)	kilogram (kg)*	M
Time	second (sec)	second (sec)	T
Velocity, linear	cm/sec	m/sec	LT^{-1}
Acceleration, linear	cm/sec²	m/sec²	LT^{-2}
Angular velocity	rad/sec	rad/sec	T^{-1}
Angular acceleration	rad/sec²	rad/sec²	T^{-2}
Force	dyne	newton	LMT^{-2}
Weight	dyne	newton	LMT^{-2}
Specific weight	dyne/cm³	newton/m³	$L^{-2}MT^{-2}$
Density	gr/cm³	kg/m³	$L^{-3}M$
Moment or torque of a force	cm-dyne	m-newton	L^2MT^{-2}
Moment of an area	cm³	m³	L^3
Moment of inertia of an area	cm⁴	m⁴	L^4
Moment of inertia of a body	gr-cm²	kg-m²	L^2M
Work	erg or cm-dyne	joule	L^2MT^{-2}
Energy	erg	joule	L^2MT^{-2}
Power	erg/sec	watt	L^2MT^{-3}
Momentum, linear	dyne-sec	newton-sec	LMT^{-1}
Angular momentum	dyne-sec-cm	newton-sec-m	L^2MT^{-1}
Impulse, linear	dyne-sec	newton-sec	LMT^{-1}
Angular impulse	dyne-sec-cm	newton-sec-m	L^2MT^{-1}
Stress, total	dyne	newton	LMT^{-2}
Stress, intensity	dyne/cm²	newton/m²	$L^{-1}MT^{-2}$

* The name "kog" has been proposed by Professor Edward Bennett as an advantageous substitute for kilogram. Kilokog would mean 1000 kogs; milikog, 0.001 kog; etc.

A3. Some Uses of Dimensions. A knowledge of dimensions is probably of most value to the beginner in mechanics as a help to a clear understanding of the different quantities of mechanics and their relations. It is useful practically in other ways, two of which are explained below.

1. To test for errors in an equation containing mechanical quantities. Such an equation if rationally and correctly deduced is homogeneous; that is, the terms in it are the same in kind. To ascertain whether terms are the same in

kind we write the equation in dimensional form, reduce its terms to their simplest forms, and compare. If they are alike, the terms in the original equation are the same in kind, and the equation is homogeneous. To illustrate let us consider Eq. 4, Art. 77,

$$T = \frac{1}{2} wa \left(1 + \frac{a^2}{16f^2}\right)^{\frac{1}{2}}$$

Two Systems of Units Based on Units of Length, Force, and Time

Name of Quantity	British (Name of Unit)	Continental (Name of Unit)	Dimensional Formula
Length	foot (ft)	meter (m)	L
Force	pound (lb)	kilogram (kg)	F
Time	second (sec)	second (sec)	T
Velocity, linear	ft/sec	m/sec	LT^{-1}
Acceleration, linear	ft/sec²	m/sec²	LT^{-2}
Angular velocity	rad/sec	rad/sec	T^{-1}
Angular acceleration	rad/sec²	rad/sec²	T^{-2}
Mass	slug (sl)	metric slug (msl)*	$L^{-1}FT^2$
Weight	pound	kilogram	F
Specific weight	lb/ft³	kg/m³	$L^{-3}F$
Density	sl/ft³	msl/m³	$L^{-4}FT^2$
Moment or torque of a force	ft-lb	m-kg	LF
Moment of an area	ft³	m³	L^3
Moment of inertia of an area	ft⁴	m⁴	L^4
Moment of inertia of a body	sl-ft²	msl-m²	LFT^2
Work	ft-lb	m-kg	LF
Energy	ft-lb	m-kg	LF
Power	ft-lb/sec	m-kg/sec	LFT^{-1}
Momentum, linear	lb-sec	kg-sec	FT
Angular momentum	lb-sec-ft	kg-sec-m	LFT
Impulse, linear	lb-sec	kg-sec	FT
Angular impulse	lb-sec-ft	kg-sec-m	LFT
Stress, total	lb	kg	F
Stress, intensity	lb/ft²	kg/m²	$L^{-2}F$

* Used here for want of a better term.

where T denotes tension (a force), w weight (a force) per unit length, and a and f lengths. (i) Using L, M, and T, the dimensional form of the equation is

$$LMT^{-2} = \frac{1}{2} \frac{LMT^{-2}}{L} L \left(1 + \frac{1}{16} \frac{L^2}{L^2}\right)^{\frac{1}{2}}$$

Each term in the parentheses is, you see, dimensionless; so are their sum and the square root of the sum. The entire right-hand member reduces to LMT^{-2},

which is just like the left-hand member; hence the original equation is correct dimensionally and homogeneous. (ii) Using L, F, and T, the dimensional form of the equation is

$$F = \frac{1}{2}\frac{F}{L}L\left(1 + \frac{1}{16}\frac{L^2}{L^2}\right)^{\frac{1}{2}}$$

which is simpler than the first form. Indeed, dimensional equations based on L, F, and T are generally the simpler in formulas of mechanics, particularly if mass does not appear in the formula.

Showing that an equation is homogeneous does not prove that it is correct but that it may be correct; showing that an equation is nonhomogeneous shows it to be incorrect. Since abstract numbers do not appear in the dimensional form of an equation, the test for homogeneity does not disclose errors in numerical coefficients and terms, or errors in signs.

2. To ascertain the name of the unit of the result of a numerical calculation. One substitutes for the quantities the names of the units in which they are expressed, and then repeats the calculation, treating the names as though they were algebraic quantities. The reduced answer is the name of the unit of the numerical answer. For example: The formula for the elongation of a rod due to a pull at each end is Pl/AE (wherein P denotes pull, l length of the rod, A area of cross section, and E Young's modulus for the material. Suppose that $P = 10{,}000$ lb, $l = 50$ in., $A = 0.5$ in.^{2}, $E = 30{,}000{,}000$ lb/in.^{2} The calculation for the elongation is

$$\frac{10{,}000 \times 50}{0.5 \times 30{,}000{,}000} = 0.033$$

The calculation for the name of the unit is

$$\frac{\text{lb} \times \text{in.}}{\text{in.}^2 \times \text{lb/in.}^2} = \frac{\text{lb} \times \text{in.} \times \text{in.}^2}{\text{in.}^2 \times \text{lb}} = \text{in.}$$

APPENDIX B

MOMENT OF INERTIA AND RADIUS OF GYRATION*

B1. General Principles, Etc. Perhaps you have observed that the effort required to start a body to rotating about a fixed axis seems to depend not only on the mass of the body but also on the remoteness of the material of the body from the axis of rotation. Figure B1 represents a simple apparatus by means of which one can roughly sense this fact. It consists of a vertical shaft S to which a grooved pulley P and cross arm A are fastened rigidly, and a heavy body B which can be clamped on the cross arm. The pull or turning effort may be applied by means of a cord wrapped about the pulley. It is shown in Art. 113 that this rotational inertia of a body is proportional to the *moment of inertia* of the body about the axis, which may be defined as follows:

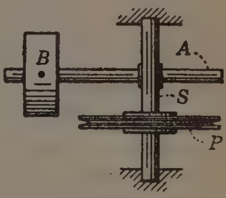

FIG. B1.

The moment of inertia of a body with respect to or about a line is the sum of the products obtained by multiplying the mass of each elementary particle of the body by the square of its distance from the line. If I = moment of inertia, $(dm)_1$, $(dm)_2$, $(dm)_3$, etc., = the masses of the particles, and r_1, r_2, r_3, etc., their distances respectively from the line or axis, then

$$I = (dm)_1 r_1^2 + (dm)_2 r_2^2 + \cdots = \Sigma dm \cdot r^2$$

or, if the body is continuous,

$$I = \int dm \cdot r^2 \tag{1}$$

where dm denotes the mass of any elementary particle and r its distance from the line about which the moment of inertia is taken.†

* Moment of inertia of bodies and of areas is treated in works on integral calculus nowadays, principally because the subject affords practical exercise in integration. Appendix B is included in this book for the student who may not have had adequate training in the subject for the understanding of Chapter X and other parts of dynamics. Students are cautioned to distinguish carefully between moments of inertia of (physical) *bodies* and of (geometrical) *areas*. The former are employed in dynamics; the latter in strength of materials, or elasticity, and to a lesser extent in other branches of applied mechanics.

† Euler (1707–83) introduced the term "moment of inertia," and he explained its appropriateness in his *Theoria motus corporum solidorum*, somewhat as follows: The choice of the name, moment of inertia, is based on analogies in the equations of motion for translations and rotations. In a translation the acceleration is proportional directly to the *accelerating force*

In applying Eq. 1, it is necessary to choose the elementary mass dm of such shape that all points of it will be equally distant from the axis or line about which moment of inertia is being calculated so that there will be no doubt as to the meaning of r. (See Ex. 1 following.)

In some problems one may conceive the body as consisting of an infinite number of parts whose moments of inertia about the line or axis specified are known from previous calculation. Then

$$I = dI_1 + dI_2 + \cdots = \int dI \tag{2}$$

(See Ex. 2 following.)

A *unit of moment of inertia* depends upon the units of mass and distance used. There is no single-word name for any unit of moment of inertia. Each unit is described by stating the units of mass and distance involved in it, and in accordance with the " make-up " of the unit. Thus, when the pound and the foot are used as units of mass and length respectively, then the unit of moment of inertia is called a pound-foot square (lb-ft^2); when the slug and the foot are used, then the unit of moment of inertia is called slug-foot square (slug-ft^2).

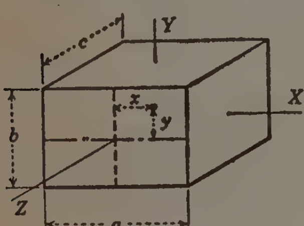

FIG. B2.

EXAMPLE 1. Required to show that the moment of inertia of a homogeneous right parallelepiped about a central axis parallel to an edge is $\frac{1}{12} m(a^2 + b^2)$, where m is the mass of the parallelepiped, and a and b are the lengths of the edges perpendicular to the mentioned axis.

Solution 1. See Fig. B2, where the z axis is the chosen central axis. We choose $dx\, dy\, dz$ as elementary volume; then $dm = \delta\, dx\, dy\, dz$, where δ denotes density. Let x, y, and z denote the coordinates of dm; then the distance ρ of the element from the z axis is $(x^2 + y^2)^{\frac{1}{2}}$. Hence $\int dm \cdot \rho^2$ or

$$I_z = \delta \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} \int_{-\frac{1}{2}b}^{+\frac{1}{2}b} \int_{-\frac{1}{2}c}^{+\frac{1}{2}c} (x^2 + y^2)\, dx\, dy\, dz = \frac{\delta c}{12} (a^3 b + ab^3) = \text{etc.}$$

---

and inversely to the mass, or *inertia*, of the moving body; and in a rotation the angular acceleration is proportional directly to the *moment* of the accelerating force and inversely to a *quantity*, Σdmr^2, depending on the mass or inertia. This quantity, to complete a similarity, we may call *moment of inertia*. Then we have for translations and rotations respectively,

$$\text{linear acceleration} = \frac{\text{force}}{\text{inertia or mass}}$$

and

$$\text{angular acceleration} = \frac{\text{moment of force}}{\text{moment of inertia}}$$

It has been suggested that the word " moment " in the term " moment of inertia " means important and that moment of inertia means importance of inertia. This is a clever justification of the appropriateness of the term, but it does not agree with the original justification.

Solution 2. We may choose a rodlike elementary volume parallel to the z axis. All points of the rod are equally distant from the z axis; the distance ρ is $(x^2 + y^2)^{\frac{1}{2}}$. For cross section of the rod we take $dx\,dy$; then $dm = \delta(c\,dx\,dy)$. Hence $\int dm \cdot \rho^2$ or

$$I_z = \delta c \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} \int_{-\frac{1}{2}b}^{+\frac{1}{2}b} (x^2 + y^2)\,dx\,dy = \frac{\delta c}{12}(a^3b + ab^3) = \text{etc.}$$

EXAMPLE 2. Required to show that the moment of inertia of a homogeneous right circular cylinder with respect to its axis is $\frac{1}{2}mr^2$, where m is the mass of the cylinder and r the radius of its base.

Solution 1. See Fig. B3, where the z axis coincides with the axis of the cylinder. We choose a rodlike elementary volume parallel to the z axis, and of a cross section $\rho\,d\theta\,d\rho$ (as indicated). As in Eq. 1, ρ is the distance from the z axis to the rod; $dm = \delta a(\rho\,d\theta\,d\rho)$ where a denotes altitude of the cylinder. Hence $\int dm \cdot \rho^2$ or

$$I_z = \delta a \int_0^r \int_0^{2\pi} \rho^3\,d\rho\,d\theta = \delta a 2\pi \frac{r^4}{4} = \frac{1}{2}mr^2$$

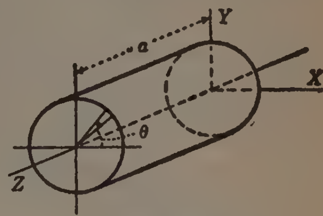

FIG. B3.

Solution 2. We may look upon the cylinder as consisting of elementary concentric shells, each of a thickness $d\rho$. All points of a shell are equally distant, ρ, from the z axis. The volume of a shell is $2\pi\rho\,d\rho \cdot a$, and so $dm = \delta 2\pi a\rho\,d\rho$. Hence $\int dm \cdot \rho^2$ or

$$I_z = \delta 2\pi a \int_0^r \rho^3\,d\rho = \delta 2\pi a \frac{r^4}{4} = \frac{1}{2}mr^2$$

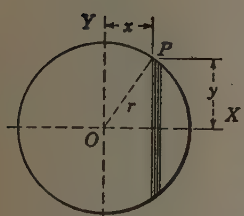

FIG. B4.

EXAMPLE 3. Required to show that the moment of inertia of a homogeneous sphere about a diameter is $\frac{2}{5}mr^2$, where m is the mass of the sphere and r its radius.

Solution. For a simple solution we look upon the sphere as consisting of elementary laminas perpendicular to the x axis (Fig. B4). Not all points of such an element are equally distant from the axis; hence it will not serve for use in Eq. 1. But, looking upon the lamina as a thin cylinder, the radius of its base is y, the altitude or thickness is dx, and the mass is $\delta(\pi y^2)\,dx$; and from Ex. 2 the moment of inertia of the lamina about the x axis is $\frac{1}{2}(\delta\pi y^2\,dx)y^2$. Hence, from Eq. 2, the moment of inertia of the sphere is

$$\int_{-r}^{+r} \tfrac{1}{2}\delta\pi y^2\,dx \cdot y^2 = \tfrac{1}{2}\delta\pi \int_{-r}^{+r} y^4\,dx = \tfrac{1}{2}\delta\pi \int_{-r}^{+r} (r^2 - x^2)^2\,dx = \tfrac{8}{15}\delta\pi r^5 = \text{etc.}$$

EXAMPLE 4. Required to show that the moment of inertia of a homogeneous right circular cylinder with respect to a line through its center and parallel to its bases is $\frac{1}{12}m(3r^2 + a^2)$ where r is the radius of its bases, a is its altitude, and m its mass.

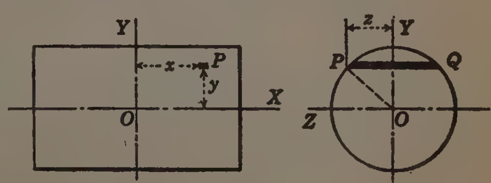

FIG. B5.

Solution. Let the z axis (Fig. B5) be the line about which the moment of inertia is required, and then consider that the cylinder

consists of elementary rods parallel to the z axis. Let x, y, and z be the coordinates of the end P of the rod PQ. The square of the distance, ρ, of the rod from the z axis is $(x^2 + y^2)$. The mass of the rod is $\delta(2z\,dx\,dy)$. Hence the moment of inertia $\int dm \cdot \rho^2$ is

$$I_z = \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} \int_{-r}^{+r} (\delta 2z\,dx\,dy)(x^2 + y^2) = 2\delta \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} \int_{-r}^{+r} (r^2 - y^2)^{\frac{1}{2}}\,dx\,dy(x^2 + y^2) = \text{etc.}$$

If the cylinder is *slender* so that $3r^2$ is negligible in comparison with a^2, then $I_z = \frac{1}{12}ma^2$. If the cylinder is *thin* so that a^2 is negligible in comparison with $3r^2$, then $I_z = \frac{1}{4}mr^2$.

B2. Radius of Gyration of a Body. Let m denote the mass of the body represented in Fig. B6 and I its moment of inertia about any given line or axis. Let P be a very dense particle such that its mass also is m, and let k denote the distance of P from the axis; then the moment of inertia of P about the axis is mk^2. If k is such that $mk^2 = I$ (of the body), then k is called the *radius of gyration* of the body with respect to or about the axis. So defined,

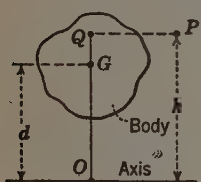

FIG. B6.

$$k = \sqrt{\frac{I}{m}} \tag{1}$$

A radius of gyration is a mere length, but sometimes it is represented or thought of as a definite line OQ through the center of gravity of the body, perpendicular to the axis, and terminating on the axis. When the radius of gyration is so localized, Q is called the *center of gyration* of the body with respect to the given axis. (See next article for the relative positions of the centers of gravity and gyration.)

Practical uses of radius of gyration are based on the definition above. But there is another way to look upon it: Consider the body as made up of elementary particles of equal mass dm; let n denote the number of particles, and r_1, r_2, r_3, etc., their distances from the given line. The mean of the squares of these distances is

$$\frac{r_1^2 + r_2^2 + \cdots}{n}$$

It can be shown readily that this mean square equals I/m or k^2. Hence

$$k = \sqrt{\text{mean (square distance)}}$$

Obviously the radius of gyration of a body with respect to a line is intermediate between the distances from the line to the nearest and most remote particles of the body. This fact will assist in estimating the radius of gyration of a body.

B3. Parallel-Axis Formulas. There is a simple relation between the moments of inertia (and the radii of gyration) of a body with respect to two parallel lines one of which passes through the mass-center of the body. By

means of this relation one can simplify many calculations of moment of inertia and avoid integrations (see examples following); it may be stated as follows:

The moment of inertia of a body with respect to any line equals its moment of inertia with respect to a parallel line passing through the mass-center plus the product of the mass of the body and the square of the distance between the lines. Or, if I = the first moment of inertia, $\bar{I}$ = the second (for the line through the mass-center), m = mass, and d = the distance between the parallel lines, then

$$I = \bar{I} + md^2 \tag{1}$$

Proof. Let O (Fig. B7) be the mass-center of the body (not shown), LL the line about which the moment of inertia is I, and OZ a parallel line (through the mass-center) about which the moment of inertia is $\bar{I}$; the distance between these parallel lines is d. For convenience we take x and y axes through O, the x axis in the plane of the two parallel lines and the y axis perpendicular to that plane.

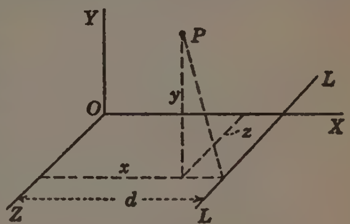

FIG. B7.

Let P be any particle of the body, x, y, and z the coordinates of P, and dm the mass of P. The square of the distance of P from LL is $(d - x)^2 + y^2$; hence

$$I = \int [(d - x)^2 + y^2]\, dm = \int (x^2 + y^2)\, dm + d^2 \int dm - 2d \int x\, dm$$

Now the first of the last three terms $= \bar{I}$, and the second one $= md^2$. And, if W denotes the weight of the body, the third term equals

$$2d \int x \frac{dW}{g} = \frac{2d}{g} W\bar{x}$$

Since $\bar{x} = 0$, the third integral equals zero. Hence $I = \bar{I} + md^2$.

The square of the radius of gyration of a body with respect to any line equals the square of its radius of gyration with respect to a parallel line passing through the mass-center plus the square of the distance between the two lines, or, if k = the first radius of gyration, $\bar{k}$ = the second, and d = the distance between the lines, then

$$k^2 = \bar{k}^2 + d^2 \tag{2}$$

This equation results at once by dividing Eq. 1 by m.

According to Eq. 2, k is always greater than d; that is, *the radius of gyration of a body with respect to a line is always greater than the distance from the line to the center of gravity of the body.* But, if the dimensions of the cross sections of the body perpendicular to the line in question are small compared to d, then $\bar{k}/d$ is small compared to 1, and k equals d approximately (see Ex. 2). In such a case the moment of inertia is approximately equal to md^2.

EXAMPLE 1. Required the moment of inertia of a prism of cast iron (weighing 450 lb/ft³) 6 by 9 in. by 3 ft with respect to one of the long edges.

Solution. The block weighs 507 lb; hence according to Ex. 1, Art. B1, the moment of inertia of the block with respect to the line through the mass-center parallel to the long edge is $507(6^2 + 9^2) \div 12 = 4940$ lb-in.² The square of the distance from a long edge to the mass-center is 29.25 in.²; hence according to Eq. 1, the moment of inertia desired is $4940 + 507 \times 29.25 = 19{,}760$ lb-in.² = 137 lb-ft² = 4.27 slug-ft².

EXAMPLE 2. Required the radius of gyration of a round steel rod 1 in. in diameter with reference to a line 12 in. from the axis of the rod.

Solution. The square of the radius of gyration of the rod with respect to its axis is $\frac{1}{2}(0.5)^2 = 0.125$ in.² (see Ex. 2, Art. B1). According to Eq. 2, the radius of gyration desired is $(0.125 + 12^2)^{\frac{1}{2}} = 12.01$ in., nearly the same as the distance from the line of reference to the mass-center of the rod.

EXAMPLE 3. It is required to show that the moment of inertia of a right circular cone with respect to a line through the apex and parallel to the base $= \frac{3}{20}\, m(r^2 + 4a^2)$, where m is the mass of the cone, r the radius of its base, and a its altitude.

Solution. We conceive the cone as made of laminas parallel to the base, find the moment of inertia of each lamina with respect to the specified line, and then add all such moments. For convenience we take the axis of the cone as the y coordinate axis, and the line for which the moment of inertia is required as the x axis (Fig. B8). The moment of inertia of the lamina indicated about a diameter is $\frac{1}{4}dm \cdot x^2$, where dm = the mass of the lamina and x = its radius (see Ex. 4, Art. B1). Hence its moment of inertia about the x axis $= \frac{1}{4}dm \cdot x^2 + dm \cdot y^2$ (see Eq. 1), and the moment of inertia of the entire cone $= \int(\frac{1}{4}dm \cdot x^2 + dm \cdot y^2)$, the limits being assigned so as to include all laminas. We choose to integrate with respect to y, and so must express dm and x in terms of y. From similar triangles in the figure, $x/y = r/a$, or $x = ry/a$; obviously $dm = \delta\pi x^2\, dy = \delta\pi(r^2y^2/a^2)\, dy$, where δ = density. Hence

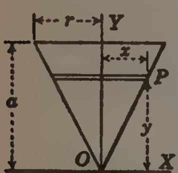

FIG. B8.

$$I = \int_0^a \frac{\pi r^4\delta y^4\, dy}{4a^4} + \int_0^a \frac{\pi r^2\delta y^4\, dy}{a^2} = \frac{\pi r^4\delta a}{20} + \frac{\pi r^2\delta a^3}{5} = \text{etc.}$$

B4. Composite Body. By this term is meant a body which one naturally conceives as consisting of finite parts, for example, a flywheel which consists of a hub, several spokes, and a rim. The moment of inertia of such a body with respect to any line can be computed by adding the moments of inertia of all the component parts with respect to that same line. The radius of gyration of a composite body does not equal the sum of the radii of gyration of the component parts. It can be determined from Eq. 1, Art. B2, where I = moment of inertia of the whole body and m = its mass.

EXAMPLE 1. A dumbbell-shaped body consists of two spheres joined by a cylinder. The spheres are 6 in. in diameter and weigh 30 lb each; the cylinder is 1½ in. in diameter, 16 in. long, and weighs 6 lb. Required the moment of inertia and the radius of gyration of the body with respect to a line L perpendicular to the axis of the cylinder at its midpoint.

Solution. The moment of inertia of either sphere about a diameter is $\frac{2}{5} \times 30 \times 3^2 = 108$ lb-in.² (Ex. 3, Art. B1). The distance of the mass-center of either sphere from the line L is 11 in.; therefore the moment of inertia of either sphere about L is $108 + (30 \times 11^2) = 3738$ lb-in.² The moment of inertia of the cylinder about L is (see Ex. 4, Art. B1),

$$\tfrac{1}{12}\, 6(3 \times 0.75^2 + 16^2) = 128.8 \text{ lb-in.}^2$$

Hence the required moment of inertia is

$$2 \times 3738 + 128.8 = 7605 \text{ lb-in.}^2$$

The radius of gyration about L is

$$[7605 \div (60 + 6)]^{\frac{1}{2}} = 10.7 \text{ in.}$$

B5. Product of Inertia. In equations pertaining to rotation (Art. 134) and in an important equation of the following article there are certain terms which we now name, explain, and discuss briefly.

By *product of inertia* of a body with respect to a pair of coordinate planes is meant the sum of all products obtained by multiplying each elementary mass of a body by its coordinates with respect to the two planes, thus

$$K_{xy} = \Sigma dm \cdot xy \qquad K_{yz} = \Sigma dm \cdot yz \qquad K_{zx} = \Sigma dm \cdot zx$$

where K_{xy} is our symbol for the product of inertia with respect to the planes from which the x and y coordinates of dm are measured (the planes YOZ and ZOX); similarly, K_{yz} and K_{zx}.

A product of inertia is the same kind of a quantity as a moment of inertia; that is, both are expressible in the same unit. Unlike a moment of inertia, a product of inertia may be negative or even zero.

For any body and any origin of coordinates there is at least one set of coordinate planes with respect to each pair of which the product of inertia of the body is zero (proved in Art. B8). The intersections of the planes, that is the coordinate axes, are of dynamical importance (see Art. 175). In general these planes, and hence their intersections, cannot be found readily. However, for a homogeneous body having more or less symmetry such planes through certain origins can be found by means of the following proposition:

If a homogeneous body has a plane of symmetry P_1, the product of inertia of the body with respect to P_1 and another plane P_2 perpendicular to P_1, both regarded as coordinate planes, is zero.

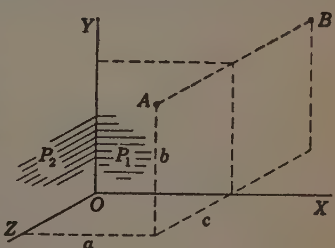

FIG. B9.

Proof. See Fig. B9, where P_1 is taken as xy and P_2 as yz plane; the origin O is anywhere on the intersection of P_1 and P_2. A and B are a symmetrical pair of particles of equal size (and hence of equal mass). The coordinates of A are a, b, c; coordinates of B are $a, b, -c$. The product of inertia of the pair with respect to planes P_1 and P_2 is

$$dm(+c)(a) + dm(-c)(a)$$

which equals zero. Hence the product of inertia of all pairs, or of the whole body, is zero.

B6. Principal Axes; Principal Moments of Inertia and Radii of Gyration. As stated in the preceding article, there are at least three coordinate planes intersecting at any given point O, such that the products of inertia of the body under consideration with respect to each pair of the planes is zero; also that the coordinate axes are of importance dynamically. Such axes are called *principal axes* of the body at or for the point or origin O. The moments of inertia and the radii of gyration of the body with respect to the principal axes are called *principal moments of inertia* and *principal radii of gyration.* If the point or origin is at the center of gravity, the terminology is *central* principal axes, moments of inertia, and radii of gyration.

It is shown in Art. B8 that one of the three principal moments of inertia (radii of gyration) at a point is in general greater or less than the moment of inertia (radius of gyration) about any other line through the point. That is, one of the principal moments of inertia (radii of gyration) is an absolute maximum and one is an absolute minimum.

PRINCIPAL AXES OF SOME HOMOGENEOUS SYMMETRICAL BODIES. Bodies having two planes of symmetry P_1 and P_2, for example a right pyramid with a rectangular base: The intersection of P_1 and P_2 is a line of symmetry. The principal axes at any point O of this line are the line itself and the intersections of P_1 and P_2 with a plane through O and perpendicular to the line.

Bodies having three planes of symmetry P_1, P_2, and P_3, for example a right parallelepiped: The intersections of the three planes are axes of symmetry. The principal axes at any point O of an axis of symmetry are parallel to the axes of symmetry.

Bodies having an axis of symmetry and many planes of symmetry containing this axis, for example a right circular cylinder: Any such body has many principal axes at any point O of the axis of symmetry. The axis of symmetry is one principal axis; any line through O and perpendicular to that axis is another.

NONHOMOGENEOUS ROTORS. Generally, the density variation of a rotor has a certain symmetry that may be described as follows: The rotor has two axial planes zx and zy (Fig. B10) such that all particles of the rotor can be grouped into fours, the particles A, B, C, and D of each four being of equal size and density, and having x, y, and z coordinates that are respectively

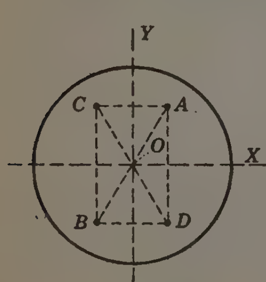

FIG. B10.

$$(a, b, c) \quad (-a, -b, c) \quad (-a, b, c) \quad (a, -b, c)$$

The products $dmyz$ for the four add up to zero; also the products $dmzx$; hence the products of inertia K_{yz} and K_{zx} of the rotor equal zero, and the z axis is a principal axis at every point of that axis.

KINETIC SYMMETRY. If two of the central principal moments of inertia of a

body are equal, say $I_2 = I_3$, the body is said to be kinetically symmetrical. It follows from Art. B8 that the moments of inertia of the body about any line through the center of gravity and in the plane of axes 2 and 3 are equal. Obviously all homogeneous solids of revolution are kinetically symmetrical. Any homogeneous right prism whose bases are regular polygons is kinetically symmetrical. Most rotors are kinetically symmetrical.

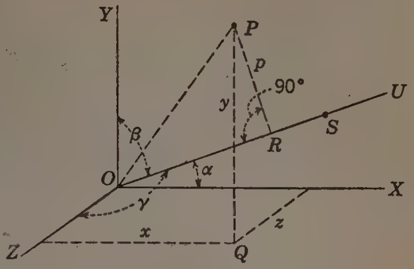

FIG. B11.

B7. Inclined-Axis Formulas. Let O (Fig. B11) be any point of a body, not shown, O-XYZ a set of coordinate axes, OU any line through O, λ, μ, and ν the direction-cosines of OU, and I the moment of inertia of the body about OU. As proved below,

$$I_u = \lambda^2 I_x + \mu^2 I_y + \nu^2 I_z - 2\lambda\mu K_{xy} - 2\mu\nu K_{yz} - 2\nu\lambda K_{zx} \qquad (1)$$

from which one can calculate I_u provided that the set of axes is such that the I's and K's are known or determinable. It is not easy, in general, to locate such a set of axes for a given point of a body.

If the chosen coordinate axes are principal axes at O, then the products of inertia are zero and the above formula becomes

$$I_u = \lambda^2 I_x + \mu^2 I_y + \nu^2 I_z \quad \text{or} \quad \lambda^2 I_1 + \mu^2 I_2 + \nu^2 I_3 \qquad (2)$$

where I_1, I_2, and I_3 more conspicuously denote principal moments of inertia.

Derivation of Eq. 1. Let P, Fig. B11, be a particle of the body, dm its mass, p its distance from OU; then $I_u = \int dm \cdot p^2$. Now

$$p^2 = (OP)^2 - (OR)^2 = (x^2 + y^2 + z^2) - (\lambda x + \mu y + \nu z)^2$$

Expanding this expression for p^2 and arranging, you find that $\int dm \cdot p^2$ or

$$I_u = \int dm[\lambda^2(y^2 + z^2) + \mu^2(z^2 + x^2) + \nu^2(x^2 + y^2) - 2\lambda\mu xy - 2\mu\nu yz - 2\nu\lambda zx]$$

which can be expanded to

$$I_u = \lambda^2 \int (y^2 + z^2)\, dm + \cdots + \cdots - 2\lambda\mu \int xy\, dm - \cdots - \cdots$$

which leads readily to Eq. 1.

B8. Momental Ellipsoid. Imagine lengths OS laid off on many lines like OU (Fig. B11), the lengths being inversely proportional to the radii of gyration of the body about the lines respectively. If r denotes the length OS and k the radius of gyration about OU

$$r = \frac{L^2}{k}$$

where L is any chosen constant length and then L^2 is a proportionality factor. All points S are on and define a surface of course; we show below that that surface is an ellipsoid whose axes coincide with the principal axes of the body at O. It is called the *momental ellipsoid* of the body at O.

In general, one of the diameters of the ellipsoid is longer and one is shorter than any other diameter. Hence, the radius of gyration and the moment of inertia about the longest diameter are minimum, smaller than those about any other line through O; the radius of gyration and the moment of inertia about the shortest diameter are maximum, larger than those about any other line through O.

If two principal moments of inertia at O are equal, say $I_2 = I_3$, then the ellipsoid is one of revolution; all diameters in the plane of axes 2 and 3 are equal, and the moments of inertia about all such lines are equal. If three principal moments of inertia at O are equal, the ellipsoid is a sphere, and the moments of inertia about all lines through O are equal.

To find the surface defined by all points S, let x', y', and z' be the coordinates of any such point with respect to coordinate axes x, y, and z (Fig. B11); then

$$\lambda = \frac{x'}{r} \qquad \mu = \frac{y'}{r} \qquad \nu = \frac{z'}{r}$$

These values of λ, μ, and ν substituted in Eq. 1 of the preceding article lead to

$$I_x x'^2 + I_y y'^2 + I_z z'^2 - 2K_{xy}x'y' - 2K_{yz}y'z' - 2K_{zx}z'x' = mL^4$$

which is the equation of an ellipsoid.

If the coordinate axes (x, y, and z) are taken to coincide with the geometrical axes of the ellipsoid, then the equation still holds, of course; but, as known from geometry, the three negative or K terms vanish for such coordinate axes. Now these terms cannot vanish for all lines OS unless each K is zero. Hence, for any body and any point O, there is a set of coordinate planes for each pair of which the product of inertia is zero. Hence also the geometrical axes of the momental ellipsoid of the body, for the point O, are the principal axes of the body, for that point.

B9. Experimental Determination of Moment of Inertia. When the body is so irregular in shape that a desired moment of inertia cannot be computed easily, an experimental method may be relatively simpler. Several such methods are available.

1. The following method is available if the body can be suspended and oscillated as a pendulum about an axis coinciding with or parallel to the line L with respect to which the moment of inertia is desired. Let T = the time of one complete (to-and-fro) oscillation, b = distance from the axis of suspension to the mass-center, W = weight of the pendulum, g = acceleration due to gravity,

and I = moment of inertia about the axis of suspension; then

$$I = \frac{Wb}{4\pi^2} T^2 \tag{1}$$

If the axis of suspension does not coincide with the line L, it remains to "transfer" I to L (see Art. B3).

Equation 1 follows from the formula for the period of a pendulum, namely,

$$T = 2\pi \sqrt{\frac{k^2}{bg}} = 2\pi \sqrt{\frac{I}{Wb}}$$

see Art. 116.

2. The next method requires the use of a suitable hinged table or platform that can be vibrated; see Fig. B12, where A is the table, and O is the hinge. One or more springs partially support the table and body, a car say, whose moment of inertia is desired. The body should be placed so that the line L to which the moment of inertia refers is parallel to the hinge. (A car and its springs should be blocked so as to be substantially a rigid body.)

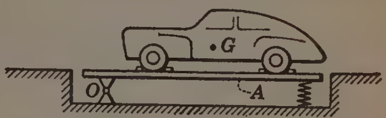

FIG. B12.

Let T_1 denote the period of one complete oscillation, up and down, of the table alone, and T_2 the period of the table with the car. Then the moment of inertia I of the car about the hinge is given by

$$I = \frac{C}{4\pi^2}(T_2^2 - T_1^2) \tag{2}$$

where C is the ratio of the moment about the hinge of any force applied to the table and the tilt in radians of the table due to that moment. C is the tilting moment per radian of tilt; it is a measure of the stiffness of the table-spring system. I having been found from Eq. 2, in any experiment, it remains to transfer I to the line L (see Art. B3).

Equation 2 follows from the formulas for the periods of the vibrations of the table alone and the table with the car, namely,

$$T_1 = 2\pi \sqrt{\frac{I_1}{C}} \quad \text{and} \quad T_2 = 2\pi \sqrt{\frac{I_2}{C}}$$

where I_1 denotes the moment of inertia about the hinge of the table alone, and I_2 that of the table with the car.

APPENDIX C

MECHANICAL VIBRATIONS

C1. Preliminary. The following are some examples of vibrations: the bodies of vehicles traveling the highways or railways and parts of ships coursing the seas; floors of buildings housing heavy machinery in operation and parts of such machines. Some of these vibrations do not matter; some do, and should have been avoided in the design of the vehicle, machine, or structure in question. Some may be so serious as to call for corrective measures. Thus various sorts of vibration problems occur in engineering.

The literature on mechanical vibrations, in books and technical papers,* is extensive. The purpose of the present appendix is to furnish a mere introduction to the subject. Only very simple problems can be considered; the principles involved apply generally in many others.

Some simple vibration systems are represented in Fig. C1. Briefly they are: (*a*) a pendulum; (*b*) a stretched elastic cord carrying a small weight; (*c*) a weighted rod floating in a vertical position; (*d*) a weight supported by a vertical coil spring; (*e*) a strip of spring steel clamped at one end. These systems are represented in their equilibrium positions. You see, without explanation, how to displace each system slightly so that when released it will vibrate. In each, the vibration takes place under the action of a force, or resultant, which always tends to restore the system to its equilibrium position. Such a force is called a *restoring force.*

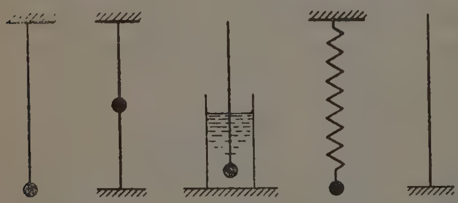
(*a*) (*b*) (*c*) (*d*) (*e*)

Fig. C1.

C2. Definitions and Illustrations. It is convenient to distinguish between kinds of vibrations. The rolling of a ship in a heavy sea is a forced vibration; the waves force the ship to keep on rolling. Arrived behind a breakwater, the ship might still be rolling; the rolling is now a *free vibration.* Because this motion dies down, it is said to be *damped.* All vibrating bodies are subjected

* *Vibration Problems in Engineering*, by Timoshenko, and *Mechanical Vibrations*, by Den Hartog, are comprehensive and practical. The *Journal of Applied Mechanics* of the American Society of Mechanical Engineers abounds in papers on vibrations.

to more or less damping, due to friction or other resistance. An undamped vibration is really hypothetical, but the concept is important because it leads up to real cases.

Damping is of various kinds: (i) The damping force may be constant, independent of velocity, like friction between dry surfaces; this is called *Coulomb damping*. (ii) It may depend on the velocity of some part of the vibrating system, like air resistance or the internal friction in the springy part of the system; this is called *velocity damping*. (iii) It may follow some other law, for example one for a combination of Coulomb and velocity damping.

Velocity damping that is proportional to the first power of velocity is common; it is called *viscous damping* because it is like the damping in a viscous-oil dashpot. This damping force may be written $-cv$, where v denotes the velocity on which the force depends and c is a *coefficient of damping*. The negative sign is necessary because the direction of the damping force is opposite to that of the velocity. A numerical value of c is expressed in a unit of force per unit of velocity, for example, pounds per foot per second.

NATURAL FREQUENCY. Many vibrations are cyclic, each repeating itself closely in all or some respects in equal intervals of time so that it has a definite period and frequency. If such a vibration is free, whether damped or undamped, the period and frequency are called *natural*.

C3. A Linear Vibration, Free and Undamped. It can be proved that system d of Fig. C1 is typical of the others shown and of still others not so simple. Moreover, all vibrating systems include a part or element that is springy. Therefore we discuss system d at length.

As you probably know, springs obey Hooke's law for very slowly changing deformations (stretches and shortenings of a coil spring). The tension (or compression) is proportional to the deformation; the force-deformation relation is a linear one. If S denotes the spring force at a moderate deformation e,

$$S = ke \qquad \text{and} \qquad k = \frac{S}{e}$$

where k is a proportionality factor, called *spring constant* or *spring factor*.* For a given spring, k is a measure of the stiffness of the spring, and is expressible in pounds per inch, etc. It may be looked upon as the spring force per unit stretch or shortening. Though springs do not obey Hooke's law when deforming rapidly, we deal at first with springs imagined to obey the law even when deforming rapidly; such springs are called sometimes *ideal* or *perfect*, for convenience.

* Note an important change of notation: Up to this point k has denoted radius of gyration, following a general usage in dynamics. In this appendix k denotes spring factor, following general usage in mechanical vibration literature, and ρ denotes radius of gyration.

Figure C2 represents the system to be discussed. A is the equilibrium position of the suspended body, e_s is the elongation of the spring for that position, and l is the natural length of the spring. B is any other position vertically above or below A; e, the elongation for that position; y is the vertical coordinate or displacement of B with respect to A; S is the spring force, W the weight, m the mass, of the body; and R is the restoring force for B.

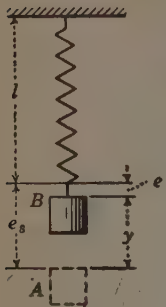

Fig. C2.

$$R = S - W = ke - ke_s = -k(e_s - e) = -ky \tag{1}$$

The final expression shows that R is proportional to the displacement of B from the equilibrium position, and that R and y are opposite in sign.

Since the acceleration a of the vibrating body is equal to R/m,

$$a = -\frac{k}{m}y \tag{2}$$

That is, the acceleration in this vibration is always proportional to the displacement y and is always directed toward the origin from which y is measured. This is the law of acceleration in simple harmonic motion; see Art. 103, where Eq. 4 is $a = -\omega^2 y$. Hence the vibration under discussion is a shm, and $\omega^2 = k/m$. Here, this frequency ω is a natural one; therefore we use a special symbol p to denote it, and so

$$p = \sqrt{\frac{k}{m}} \tag{3}$$

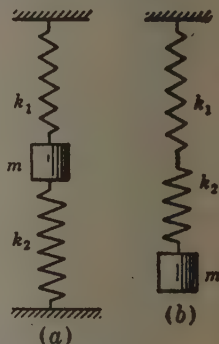

Fig. C3.

Equations 2 and 3 hold not only for the system represented in Fig. C2 but also for any system whose undamped free vibration is a shm. For example, consider the systems represented in Fig. C3. In each, the restoring force varies as the displacement from the equilibrium position, and the acceleration also; hence the vibration is a shm. For (a), $k = k_1 + k_2$, and for (b), $k = (k_1k_2) \div (k_1 + k_2)$, where k_1 and k_2 denote the spring factors for the individual springs as indicated. (You should supply proofs.)

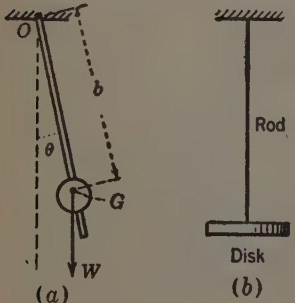

Fig. C4.

C4. An Angular Vibration, Free and Undamped. Figure C4 represents two systems that can be given angular or rotary vibration. The moment, about the axis of rotation, of the forces acting on such a

vibrating body is a restoring moment; our symbol is M. In these and similar cases without damping, M is proportional to the angular displacement θ from the equilibrium position; then

$$M = -k'\theta \tag{1}$$

where k', like k, is a proportionality factor; it is the restoring moment per unit displacement, and it is expressed in foot-pounds per radian, etc. The minus sign appears in Eq. 1 because M and θ are always opposite in sense.

To find k' in any given case imagine any force or couple that would give the system an angular displacement; calculate the moment of the force or couple about the axis of rotation, the angular displacement produced by the force or couple, and finally the ratio of the moment and the displacement; this ratio is k'.

For any case in which Eq. 1 holds, the angular acceleration α of the vibrating body is M/I (Art. 113); hence

$$\alpha = -\frac{k'}{I}\theta \tag{2}$$

where I denotes moment of inertia of the body about the axis of rotation. Since this acceleration varies as displacement, just as the acceleration in a shm varies, this angular vibration also is called a shm.

Since Eq. 2 above and Eq. 2 of Art. C3 are perfectly analogous, Eq. 3 of Art. C3 indicates that the analogous frequency formula for angular vibration is

$$p = \sqrt{\frac{k'}{I}} \tag{3}$$

The gravity pendulum (Fig. C4*a*). Let W denote the weight, ρ the radius of gyration with respect to the axis of suspension, and b the distance from the axis to the center of gravity. $M = -Wb \sin\theta$; for small angles, $M = -Wb\theta$ and $k' = Wb$; $I = (W/g)\rho^2$. Hence, from Eq. 3,

$$p = \sqrt{\frac{bg}{\rho^2}} \qquad T = 2\pi\sqrt{\frac{\rho^2}{bg}}$$

The torsion pendulum (Fig. C4*b*). The spring factor can be determined by calculation or by experiment (see "Strength of Materials"). It can be expressed as $k' = \kappa/l$, where κ is the spring factor of a rod of unit length. $I = \frac{1}{8}(W/g)d^2$, where W and d, respectively, denote weight and diameter of the disk (see Art. B1, App. B). Hence, from Eq. 3,

$$p = \sqrt{\frac{8\kappa g}{Wld^2}} \quad \text{and} \quad T = \pi\sqrt{\frac{Wld^2}{2\kappa g}}$$

EXAMPLE 1. The body OA, Fig. C5a, represented in its equilibrium position, is hinged at O; C is its center of gravity. Required: the natural frequency of the system.

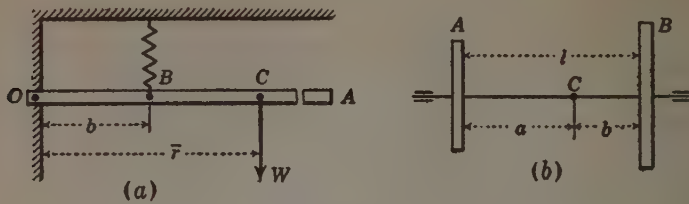

FIG. C5.

Solution. Let S_0 be the spring force for the position shown; $S_0b = W\bar{r}$. Let S be the spring force for a slight upward displacement θ of OA; $S = S_0 - kb\theta$. Since $M = -W\bar{r} + Sb = -kb^2\theta$, $k' = kb^2$; and

$$p = \frac{b}{\rho}\sqrt{\frac{k}{m}}$$

where ρ denotes the radius of gyration of the body about the hinge. (What is the physical meaning of $\sqrt{k/m}$?)

EXAMPLE 2. The system represented in Fig. C5b consists of a shaft in two frictionless bearings and two disks keyed to the shaft. Suppose that A is held fast and B is given a small angular displacement, and then both disks are released simultaneously. The ensuing motions of A and B are to be investigated.

At the instant of release the angular momentum of the system is zero. Since the external forces acting on the system have no moment about the axis of the shaft, the angular momentum of the system about that axis remains zero during the motion. Therefore, at each instant of the motion, the disks are rotating in opposite directions; A twists a part of the shaft in one direction, and B twists another part in the opposite direction. Let C denote the node or section of no motion; the part of the system on each side of C is like a torsional pendulum. The frequencies of the parts are

$$p_1 = \sqrt{\frac{k_1'}{I_1}} \qquad \text{and} \qquad p_2 = \sqrt{\frac{k_2'}{I_2}}$$

where k_1' and k_2' are the spring constants of the parts a and b, and I_1 and I_2 are the moments of inertia of A and B about the axis of the shaft. Now

$$\frac{k_1'}{k_2'} = \frac{b}{a} \qquad \text{and} \qquad p_1 = p_2 \qquad \text{hence} \qquad I_1a = I_2b$$

And, since $a + b = l$,

$$a = \frac{I_2}{I_1 + I_2}l \qquad \text{and} \qquad b = \frac{I_1}{I_1 + I_2}l$$

Finally let κ denote the spring factor of a portion of the shaft of unit length; then

$$k_1' = \frac{\kappa}{a} \qquad \text{and} \qquad k_2' = \frac{\kappa}{b}$$

and the circular frequency of A or B is

$$p = \sqrt{\frac{\kappa/a}{I_1}} = \sqrt{\frac{\kappa}{l}\,\frac{I_1 + I_2}{I_1 I_2}}$$

C5. A Free Vibration without Damping, Continued. The following analysis may be regarded as an alternative for a part of Art. C3. It is intended also as a preliminary to Art. C6, which deals with a free vibration with damping.

Since the restoring force $R = -ky$ (Art. C3), the equation of motion $R = ma$ becomes

$$-ky = m\frac{d^2y}{dt^2} \qquad \text{or} \qquad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \tag{1}$$

The solution of Eq. 1 is (see " Differential Equations ")

$$y = A_1 \sin\sqrt{\frac{k}{m}}\,t + A_2 \cos\sqrt{\frac{k}{m}}\,t \tag{2}$$

(This solution may be verified by substituting for y and d^2y/dt^2 in Eq. 1 their values, namely: y as given by Eq. 2 and d^2y/dt^2 obtained from Eq. 2 by two successive differentiations.) A_1 and A_2 are constants that depend on the initial conditions, that is, on values of y and dy/dt at $t = 0$. Differentiating Eq. 2,

$$\frac{dy}{dt} = A_1\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}\,t - A_2\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}\,t \tag{3}$$

Suppose that the vibration is started by releasing B, Fig. C2, from an elevation h above its equilibrium position, and that time is reckoned from the instant of release; then, when $t = 0$, $y = h$ and $dy/dt = 0$. For $t = 0$, Eqs. 2 and 3 respectively become

$$h = 0 + A_2 \qquad \text{and} \qquad 0 = A_1\sqrt{\frac{k}{m}} + 0$$

hence $A_1 = 0$ and $A_2 = h$. Equations 2 and 3 become

$$y = h\cos\sqrt{\frac{k}{m}}\,t \qquad \text{and} \qquad \frac{dy}{dt} = -h\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}\,t \tag{4) (5}$$

$$\frac{d^2y}{dt^2} = -h\frac{k}{m}\cos\sqrt{\frac{k}{m}}\,t \tag{6}$$

Comparing Eqs. 4, 5, and 6 respectively with Eqs. 1, 2, and 3 of Art. 103 you see that they differ only in the corresponding symbols h and y and $\sqrt{k/m}$ and ω. But both h and y stand for amplitude, and, as shown in Art. C3, $k/m = \omega^2$; and so the equations agree as one would expect.

C6. A Free Vibration with Viscous Damping. Figure C2, in part, is repeated in Fig. C6*a*. The spring in Fig. C6*a* is supposed to be a real one having damping when the system is vibrating. As in Art. C3, the restoring force for any instant may be written $R = S - W$, where now S denotes the force exerted by the real spring at that instant.

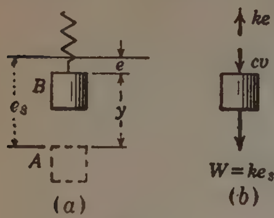

FIG. C6.

It is convenient to imagine the real spring replaced by an ideal one and a dashpot that supplies a damping force to the suspended body, equivalent to the real damping. In Fig. C6*b*, ke is the ideal spring force, and cv is the damping force, indicated for an upward velocity. The force S is $ke - cv$; the restoring force is

$$R = ke - cv - ke_s = -ky - cv \tag{1}$$

The equation of motion $R = ma$ becomes $-ky - cv = m\, d^2y/dt^2$ or

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = 0 \tag{2}$$

which is the differential equation of the motion. The solution of Eq. 2 depends on the relative values of $(\frac{1}{2}c/m)^2$ and (k/m), which are respectively the square of half the coefficient of dy/dt in Eq. 2 and the coefficient of y.

If $(\frac{1}{2}c/m)^2 < k/m$, or otherwise stated if $c < 2\sqrt{km}$, the solution of Eq. 2 is (see " Differential Equations ")

$$y = e^{-ut}(B_1 \sin qt + B_2 \cos qt) \tag{3}$$

where e is the Napierian base, and u and q are abbreviations defined by

$$u = \frac{1}{2}\frac{c}{m} \quad \text{and} \quad q = \left[\frac{k}{m} - \left(\frac{1}{2}\frac{c}{m}\right)^2\right]^{\frac{1}{2}} \tag{4}$$

B_1 and B_2 are constants whose values depend on initial conditions of the motion. (Since $k/m = p^2$, Eq. 4 can be written

$$p^2 = q^2 + u^2 \tag{5}$$

which shows that q and u are circular frequencies.)

Equation 3 can be put into the following form (see " Trigonometry "), which is more suitable for subsequent discussions:

$$y = e^{-ut}B \cos (qt - \alpha) \tag{6}$$

B and α, like B_1 and B_2, are constants depending on initial conditions. $B = (B_1^2 + B_2^2)^{\frac{1}{2}}$ and $\alpha = \tan^{-1}(B_1/B_2)$.

You see that the displacement y at any time t is equal to the product of two factors or functions, e^{-ut} and $B \cos (qt - \alpha)$. The first decreases continuously

from its initial value, 1 (when $t = 0$), to 0 (when $t = \infty$); the second represents a shm $y' = B \cos (qt - \alpha)$ whose amplitude is B, circular frequency q, and period $2\pi/q$. The dashed line of Fig. C7 is the graph of y'; the initial value of y' (when $t = 0$) is $B \cos \alpha$. The solid line is a graph of y for the conditions: $y = h$ and $v = 0$ when $t = 0$. Because $e^{-ut}B$, the amplitude, continually decreases, the motion is described as a vibration of *diminishing* amplitude.

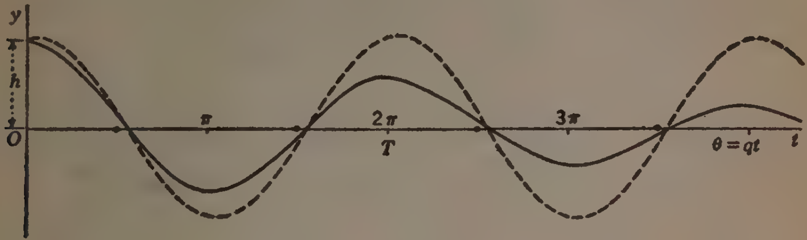

FIG. C7.

Since the two curves cross on the t axis, the periods of y and y' are equal; hence q is the frequency of y also. That is, q, originally defined by Eq. 5, is now found to be the circular frequency of the damped free vibration under discussion.

Critical damping of a system. The solutions of Eq. 1 for the cases $c = 2\sqrt{km}$ and $c > 2\sqrt{km}$ show that motion from any initial position above or below the equilibrium position is, invariably, a slow creep to the equilibrium position. Vibration ensues only if c is less than $2\sqrt{km}$. This limiting value of c is known as the *coefficient of critical damping;* denoting it by c_c,

$$c_c = 2\sqrt{km} = 2mp = \frac{2k}{p} \tag{7}$$

For any given system c_c depends, you see, only on k and m for the system. If, for example, $k = 50$ lb/in. $= 600$ lb/ft and $m = 100$ lb $= 3.11$ slugs, $c_c = 2\sqrt{600 \times 3.11} = 86.4$ lb per ft/sec. The values of c for systems encountered in practice are generally less than $0.2c_c$.

Effect of damping on natural frequency; relation between p and q. Equation 4 can be transformed to

$$q^2 = \left(1 - \frac{c^2}{4km}\right)\frac{k}{m} = \left(1 - \frac{c^2}{c_c^2}\right)p^2$$

hence

$$q = \sqrt{1 - \left(\frac{c}{c_c}\right)^2}\, p \tag{8}$$

This shows that increase of damping decreases natural frequency q. The decrease is slight for low values of c/c_c (see Fig. C8, a graph of Eq. 8). For common damping, q is practically equal to p; their difference is only 2 per cent for $c/c_c = 0.2$. (The ratio c/c_c is sometimes called *damping ratio.*)

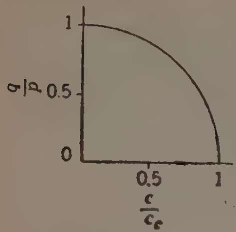

FIG. C8.

Decrease of amplitude. Let y_1 denote any maximum y, y_2 the next one, and t_1 and t_2 the corresponding times. Then

$$y_1 = e^{-ut_1} B \cos (qt_1 - \alpha)$$

and

$$y_2 = e^{-ut_2} B \cos (qt_2 - \alpha)$$

Since the angles, in parentheses, differ by 2π, the cosines are equal and

$$\frac{y_1}{y_2} = e^{u(t_2-t_1)} = e^{uT} \tag{9}$$

That is, all such ratios for a given free vibration with viscous damping are equal. Moreover, since

$$\log y_1 - \log y_2 = uT$$

all such "logarithmic decrements" are equal. They are generally denoted by δ; thus $\delta = uT$. Since $u = \frac{1}{2}c/m$ and $T = 2\pi/q$,

$$\delta = \frac{\pi c}{mq} = \frac{2\pi c/c_c}{\sqrt{1 - (c/c_c)^2}} \approx 2\pi \frac{c}{c_c} \tag{10}$$

The approximation is less than 2 per cent in error if c/c_c is less than 0.2.

The ratio of the first and last maximums in a succession of n consecutive cycles is equal to $e^{n\delta}$; hence

$$\log_e \frac{y_1}{y_{n+1}} = n\delta = \frac{2\pi n c/c_c}{\sqrt{1 - (c/c_c)^2}} \approx 2\pi n \frac{c}{c_c} \tag{11}$$

Any one of the three quantities n, c/c_c, and y_{n+1}/y_1 may be computed from Eq. 11 if the other two are given. For example, suppose that in nine successive cycles of a recorded vibration y_1/y_{10} was 2.22. Then

$$\frac{c}{c_c} = \frac{1}{2\pi 9} \log_e 2.22 = 0.014$$

C7. A Forced Vibration with Viscous Damping. Figure C9a is a part of Fig. C6 with the addition of a double arrow to indicate a periodic excitation (force P) applied to the suspended body or "mass." P is supposed to vary simple-harmonically so that

$$P = P_0 \sin \omega t \tag{1}$$

P_0 is the maximum value (or amplitude) of P, and ω the circular frequency of P; the time t is dated from the instant when P was zero and increasing. Figure C9b represents the sort of motion P would produce. The early part is quite erratic; soon the motion is less so; and eventually it is steady. The duration of the erratic part depends principally on the damping; this part, being short lived, is called the *transient vibration.* The remaining part is called the *steady vibration.*

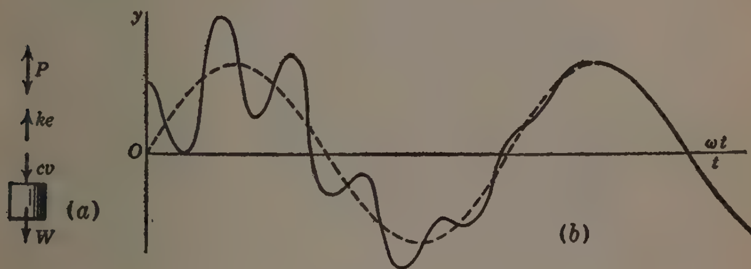

FIG. C9.

It is shown below that the steady vibration is a shm, that its frequency is equal to that of the excitation, and that it always lags the excitation.

The forces acting on the vibrating body at any instant are P, W, and the spring force S. The resultant of S and W is $-ky - cv$ (see Art. 6). Hence, the equation of the motion is

$$P_0 \sin \omega t - ky - c\frac{dy}{dt} = m\frac{d^2y}{dt^2} \tag{1}$$

Or, writing Eq. 1 as a differential equation in standard form,

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{P_0}{m}\sin \omega t \tag{2}$$

The general solution of this equation is (see any textbook on differential equations)

$$y = [e^{-ut}(B_1 \sin qt + B_2 \cos qt)] + [C \sin \omega t + D \cos \omega t] \tag{3}$$

B_1, B_2, u, and q have meanings as in Art. C6; C and D are constants, depending on certain physical characteristics of the system and the excitation.

The square brackets in Eq. 3 indicate two components of the right-hand member which, for convenience, we denote by y' and y'', respectively, so that $y = y' + y''$. The first component y' is, you see, a damped free vibration; in due time the vibration y' vanishes, and thereafter the vibration is $y = y''$. The first part of the motion, while y' is appreciable, is the transient vibration; and the second part, while $y = y''$, is the steady vibration. The equation for the steady vibration can be put into the form

$$y = y_0 \sin(\omega t - \phi) \tag{4}$$

From Eq. 4 you see that, as stated, *the steady vibration is a simple harmonic one, that its frequency is just the same as that of the excitation, and that it lags the excitation P by the amount ϕ.* See Eq. 5.

To find formulas for the amplitude y_0 and the lag angle ϕ, we employ Eq. 4, writing $\theta = \omega t - \phi$ for convenience; then in the steady vibration

$$y = y_0 \sin\theta \qquad \frac{dy}{dt} = y_0\omega \cos\theta \qquad \frac{d^2y}{dt^2} = -y_0\omega^2 \sin\theta$$

$$P_0 \sin(\theta + \phi) - ky_0 \sin\theta - cy_0\omega \cos\theta = -my_0\omega^2 \sin\theta$$

Since the last equation holds for any value of t, in the steady vibration, it holds for any value of θ. For $\theta = 0$ and $\theta = 90°$, respectively, it becomes

$$P_0 \sin\phi - cy_0\omega = 0 \qquad \text{and} \qquad P_0 \cos\phi - ky_0 = -my_0\omega^2$$

Solved simultaneously, these give

$$\phi = \tan^{-1} \frac{c\omega}{k - m\omega^2} \tag{5}$$

$$y_0 = \frac{P_0}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} \tag{6}$$

Since $c_c = 2\sqrt{km}$, see Art. C6, Eq. 5 can be transformed into

$$\phi = \tan^{-1} \frac{2(c/c_c)(\omega/p)}{1 - (\omega/p)^2} \tag{7}$$

Let e_s denote the elongation of the spring due to a steady force equal to P_0; then $P_0 = ke_s$. Equation 6 can now be transformed into

$$\frac{y_0}{e_s} = \frac{1}{\sqrt{[1 - (\omega/p)^2]^2 + [2(c/c_c)(\omega/p)]^2}} = \beta \tag{8}$$

where β stands for the ratio y_0/e_s; it is called *magnification factor.* Then $y_0 = \beta e_s$, from which one calculates amplitude of a steady-state forced vibration.

The right-hand members of Eqs. 7 and 8 contain two ratios. The damping

ratio c/c_c, we remind you, is the quotient of the actual damping factor and the critical damping factor of the vibrating system. The second ratio is the quotient of the frequency of the excitation P and the frequency of the free vibration of the system without damping. This quotient may be computed from circular or ordinary frequencies, of course. It is often called *frequency ratio* for convenience.

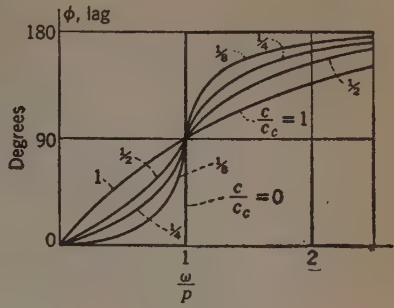

Fig. C10.

Figures C10 and C11 show some graphs of Eqs. 7 and 8; they disclose some important facts. From Fig. C10 you see that: at low frequency of excitation, say $\omega < 0.2p$, the vibration is nearly in phase with the excitation; if $\omega = p$, the vibration lags the excitation 90° or one-quarter period; if $\omega > 3p$, the vibration lags behind the excitation about 180° or one-half period. From Fig. C11 you see that: at low frequencies of excitation, say $\omega < 0.2p$, β is nearly 1; if $\omega = p$, then β is near a maximum; and if $\omega > 3p$, then β is nearly 0.

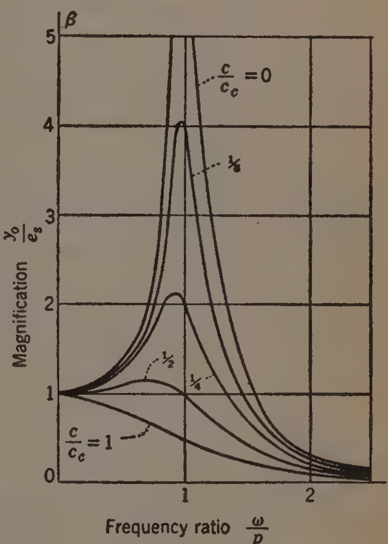

Fig. C11.

Any vibration of maximum amplitude is a *resonance phenomenon;* the vibrating system and the exciter or agent producing the vibration are said to be *in resonance*, and the vibration and the frequency are resonant. If a running machine is producing a resonant vibration, the running speed is called a *critical speed* of the machine, particularly if the vibration is harmful or objectionable.

An interesting illustration of resonance is afforded by a frequency meter known as Frahm's tachometer (see Fig. C12). It consists of a row of strip steel cantilever springs embedded in a base A; each is loaded with a weight at its free end. The natural frequency of each loaded spring is indicated on the instrument. Suppose that the instrument is given a shm perpendicular to the plane of the springs. If the frequency of this motion is within the frequency range of the instrument, one or more springs vibrate conspicuously. The vibration

Fig. C12.

of the one whose amplitude is greatest is at resonance or nearly so; and its indicated natural frequency is approximately the same as the frequency of the base.

Consider now a system (Fig. C9a) for which the frequency ratio ω/p is large, say 5 or larger. (The excitation is rapid, the spring is soft, and the mass is small.) The magnification factor β is very small, see Fig. C11; hence the suspended body stands still practically, and the force exerted by the spring on its support is practically constant. These circumstances obtain in the motor systems of some electric refrigerators. Not being in perfect dynamic balance, the rotors develop rapid excitations; the motors are supported by soft springs. The overall effect is nearly constant spring forces and little or no induced vibration and noise in the refrigerator.

Consider next a system for which ω/p is very small, so that practically $\phi = 0$ and $\beta = 1$. Hence $y_0 = e_s = P_0/k$, and $y = (P_0/k)\sin \omega t$. And, since $P = P_0 \sin \omega t$, $y = P/k$; that is, displacements y respectively are proportional to forces P. This fact suggests that the simple vibration apparatus, suitably supplemented, could be used to record the variation in a periodic force such as the pressure in a cylinder of a running steam engine. Indeed it is so employed in engine indicators. (Perhaps it occurs to you that the theory above presupposes a force P that varies simple-harmonically. It is shown in Art. C10 that the instrument functions also for periodic forces that vary otherwise.)

C8. A Forced Vibration with Damping, Continued. Figure C13 represents a system consisting of a spring and a weight W. For purposes of description it is shown supported in a box. Suppose that (a) represents the equilibrium position of the system; AB on the box is a datum or reference line; the tension in the spring is $S_0 = ke_s = W$. Suppose now that the box is given a vertical shm, $z = z_0 \sin \omega t$. Suppose that (b) represents the system in a displacement z; x is the displacement of the weight with respect to the box. The absolute displacement of the weight is $y = z + x$. The stretch of the spring is $e_s - x$, and the spring force S is $k(e_s - x)$; the restoring force, without damping, is $S - W = -kx$. We take the damping force as proportional to the relative velocity of W; thus the equation of motion of W is

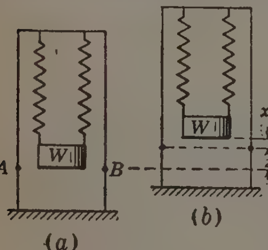

FIG. C13.

$$-kx - c\frac{dx}{dt} = m\frac{d^2y}{dt^2}$$

This can be transformed readily into

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = z_0\omega^2 \sin \omega t \qquad (1)$$

which is quite analogous to

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{P_0}{m}\sin\omega t \tag{2}$$

from Art. C6. You see that $z_0\omega^2$ corresponds to P_0/m. And, by analogy, the equation of the steady state of the x vibration (the one here under discussion) is

$$x = x_0 \sin(\omega t - \phi) \tag{3}$$

where ϕ is the lag of the x vibration with respect to the z vibration. The formula for ϕ is just like Eq. 7, Art. C6; and Fig. C10 holds here also.

In the steady-state vibration of Art. C6, $y_0 = \beta P_0/k$; therefore

$$x_0 = \beta\frac{mz_0\omega^2}{k} = \beta\left(\frac{\omega}{p}\right)^2 z_0$$

or

$$\frac{x_0}{z_0} = \left(\frac{\omega}{p}\right)^2 \beta = \beta' \tag{4}$$

where β' denotes magnification factor for the x vibration.

Figure C14 shows graphs of Eq. 4 for several values of damping ratio c/c_c. For larger and larger values of frequency ratio ω/p, β' approaches 1 and ϕ approaches π. For $\omega/p = 5$ or more, $\beta' \approx 1$, whence $x_0 = z_0$, and $\phi \approx \pi$; therefore $x = x_0 \sin(\omega t - \pi) = -x_0 \sin \omega t = -z_0 \sin \omega t$. And, since $y = x + z$, $y = -z_0 \sin \omega t + z_0 \sin \omega t = 0$, that is, the weight (Fig. C13) does not move relative to its equilibrium position; it "stands still in space." Hence the suspended weight serves as a fixed base or datum to which the vibration z of the box may be referred.

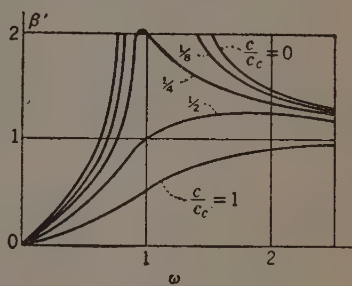

Fig. C14.

The system (box, soft springs, and heavy weight) serves as a vibration apparatus. Supplemented with a dial that indicates amplitudes (z_0), it is a *vibrometer;* supplemented with a device that records $z = z_0 \sin \omega t$, it is a *vibragraph.* (Perhaps it occurs to you that the theory underlying this instrument presupposes shm. It is shown in Art. C10 that the instrument functions for other vibrations also.)

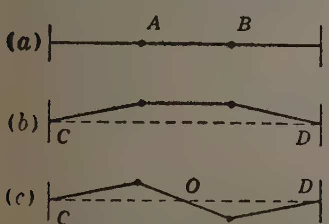

Fig. C15.

C9. Systems with Several Degrees of Freedom. The systems discussed in preceding articles have a single degree of freedom or were so considered; a single coordinate served to define the position of the vibrating body. A two-degree-of-freedom system is represented in Fig. C15. It con-

sists of a cord tightly stretched between two supports and two equal masses fastened to the cord at its third points. Considering only small displacements of A and B in the plane of the paper, two coordinates (y_1 and y_2) of A and B respectively describe the position or configuration of the system. It has two degrees of freedom.

Suppose that the system is released from a symmetrical configuration (Fig. C15, b or c). It is obvious that A and B move quite alike; the pattern of the configuration is preserved, and the motion of the system is called a ***normal mode*** of vibration. In the first mode there are two ***nodes*** or points of rest (C and D); in the second there are three nodes (C, O, and D).

The circular frequencies of these normal vibrations respectively are

$$p_1 = \sqrt{\frac{3T}{lm}} \quad \text{and} \quad p_2 = \sqrt{\frac{9T}{lm}} \tag{1}$$

where T denotes tension in the cord, l length, and m mass of the cord. (These formulas are derived in Example 1 below.)

The foregoing suggests that a system of three equal masses equally spaced on a stretched cord has three normal modes of vibration. And since the mass of a bare cord may be looked upon as consisting of an infinite number of parts it appears that a cord stretched between end supports has an infinite number of normal modes of vibration.

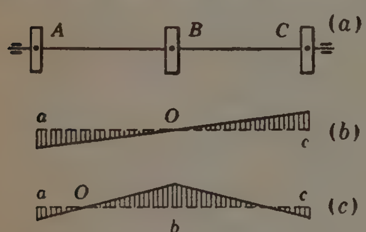

FIG. C16.

The torsional system represented in Fig. C16a consists of three like disks equally spaced on a shaft resting in end bearings. It has two normal modes of vibration; their patterns are indicated in (b) and (c). In the first mode, the middle disk does not move; the outer ones have equal and opposite velocities at each instant; there is one node. In the second mode, the outer disks move exactly alike; at each instant the motion of the middle disk is opposite to that of the outer ones; there are two nodes. The circular frequencies of these two vibrations respectively are

$$p_1 = \sqrt{\frac{k'}{I}} \quad \text{and} \quad p_2 = \sqrt{\frac{3k'}{I}} \tag{2}$$

where k' is the spring factor of a portion of the shaft between disks A and B or B and C, and I is the moment of inertia of one disk about the axis of the shaft. These formulas do not take the inertia of the shaft into account; they are derived in Example 2 below.

Figure C17 indicates the first two normal modes of vibration of a flexible beam or rod. In (*a*) the beam is freely supported at its ends, so that it can swivel about the supports; in (*b*) the beam is supported along its length as by a frictionless table.

(*a*) (*b*)

FIG. C17.

The foregoing illustrations suggest that there are many systems having more than one natural frequency of vibration, one for each normal mode of vibration. A multithrow crankshaft is a practical example. If the shaft has more than three throws, its natural frequencies of vibration (torsional) are found by a trial-and-error method, quite satisfactory for one experienced in such calculations.

EXAMPLE 1. It is required to derive Eq. 1 above. For small displacements, T remains practically constant, and the restoring force on A in (*b*) is $-T(y \div \frac{1}{3}l) = -(3T \div l)y$, where y denotes the displacement of A. It follows that the vibration of A is a shm, and $p^2 = (3T/l) \div m$. The restoring force for A in (*c*) is

$$-T(y \div \tfrac{1}{3}l) - T(y \div \tfrac{1}{6}l) = -9(T \div l)y$$

Hence the vibration of A (and of B) is a shm, and $p^2 = (9T/l) \div m$.

EXAMPLE 2. It is required to derive Eq. 2 above. In the first mode of vibration, the system consisting of A and the part of the shaft between A and B may be looked upon as a torsional pendulum (see Art. C4). Hence the natural frequency of A is $\sqrt{k'/I}$. In the second mode of vibration the system consisting of Oa and disk A may be looked upon as a torsional pendulum; also Ob and half of B. Let $l = ab$, $l_1 = Oa$, $l_2 = Ob$, and let k_1' and k_2' be the spring factors of Oa and Ob; then

$$p^2 = \frac{k_1'}{I} = \frac{k_2'}{\frac{1}{2}I} \qquad \text{and} \qquad k_1' = 2k_2'$$

It follows that $l_1 = \frac{1}{2}l_2$, $l_1 = \frac{1}{3}l$, and $l_2 = \frac{2}{3}l$; hence, $k_1' = 3k'$, $k_2' = 1.5k'$, and $p = \sqrt{3k'/I}$.

C10. Nonsimple Harmonic Excitations. The excitations dealt with in Arts. C7 and 8 were assumed to be simple harmonic; not all are such. However, the results arrived at in those articles can be utilized in cases of periodic excitations. We indicate the method presently, but first it is necessary to consider a subject called harmonic analysis.

HARMONIC ANALYSIS. It can be shown that any given periodic vibration $y = f(t)$ can be resolved into components such that

$$y = y_a + y_1 + y_2 + y_3 + \cdots$$

where y_a is the average value of y over the interval of one period or cycle of y, and $y_1, y_2, y_3 \cdots$ are defined by

$$y_1 = A_1 \sin \omega t + B_1 \cos \omega t = C_1 \sin (\omega t + \alpha_1)$$

$$y_2 = A_2 \sin 2\omega t + B_2 \cos 2\omega t = C_2 \sin (2\omega t + \alpha_2)$$

where ω denotes the circular frequency of y. The components y_1, y_2, etc., are

shm's; their amplitudes are C_1, C_2, etc., and their frequencies are ω, 2ω, etc. In a given case, one or more of these components may be absent. The harmonic components are designated as first, second, etc.; the first is also called the fundamental harmonic and the others higher harmonics. Equation 1 with the given expressions for y_1, y_2, etc., inserted is known as a *Fourier series*.

The process of resolving y into harmonic components is called an *harmonic analysis* of y. If the equation $y = f(t)$ is known in a given case, the harmonic analysis can be made quite directly by means of available formulas. If, as generally happens, only a more or less accurate graph of y is known, an approximate analysis can be made by means of one of several practical calculation programs or schedules that have been devised; and special calculating machines, harmonic analyzers, have been contrived for such cases. Any one of these methods determines the A's and the B's, from which the C's and α's can be computed readily.

Although the foregoing pertains to periodic *motions*, it is obvious that any quantity, force, couple, etc., that varies periodically can be analyzed in the manner just indicated. The components y_1, y_2, etc., of such quantities too are called harmonics. The turning moment or torque on a crank of a running engine is a practical example of a periodic excitation. (See "Harmonic Coefficients of Engine Torque Curves" by F. P. Porter, *Trans. A.S.M.E.*, Vol. 65, 1943. The paper gives results, coefficients A, B, and C, of more than 100 harmonic analyses.)

The foregoing is now employed in three examples of nonsimple harmonic excitation. 1. The first example pertains to the engine indicator already noted in Art. C7. Since an engine is indicated under conditions of constant load, speed, and boiler pressure, the steam pressure in a cylinder is periodic, and the total pressure P on the piston of the indicator can be expressed by a Fourier's series. It is assumed that the highest significant component of P has a frequency that is very small in comparison with the natural frequency of the instrument so that the magnification factor β for each component is 1 and the lag angle ϕ is 0. It follows from Art. C7 and the principle of superposition that

$$y_1 = \frac{P_1}{k} \qquad y_2 = \frac{P_2}{k} \qquad y_3 = \frac{P_3}{k} \qquad \text{etc.}$$

and since $y = y_1 + y_2 + y_3 + \cdots$

$$y = \frac{P_1 + P_2 + P_3}{k} = \frac{P}{k}$$

That is, all displacements y of the recording pencil are proportional to the forces P respectively and so the "indicator card" (graph of y) gives the desired information about the steam pressure in the cylinder.

2. The second example pertains to a vibrometer already noted in Art. C8. It is assumed that the vibration z of the housing or box of the vibrometer

(Fig. C13) is periodic; then $z = z_1 + z_2 + \cdots$, where z_1, z_2, etc., are harmonic components of z. Let ω denote the frequency of z (and of z_1) and p the natural frequency of the instrument. It is assumed also that the frequency ratio of ω/p is very large, so that (see Fig. C14) the magnification factor β' for z_1 is practically 1; the factors β' for upper harmonics are still nearer 1. Hence, the suspended mass of the instrument "stands still" for all harmonic components, and serves as base or datum, fixed in space, to which the vibrations of the box or housing are referred.

3. The third example pertains to the simple system represented in Fig. C2, which has a natural frequency p and is subjected to an excitation P; P has a frequency ω, and three harmonic components, namely,

$$P_1 = P_{01} \sin (\omega t + \alpha_1)$$
$$P_2 = P_{02} \sin (2\omega t + \alpha_2)$$
$$P_4 = P_{04} \sin (4\omega t + \alpha_4)$$

Each component is in resonance when its frequency is equal to p; that is,

P_1 is in resonance when $\omega = p$
P_2 is in resonance when $2\omega = p$, or $\omega = \frac{1}{2}p$
P_4 is in resonance when $4\omega = p$, or $\omega = \frac{1}{4}p$

P is said to be in resonance when its frequency equals p, $\frac{1}{2}p$, or $\frac{1}{4}p$. If P were acting on a system having several degrees of freedom, it would have still more resonant frequencies.

If a system has several normal modes of vibration, and is subjected to several excitations each of which has several harmonic components, the determination of the vibrations which these excitations produce is obviously very complicated. The crankshaft of a multicylinder engine is such a system. In designing such a crankshaft a torsional analysis must be made to determine especially whether at running speeds of the engine all vibrations and stresses in the shaft are safe. See *Practical Solution of Torsional Vibration Problems* (two volumes), by W. Ker Wilson.

PROBLEMS

CHAPTER II

Art. 13. Composition of Two Concurrent Forces

1. Figure 1 represents a wood frame 8 ft square to which forces are applied as indicated; they are applied by means of ropes tied to nails in the frame. By inspection, estimate where you might drive the nail and how hard and in what direction you would pull the rope to get, from a single force, the same effect as the 50- and 110-lb forces give. Do the same for the 100- and 90-lb forces; for the 60- and 40-lb forces.

2. By means of the parallelogram construction, determine completely the resultant of each of the pairs of forces named in Prob. 1. (To describe the line of action of a resultant, state where it cuts an edge of the frame.) Compare the resultants so found with the estimated resultants. *Ans.* *R of* 100- *and* 90-*lb forces* = 134.5 *lb, left and up at* 48° *to horiz.*

3. Figure 2 shows a rigid cross acted on by several forces. Make a good freehand sketch of this cross, estimating distances on the sketch at a scale of 1 in. = 4 ft. Using your best judgment, indicate the line of action of the resultant R of the 40- and 50-lb forces, and record the distance from G to the intersection of R with the floor line. Indicate also the sense of R in the usual way.

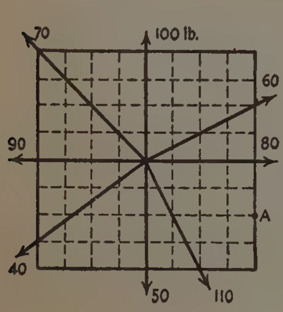

Fig. 1.

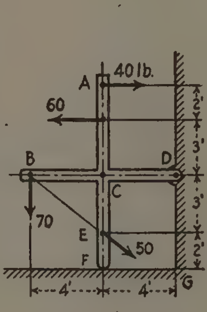

Fig. 2.

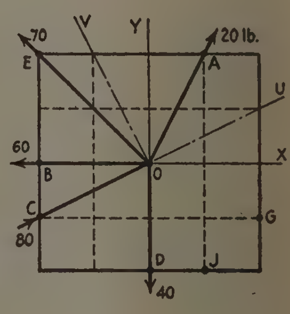

Fig. 3.

4. Make a good scale drawing of the cross of Fig. 2. (Suggested scale 1 in. = 4 ft.) By parallelogram construction (suggested scale 1 in. = 10 lb) determine the resultant R of the 50- and 60-lb forces completely. Record the magnitude of R and the distance from G to the intersection of R with the floor line; also indicate the sense of R. Do the same for the 40- and 50-lb forces, and compare with the estimated results obtained in Prob. 3. *Ans.* *R of* 40- *and* 50-*lb forces* = 85.4 *lb, right and down, cutting floor line* 12 *ft to the right of G.*

5. Compound the 50- and 60-lb forces of Fig. 1 by means of the triangle law. (Make the vector diagram separate from the space diagram.)

6. Using the cross you made for Prob. 4, determine the resultant R of the 50- and 70-lb forces by the triangle construction completely. Record the magnitude of R, and the distance from G to the intersection of R with the floor line; also indicate the sense of R. *Ans.* R = 108 *lb; cuts floor line* 6 *ft to the left of G.*

7. Compound the 60- and 70-lb forces of Fig. 1 algebraically. (Specify the direction of the resultant by means of the angles between it and the two given forces.)

8. Compound the 50- and 90-lb forces of Fig. 1 algebraically.

9. Figure 3 represents a frame 4 ft square on which a number of forces act as shown. Determine the resultant of the 60- and 40-lb forces: (*a*) by inspection, (*b*) graphically, (*c*) algebraically.

10. What is the least force P which, acting through G, Fig. 3, will make the resultant of P and the 20-lb force a horizontal force? Determine this resultant.

11. What force Q, acting through G, Fig. 3, will make the resultant of Q and the 20-lb force equal to 80 lb acting down and to the right at 30° to the vertical? Locate the line of action of this resultant.

12. The angle between the lines of action of two concurrent forces P and Q is θ. How large is the resultant R when $\theta = 0$? 90°? 180°? For what angle is the resultant greatest? least? Suppose that P and Q are equal, say of magnitude F. For what value of θ does R equal F?

Art. 15. Resolution of a Force into Two Concurrent Components

1. Resolve the 40-lb force of Fig. 1 into two components, one parallel to the 70-lb force and one vertical, by a graphical method. *Ans. Vertical component* = 55 *lb down.*

2. Resolve the 100-lb force of Fig. 1 into two components, one of which acts in the lower edge of the square and the other through the upper right corner.

3. Resolve the 80-lb force of Fig. 3 into two components, one through E, the other along the lower edge of the square.

4. Resolve the 80-lb force of Fig. 3 into two components, one through D, the other horizontal and equal to 100 lb.

5. Resolve the 80-lb force of Fig. 3 into two equal forces, one of which acts along the left edge of the square.

6. Resolve the 60-lb force of Fig. 1 into three components, applied along the two sides and bottom of the square. *Ans. Component along right side* = 13 *lb down.*

7. Draw a square and number the corners 1, 2, 3, and 4, beginning at the upper left-hand corner and proceeding in a clockwise direction. Imagine a force of 100 lb to act in $\overline{12}$ and in the direction $\overline{12}$. Resolve it into components acting in the other three sides.

Art. 16. Rectangular Components of a Force and of a System

1. Resolve the 60-lb force of Fig. 1 into two components, one horizontal and one vertical. *Ans. Horiz. comp.* = 53.6 *lb to right.*

2. Determine the x and y components of the force system shown in Fig. 1. *Ans. x comp.* = 11.3 *lb to right.*

3. Determine the horizontal and vertical components of the force system represented in Fig. 3.

4. Determine the u and v components of the force system of Fig. 3. Do this in each of two ways, first by getting the u and v components of each force and adding, second by resolving the x and y components of the system (found in Prob. 3) along the u and v axes.

Art. 17. Resolution of a Force into x, y, and z Components

1. Resolve the 30-lb force of Fig. 4 into x, y, and z components. (Figure represents a 3 ft cube.) *Ans. x comp.* = *z comp.* = -19.2 *lb.*

2. The table in Fig. 5 is 6 by 3 and $2\frac{1}{2}$ ft high; force $F = 100$ lb. Determine the x, y, and z components of F. First Method: Resolve F into two rectangular components parallel and perpendicular to a coordinate axis; then resolve the perpendicular component into two that are parallel to the other two axes. In a copy of the figure, represent the lines of action of all

these components. Second Method: Multiply F in succession by its "direction-cosines" (cosines of the angles between F and the coordinate axes).

Art. 18. Moment or Torque of a Force

1. Compute the moment of the 60-lb force of Fig. 1 about point A. *Ans.* -215 *ft-lb.*

2. With reference to Fig. 2, compute the moments of the 60-lb force with respect to A, B, and G; also the moment of the 50-lb force with respect to D.

3. Calculate the moment about C of the five-force system of Fig. 3.

Art. 19. Law of Moments

1. Figure 6 represents a 4 by 4 ft board and a force F = 100 lb applied at B. (*a*) Calculate the moment of the force F about C (force times perpendicular arm). (*b*) Resolve F into x and y components at D, and calculate the moment of each component about C; add these moments algebraically, and compare this sum with the result obtained in (*a*). (*c*) Do as just directed but for components applied at A; at B. (*d*) Is there any choice among the calculations?

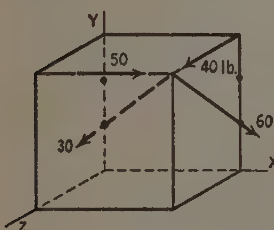

FIG. 4.

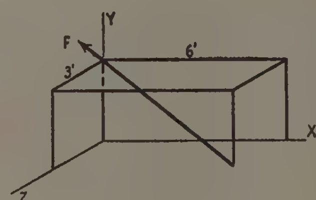

FIG. 5.

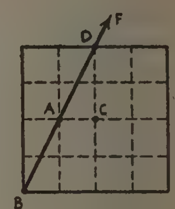

FIG. 6.

2. There is a point X on the floor line of Fig. 2 about which the moment of the 40- and 70-lb forces is zero. Locate this point. How large is the moment about X of the resultant R of the two forces? What can you say about the line of action of R?

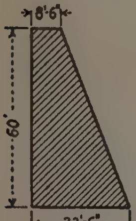

FIG. 7.

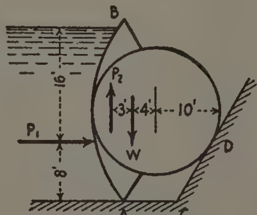

FIG. 8.

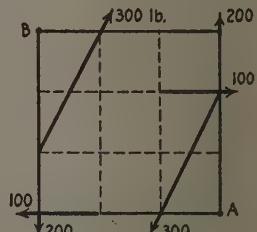

FIG. 9.

3. Figure 7 represents the cross section of a masonry dam. It weighs 150 lb/ft³, and the water pressure against it is 112,500 lb per foot length of dam. The resultant pressure acts at right angles to the face of the dam and 20 ft above its base. The center of gravity of the cross section is 11.46 from the face of the dam and 24 ft above the base. Find where the resultant of the weight and the pressure cuts the base, using the principle of moments. *Ans.* 23.7 *ft to right of upstream (left) face.*

4. A certain chimney is 150 ft high and weighs 137,500 lb. Suppose that it is subjected to a horizontal wind pressure of 54,000 lb, uniformly distributed along its height. Determine where the line of action of the resultant of the weight and pressure cuts the ground, using the principle of moments. *Ans.* *29.45 ft from center of base.*

5. Figure 8 is a cross section of a "rolling dam." AB is the sheath rigidly fastened to the cylinder which can be rolled upward on two inclined racks CD, one at either end of the dam. The figure shows the dam resting on the bed at A and against the rack at D. In that position the horizontal and vertical components, P_1 and P_2, respectively, of the water pressure are 180 and 30 tons. The weight W is 70 tons. The racks CD are inclined at 55° to the horizontal, and the distance C to D is 9 ft. Without computing the resultant of these three forces, ascertain how far from D its line of action intersects the line CD, using the principle of moments. *Ans.* *1.67 ft below D.*

Art. 20. Moment of a Couple

1. Represent a force applied at E, Fig. 2, equal and opposite to the 60-lb force, and then calculate the moments about A, B, and G of the pair of 60-lb forces. *Ans.* *About B,* $M = +360$ *ft-lb.*

2. Represent a force F_6 acting through C, Fig. 3, equal, parallel, and opposite to the 60-lb force. Compute the (combined) moment of the two forces about A; about B; about C; about D; and about a point midway between B and C.

3. The board of Fig. 9 is 3 ft square. Calculate the moment or torque of the three couples about A; about B.

Art. 24. Moment of a Force and of a Couple (Continued); General Case

1. Figure 10 represents three forces applied at A, rigidly connected to a stiff coordinate frame. Required the moments of each force about each axis. (Give signs to the moments viewing the rotations respectively from X, Y, and Z.) *Ans.* *Moment of 10-lb force about axis* $Z = -20$ *ft-lb.*

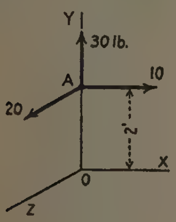

FIG. 10.

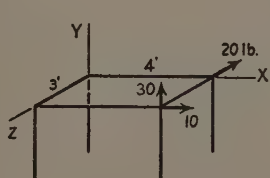

FIG. 11.

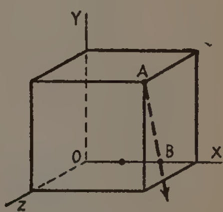

FIG. 12.

2. Figure 11 represents a table 4 by 3 by $2\frac{1}{2}$ ft high and three forces applied to it as shown. Required the moment of each force about each coordinate axis.

3. Figure 12 represents a 3-ft cube and a force $F = 100$ lb applied at A. The line of action of F passes through B, 2 ft from O. (*a*) Calculate the moments of F about each of the coordinate axes, assuming the components of F applied at A. (*b*) Solve, assuming the components applied at B, and compare results with those arrived at in (*a*).

4. Compute the moments of each of the forces represented in Fig. 4 about the x, y, and z axes. *Ans.* *Moment of 30-lb force about axis* $X = -19.2$ *ft-lb.*

5. What is the moment of the force system of Fig. 4 about each axis?

6. Figure 13 represents a wedge-shaped block in the sloping face of which a couple of 100 ft-lb acts as shown. What is the moment of this couple about the y axis? the z axis? a line drawn from A to O which slopes at 30° to the x axis?

Art. 25. Composition of Couples (Continued), and Resolution

1. Figure 14 represents a vertical wall and a horizontal floor. In the plane of the wall there are two couples, one of 100 ft-lb and one of 50 ft-lb, both clockwise viewed from the right. In the plane of the floor are three couples, one of 80 ft-lb and one of 100 ft-lb, both clockwise viewed from above, and one of 300 ft-lb counterclockwise viewed from above. Determine the resultant of these five couples.

2. Resolve the 100-ft-lb couple shown in Fig. 13 into two component couples, one in a vertical plane and one in a horizontal plane.

Art. 26. Concurrent System; Graphical Composition

1. Compound the 70-, 90-, 100-, and 110-lb forces of Fig. 1 graphically. (Specify the direction of the resultant by means of the angle θ between it and the horizontal.) *Ans.* $R =$ 103.8 *lb;* $\theta = 29.6°$.

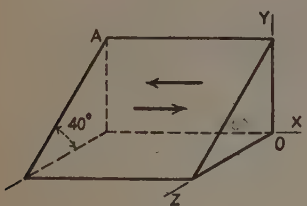

FIG. 13.

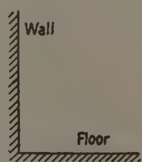

FIG. 14.

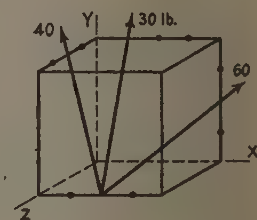

FIG. 15.

2. Compound the 40-, 50-, 60-, and 70-lb forces of Fig. 1 graphically. (Specify the direction of the resultant by telling where it cuts one edge of the square, extended if necessary.)

3. With reference to Fig. 3, determine the least force which, added to the five-force system shown, will make the resultant of the six-force system vertical, and determine this vertical resultant.

Art. 27. Concurrent System; Algebraic Composition

1. Solve Prob. 1 of Art. 26 algebraically.
2. Solve Prob. 2 of Art. 26 algebraically.
3. Solve Prob. 3 of Art. 26 algebraically.
4. Find the resultant of the five forces of Fig. 3.
5. Compound the 30-, 40-, 50-, and 60-lb forces of Fig. 4. *Ans.* $R =$ 62.6 *lb, toward right, downward, and backward at angles of* 60.5°, 42.5°, *and* 62.3° *to the x, y, and z axes respectively.*
6. Find the resultant of the three forces which act on the 4-ft cube of Fig. 15.

Art. 28. Coplanar Noncurrent System; Graphical Composition

1. Figure 16 represents a board in the form of a 3-ft equilateral triangle, along each side of which a force acts as shown. Determine the resultant of the three forces, and tell where it cuts the base of the board (extended if necessary).

2. Figure 17 represents a drawing board 8 ft square, and five forces applied to it. Their resultant is to be completely determined graphically. (Observe that the magnitudes and direction of the forces respectively are just like those of Prob. 4 of Art. 27, and that one system is concurrent and the other is nonconcurrent. Hence the resultants of the two systems are alike in magnitude and in direction; but the lines of action of the resultants are in general different. The line of action of the resultant of the concurrent system obviously contains the point of concurrence of the given forces. The line of action of the resultant of the noncurrent system is not obvious; it must be "determined." See Art. 28, where two methods of solution are explained.)

3. Determine graphically the resultant R of the four forces of Fig. 2. Record the magnitude of R and the intercepts of the line of action of R on the floor and wall lines. *Ans.* $R =$ 102 *lb; cuts floor* 2.8 *ft to left of F.*

4. Compound the four forces (wind pressures) represented in Fig. 18. (Be prepared to give the inclination of the resultant and the point where the line of action cuts the floor.) *Ans.* $R =$ 30,700 *lb, downward and toward the right at* 28° *to horizontal, cutting floor at* 75 *ft from left side of building.*

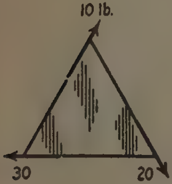

FIG. 16.

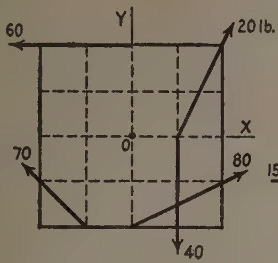

FIG. 17.

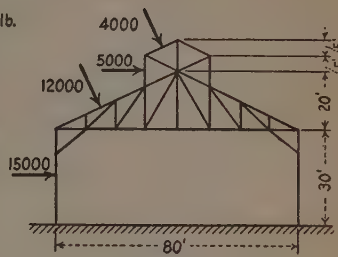

FIG. 18.

5. Figure 19 is a half-section of a building. The four forces are wind pressures, perpendicular to and applied at midpoints of the portions $\overline{01}$, $\overline{12}$, etc. $P = 10$, $P_1 = 7$, $P_2 = 4$, and $P_3 = 2$ tons. Determine completely the resultant of the four forces graphically. Note where the line of action cuts the coordinate axes. (The figures in parentheses are coordinates of the points 1, 2, and 3, with respect to O.)

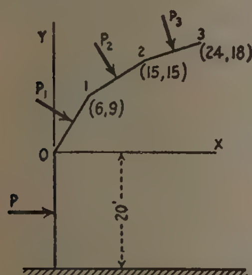

FIG. 19.

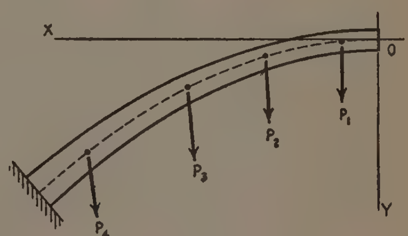

FIG. 20.

6. Figure 20 represents one-half of an arch and certain loads applied to it. $P_1 = 4000$, $P_2 = 5000$, $P_3 = 6000$, and $P_4 = 10{,}000$ lb; their inclinations are 0°, 3°, 8°, and 12°, respectively; the coordinates of their points of application are (1.6, 0.1), (4.9, 0.7), (8.4, 2.1), and (12.8, 4.8), all in feet. Compound the four loads by the funicular-polygon method. (Specify the line of action of the resultant by means of the angle between it and the x axis, and the intercept on that axis.) *Ans.* $R =$ 24,900 *lb; cuts x axis* 8.8 *ft to the left of O;* $\theta = 82.5°$.

Art. 29. Coplanar, Parallel System; Algebraic Composition

1. Figure 21 represents a 12-ft board on which six parallel forces act. What is the magnitude of the resultant R of the six forces? Does R act up or down? What is the magnitude of the moment of the system about A? Is the moment clockwise or counterclockwise? Is the

moment of R about A clockwise or counterclockwise? On which side of A is R? How large is the moment of R about A? What is the arm of R with respect to A? Indicate R in your copy of the figure.

2. Check your answer or result for the preceding problem by calculating the moment of the given system (six forces) about a point on the determined line of action of R. What should this calculated moment be?

3. The board represented in Fig. 22 is 16 ft long. Determine the resultant and the equilibrant (*a*) of the first six forces, counting from the left; (*b*) of the third to seventh force inclusive; (*c*) of the entire system.

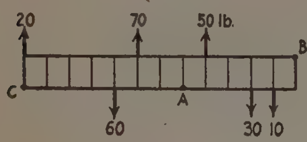

FIG. 21.

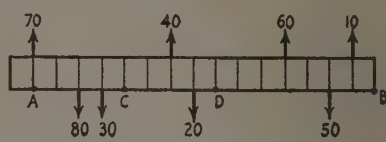

FIG. 22.

4. Determine the resultant of the locomotive wheel-loads in Fig. 23. *Ans.* $R = 184{,}000$ *lb, vertically downward at* 16 *ft* 10 *in. to right of first wheel.*

5. On a horizontal line mark off, from left to right and at intervals of 1 in., points designated as A, B, C, D, and E. Consider vertical forces to act through A, B, and D as follows: A, 10 lb up; B, 40 lb down; D, 20 lb up. Determine the resultant of the forces at A and D; at A and B.

6. Resolve the 40-lb force of the preceding problem into components applied at A and E; at C and E.

7. Replace the three forces of Prob. 5 by two forces, one acting through A and the other through E.

Art. 30. Coplanar Noncurrent Nonparallel System; Algebraic Composition

1. For a certain system of forces acting in the plane of a square board whose center is O it is known that

$$\Sigma F_x = +10 \text{ lb} \qquad \Sigma F_y = -20 \text{ lb} \qquad \Sigma M_0 = +30 \text{ ft-lb}$$

What is the general direction of the resultant R? On which side of O does R lie? How far is R from O? Represent R in a sketch of the board.

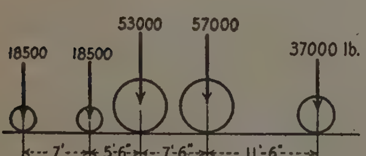

FIG. 23.

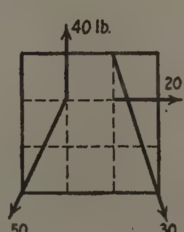

FIG. 24.

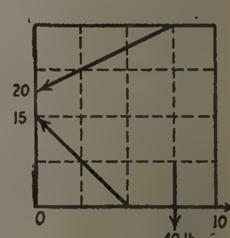

FIG. 25.

2. Change $+30$ in the preceding problem to -30, then solve the problem.

3. The board of Fig. 24 is 3 ft square. Determine the magnitude, direction, and line of action of the resultant R of the applied forces. (Employing the method explained, determine the magnitude and sense of R_x and R_y. From these, ascertain the magnitude and direction of R. Next determine the moment or torque of the given system about some selected origin

or center of moments, denoted by O. This is also the moment or torque of R about that point. Knowing now the sense of the torque of R and the direction of R, ascertain on which *side* of the point R lies; finally calculate the arm of R with respect to the point, and so complete the location of the line of action of R.) *Ans.* $R = 33.8$ *lb; acts down and right at* 77° *to horiz.; line of action is* 1.12 *ft from point where* 40- *and* 50-*lb forces intersect.*

4. Determine completely the resultant of the four forces described in Fig. 25. (Locate the line of action of the resultant in this way: Determine the moment of the system about the lower left corner of the square O. Then imagine the resultant to be resolved into its components R_x and R_y, at the point where it cuts the lower edge of the square. Since R_x passes through O, the moment of R_y about O must equal the moment of the system. Making use of this fact, determine the arm of R_y and thus ascertain where it cuts the lower edge of the square.)

5. Determine algebraically the resultant of the loads described in Prob. 6 of Art. 28.

6. Imagine a clockwise couple of 2 ft-lb to act on the square board of Fig. 1. Then compound the couple and the 40-lb force.

7. Resolve each of the forces shown in Fig. 26 into a force applied at the center of the pulley and a couple. Take the diameter of the pulley as 3 ft.

8. Solve Prob. 4 as follows: Resolve each of four given forces into a force acting through O and a couple, thus replacing the given system by a set of four concurrent forces and a set of four couples. Then find the resultant of the four concurrent forces and the resultant of the four couples. Finally, compound the resultant force and the resultant couple, and thus determine the resultant of the original system.

Art. 31. Noncoplanar Parallel System; Composition

1. The six forces, Fig. 27, act on a board 4 ft square as shown. Determine the resultant (*a*) of the smallest five forces; (*b*) of the largest five; (*c*) of all six.

2. Determine the resultant of all except the 300-lb forces acting on the 4-ft cube represented in Fig. 28. *Ans.* $R = 100$ *lb forward; acts* 4 *ft above x-axis and* 19 *ft to left of y-axis.*

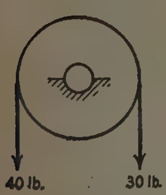

FIG. 26.

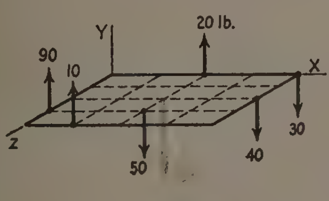

FIG. 27.

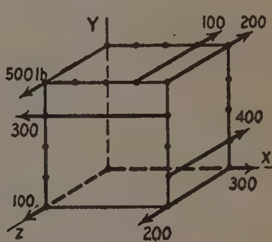

FIG. 28.

3. What seventh force, added to the system of Fig. 27, would make the resultant of the seven-force system act downward through the center of the table and be equal to 60 lb?

4. A, B, C, and D denote the corners of a horizontal square table top. At each of the corners A, B, and C is applied a downward force of 50 lb. What fourth force would make the resultant of the four-force system pass through D?

Art. 32. Noncoplanar Nonconcurrent Nonparallel System; Composition

1. Compound the six forces which act on the 4-ft cube of Fig. 29 into a force R at the origin and a couple C. *Ans.* $R = 136$ *lb;* $C = 496$ *ft-lb.*

2. Figure 30 represents a three-throw crankshaft, consecutive cranks being 120° apart. $AB = BC = 9$ in. When the shaft is made to rotate rapidly, as in a balancing machine, the

cranks are subjected to equal air pressures. Assume that these are perpendicular to the cranks respectively and have (equal) 4-in. arms; then reduce these three forces to a force applied at B and two couples.

3. The pulleys of Fig. 31 are out of balance so that when running at a certain speed they exert centrifugal pulls of 6, 8, 10, and 5 lb respectively and in the directions indicated. The coordinate axes rotate with the shaft. (*a*) Determine the resultant of these four forces. (*b*) Reduce this resultant, or the system, to a force at the left bearing and a couple. (You are to determine the magnitude and direction of the force, and the moment and plane of the couple.)

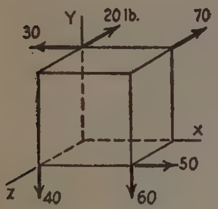

FIG. 29.

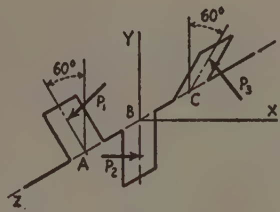

FIG. 30.

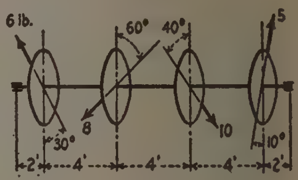

FIG. 31.

4. Determine the resultant of the eight forces acting on the 4-ft cube represented in Fig. 28. (Reduce to a force through the origin of coordinates and a couple.)

5. Determine the resultant of the three couples acting on the wedge-shaped block shown in Fig. 32.

6. Determine the resultant of all the forces acting on the 4-ft cube shown in Fig. 33.

General

1. (*a*) The force polygon for a coplanar concurrent set of forces closes. What can you say about the resultant of the set? Suppose that the set is nonconcurrent; then answer. (*b*) The x and y components of a certain set of coplanar concurrent forces are zero. What can you say about the resultant of the set? Suppose that the set is noncurrent; then answer. (*c*) The moment of a certain set of coplanar concurrent forces about a point in the plane is zero. What can you say about the resultant? Suppose that the set is nonconcurrent, then answer. (*d*) The x component of a certain set of concurrent coplanar forces is zero. What can you say

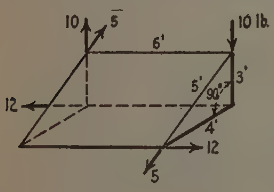

FIG. 32.

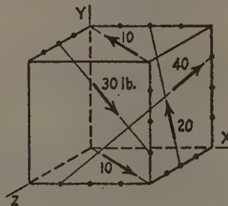

FIG. 33.

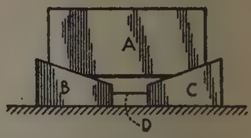

FIG. 34.

about the resultant? Suppose that the set is nonconcurrent; then answer. (*e*) The algebraic sum of the forces of a certain parallel set is zero. What can you say about the resultant? (*f*) What can you say about the resultant of a set of couples? about the resultant of a coplanar force and couple? about the resultant of a noncoplanar force and couple?

2. Can a system of noncoplanar, nonconcurrent, nonparallel forces be reduced to two forces through any two arbitrarily selected points? Can it be reduced to a force through an

arbitrary point and another force having any arbitrary direction? Can it be reduced to a force through an arbitrary point and another force having any arbitrary magnitude?

CHAPTER III

NOTE: In all the problems of this chapter it is to be assumed, unless otherwise stated, that surfaces in contact are smooth and that the weights of cords, chains, pins, etc., when not given, can be disregarded.

Art. 35. The Free-Body Diagram

1. A, Fig. 34, is a heavy body resting on B and C, which in turn rest on a floor. B and C are connected by a chain D. A, B, and C weigh 300, 200, and 100 lb respectively. Make a fbd for A; for B; for C; for A, B, C, and D; and for B, C, and D.

2. AC and BD, Fig. 35, are bars pinned to a ceiling at A and B and to each other at C. W is a heavy body. Make a fbd of AC; of BD; of AC and BD.

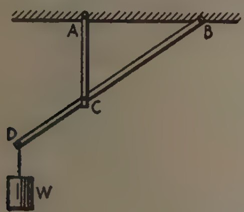

FIG. 35.

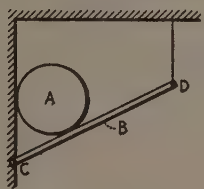

FIG. 36.

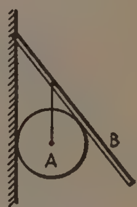

FIG. 37.

3. The cylinder A, Fig. 36, rests on a bar B and against a wall. B is supported by a rope and a pin at C. A and B each weigh 100 lb. Make a fbd for A; for B; for A and B.

4. The cylinder A, Fig. 37, rests against a wall and is supported by a rope fastened to the bar B. A and B weigh 100 and 200 lb respectively. Make a fbd for A; for B; for A and B.

5. AB, Fig. 38, is a bent bar hinged to a wall at A. C is a heavy cylinder resting as shown. Make a fbd of C; of the bar; of C and the bar.

Art. 36. Conditions of Equilibrium

1. (*a*) What do you think would happen if a horizontal force P was made to act through the center of a small block at rest on top of a table? (*b*) What would happen if two collinear forces P, P, equal in magnitude but opposite in sense, acted on the block? (*c*) What would happen if two noncollinear forces P, P, equal in magnitude but opposite in sense, acted on the block?

2. (*a*) Two couples are in equilibrium; what can you say about them? (*b*) Can a force and a couple be in equilibrium? (*c*) Three parallel forces are in equilibrium; what can you say about the middle force? (*d*) Is it possible for four forces to be in equilibrium if three are parallel? (*e*) Four forces are in equilibrium, and two are known to constitute a couple; what can you say about the other two? (*f*) What can you say about the resultant of a set of coplanar nonconcurrent forces whose force polygon closes?

3. Is it possible to apply three forces along the sides of a triangular board so that they will balance? Is it possible to apply four unequal forces along the sides of a square board so that they will balance?

4. A, Fig. 39, rests on a floor and against a wall; B rests on A and against another wall. A weighs 100 lb and B 200 lb. Make a fbd for A; for B; for A and B as one body. From inspection of the diagrams you made, and without calculations, compare: the pressures at the

walls; the pressure between A and B and the weight of B; the pressure at the floor and the weight of A and B. What can you say about the lines of action of the three forces that act on B?

5. Refer to the fbd's drawn for Prob. 4 of Art. 35. How does the pull of the rope on the cylinder compare with the weight of the cylinder? How does the pressure of the wall on the cylinder compare with the pressure of the bar on the cylinder? Suppose the bar to have no weight; what would be the direction of the force on the bar at the wall?

Art. 38. Coplanar Concurrent System in Equilibrium

1. A certain pin of a bridge truss, Fig. 40, joins four members of the truss and is subjected to forces exerted by those members. Two of the forces are wholly known; two, F_1 and F_2, are known only as to line of action. The magnitude and sense of F_1 and of F_2 are required. *Ans.* $F_1 = 5840$ *lb down.*

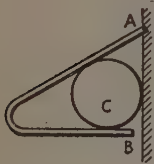

Fig. 38.

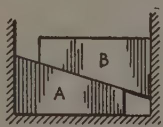

Fig. 39.

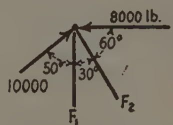

Fig. 40.

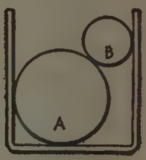

Fig. 41.

2. A ring has attached to it three cords A, B, and C; all lie in the vertical plane, and their directions are respectively up and to the right at 40° to the horizontal, up and to the left at 20° to the horizontal, and down and to the left at 60° to the horizontal. Determine the direction and the magnitude of a pull which, applied to the ring in the vertical plane, will balance tensions in A, B, and C, respectively, of 20, 60, and 90 lb.

3. A right circular cylinder of weight W lies in a trough formed by two plane surfaces whose line of intersection is horizontal. The left side A and right side B of the trough are respectively inclined at angles α and β to the horizontal. Determine the pressures exerted by the sides of the trough against the cylinder for each of the following cases: (*a*) $\alpha = 65°, \beta = 40°$; (*b*) $\alpha = 90°, \beta = 40°$; (*c*) $\alpha = 115°, \beta = 40°$.

4. For the arrangement described in the preceding problem: (*a*) determine values of α and β that will make the pressure of A and B each equal to W; (*b*) determine values of α and β that will make the pressure of A equal to $0.6W$ and the pressure of B equal to $0.9W$.

5. Two right circular cylinders are supported in a box 18 in. wide as shown in Fig. 41. A weighs 160 lb and is 16 in. in diameter; B weighs 100 lb and is 8 in. in diameter. Find all the forces acting on each cylinder, and represent them fully on separate sketches. *Ans. Pressure between A and B* = 115.4 *lb.*

6. The smaller cylinder, Fig. 42, is 6 ft in diameter and weighs 200 lb; the larger cylinder is 10 ft in diameter and weighs 100 lb; the angle between the inclined plane and the horizontal is 30°. Determine the pressures of the planes on the cylinders and the pressure between the cylinders. *Ans. Pressure between cylinders* = 103.3 *lb.*

7. Refer to Prob. 5 of Art. 35 and take the weight of the cylinder as 100 lb and the angle of the bent bar as 25°. Calculate the pressures on the cylinder. What is the maximum angle for which the system can be in equilibrium?

8. A, Fig. 43, is prevented from sliding down the plane by means of a rope as shown. A weighs 100 lb. The angle between rope and plane is 30°. (*a*) Determine the pull of the rope and the pressure between A and the inclined plane. (*b*) Suppose that the rope is parallel to the incline; then solve. (*c*) Suppose that the rope is horizontal; then solve.

9. The bell-crank ABC, Fig. 44, is pinned to a wall at A; a cylinder G is suspended by means of a cord from D as shown; $BD = 4$ in. The cylinder weighs 80 lb and is 20 in. in diameter. Determine all the forces which act upon the bell-crank. *Ans. Tension in cord = 94.3 lb.*

10. Figure 45 represents two wedges. The inclined planes of B make angles of 20° with the vertical. A push P of 1000 lb can balance what load Q? *Ans.* $Q = 728$ *lb.*

FIG. 42.

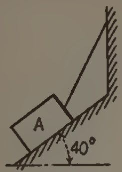

FIG. 43.

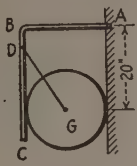

FIG. 44.

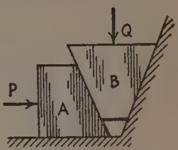

FIG. 45.

11. A body of weight W is suspended from a small ring C, Fig. 46, which is supported by two ropes AC and BC, equal in length. Calculate the tension in each rope, and note how it depends on the angle θ. For what value of θ is the tension greatest? least? For what value of θ is each tension equal to W?

12. Suppose that the lengths of the ropes of the preceding problem are unequal. Which of the two ropes is subjected to the larger pull, the longer or the shorter?

13. Suppose that the ring and cords of Fig. 46 are arranged as shown but that the suspended load is removed. Cord AC is inclined at 20° to the horizontal and will sustain a tension of 50 lb; cord BC is inclined at 50° to the horizontal and will sustain a tension of 80 lb. (*a*) In what direction and with what force would you have to pull on the ring to break both cords simultaneously? (*b*) What are the direction and magnitude of the least force which, applied to the ring, would break AC? Answer the same question for BC.

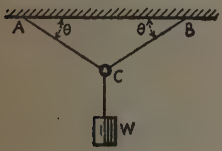

FIG. 46.

FIG. 47.

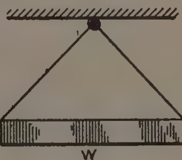

FIG. 48.

FIG. 49.

14. A ring is suspended by a vertical cord 3 ft long and has attached to it a weight of 100 lb. What are the direction and magnitude of the least force which, applied to the ring, would hold it in a position 1 ft higher than its initial position?

15. Figure 47 represents a segment of cast-iron pipe supported by a rope looped under it and passed over a hook. The pipe is 4 ft in diameter and weighs a ton; the rope will sustain a tension of 1500 lb. What is the shortest length of rope that can be used? *Ans.* 13.23 *ft.*

16. The body W, Fig. 48, weighs 2 tons. It is supported by two chains each 5 ft long; the lower ends are 8 ft apart. What is the tension in each chain? If the safe pull for each chain is 3 tons, how far apart may the lower ends be?

17. Figure 49 represents a rubber cord which, when fastened at supports A and B, is straight but not stretched. This cord will stretch 2 per cent of its length per pound of tension. (*a*) If a horizontal force $P = 10$ lb is applied at mid-length as shown, what is the resulting tension in the rubber cord? (*b*) Suppose that this cord will break under a tension of 100 lb. What force P, applied as shown, is required to break it? *Ans.* (*a*) 9.3 *lb.*

18. Figure 50 represents a rectangular plank of uniform thickness, weighing 80 lb, which is supported by a pin at the upper left corner and a horizontal rope attached at the lower right corner as shown. Determine the tension in the rope and the reaction at the pin. At what angle should the rope be inclined in order that the tension in it shall be a minimum, and what is the minimum tension?

19. A uniform beam 20 ft long weighs 50 lb. At the upper end it rests against a vertical wall and is held at the lower end by a rope 30 ft long which is also attached to the wall. Find the angle between the beam and the wall, and determine the pull in the rope. *Ans.* *Angle* = 49.8°.

20. The links in Fig. 51 are pinned at A, B, and C; A and C are rigid supports; $P = 100$ lb. The horizontal distance A to B is 10 ft. Find the reactions at A and C.

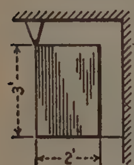

FIG. 50.

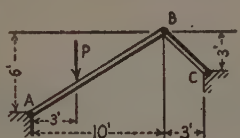

FIG. 51.

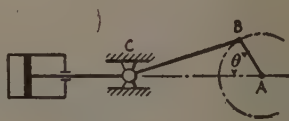

FIG. 52.

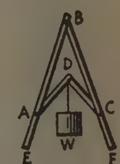

FIG. 53.

21. The cylinder of the steam engine, Fig. 52, is 10 in. in diameter, the crank AB is 5 in. long, and the connecting rod BC is 15 in. long. Assume the engine to be stalled in the position shown, $\theta = 60°$, and the steam pressure 150 lb/in.² Determine the push on the connecting rod BC and the pressure against the cross-head guide.

22. The symmetrical frame shown in Fig. 53 is made up of four rigid bars pinned at A, B, C, and D; it rests on smooth surfaces at E and F and carries a load applied as shown to the pin at D. Determine, by graphical construction, the slope that the supporting surfaces must have.

Art. 39. Coplanar Parallel System in Equilibrium

1. The beam, Fig. 54, is supported and loaded as shown. The beam weighs 50 lb per linear foot. Determine each supporting force, or reaction. *Ans.* *Left reaction* = 2540 *lb.*

2. The diagrammatic sketch in Fig. 55 represents a lever system for a scales. Assume that $W = 250$ lb; $a = 12$ in.; $b = 8$ in.; $c = g = 3$ in.; $d = h = 10$ in.; $e = 1\frac{1}{2}$ in.; $f = 15$ in. Determine P.

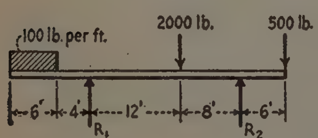

FIG. 54.

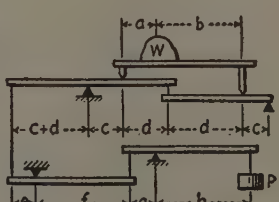

FIG. 55.

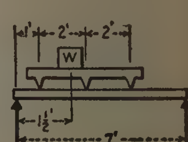

FIG. 56.

3. Figure 56 represents a beam with three points of support resting upon a second beam that is supported at its ends. A load W of 100 lb is placed upon the first beam as shown. Determine the reactions at the ends of the lower beam.

4. The bent bar, Fig. 57, is supported in a vertical plane by means of a floor and a rope, all as shown. The bar weighs 100 lb/ft. $AD = DB = 6$ ft. (*a*) Calculate the pressure at A and the tension in the rope, CB being 2 ft. (*b*) Is it possible to choose the point of attachment C (for a vertical rope) so that the pressure at A will be zero? If so, ascertain the position of C.

5. Two men of unequal strength are to carry a slender uniform beam 20 ft long. If one man is to carry twice as great a portion of the load as the other, how far apart can the men be? Does it make any difference, in the distribution of load, whether the beam is horizontal or inclined?

6. Two men carry a long steel rail up a stairway inclined at 30° to the horizontal. Which one supports the greater share of the weight? Suppose that they carry a trunk, each grasping it by the lower corners at one end. Which then carries the greater share of the weight?

7. A uniform rail 20 ft long weighing 900 lb is supported in a horizontal position by a pin at one end and by a vertical post at some point between the ends. Where should the post be placed in order that the load on it will be 600 lb? 1200 lb? 900 lb?

8. A horizontal beam 10 ft long weighs 60 lb/ft. It sustains a concentrated load of 400 lb 3 ft from the left end, is supported at the left end by a vertical rope, and at the right end has a 2-ft bearing on the horizontal top of a wall. If the tension in the rope is 510 lb, what is the total upward pressure of the wall, and where is the center of this pressure? *Ans. Center of pressure is* 8.57 *ft to right of left end.*

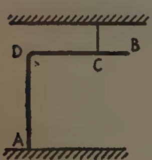

Fig. 57.

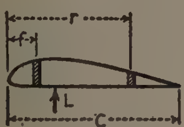

Fig. 58.

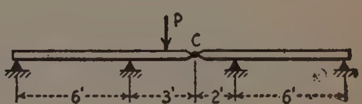

Fig. 59.

9. Figure 58 shows the section of an airplane wing with two spars. L is the resultant lift or vertical air force. The farthest forward position of L is $0.31c$ from the leading edge; the farthest rearward position is $0.50c$. It is desired to place the spars so the front spar gets 70 per cent of L when L is farthest forward and so rear spar gets 70 per cent of L when L is farthest back. Determine r and f to meet this requirement.

10. Figure 59 represents two uniform beams; the one on the left weighs 200 lb, the one on the right weighs 300 lb. Each rests on smooth supports, and they are pinned together at C. How large a load P can be applied as shown 1 ft from C without disturbing the equilibrium of the beams? For any load P less than this maximum, is it possible to determine the reactions of the supports and the force exerted by one beam on the other?

Art. 40. Coplanar Nonconcurrent Nonparallel System in Equilibrium

1. The bar AB, Fig. 60, is pinned at A to a support, and rests at B against a wall. The bar weighs 0.4 lb/in. Determine the pin pressure. *Ans. Horiz. comp.* = 96 *lb; vert. comp.* = 86.4 *lb.*

2. Refer to Prob. 4 of Art. 35 and determine all the forces on the bar and on the cylinder. The cylinder weighs 100 lb and is 4 ft in diameter; the bar weighs 40 lb, is 10 ft long, and is inclined at 40° to the wall.

3. AB, Fig. 61, is a bar 12 ft long fastened to the floor at A by a pin and resting at C on a cylinder 4 ft in diameter. The center of the cylinder is 6 ft to the right of A and is connected by a horizontal cord to the bar at D. A weight of 100 lb is hung on the free end of the bar.

Determine the pressure between the bar and the cylinder, the pressure between the cylinder and the floor, the tension in the cord, and the pressure exerted by the pin on the bar A. *Ans. Tension in cord* = 120 *lb.*

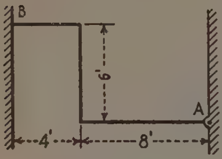

FIG. 60.

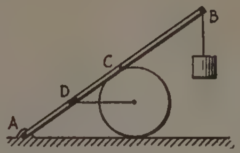

FIG. 61.

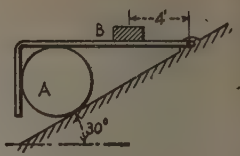

FIG. 62.

4. Calculate all the pressures on the cylinder A, Fig. 62, due to the weight of A and B, 1000 and 2000 lb, respectively. The diameter of A is 8 ft. (Make a fbd for the "whole thing"; then of A or bar.)

5. $ABCD$, Fig. 63, is a rigid bent bar pinned to the incline at A. E is a cylinder whose diameter is 3 ft and weight 200 lb. The inclination of the plane is 30°, and BC is 4 ft. Calculate the pin pressure and all pressures on the cylinder. *Ans. Pressure of plane on cylinder* = 233.2 *lb.*

6. In Fig. 64, AB = 4 ft, $AC = CD$ = 8 ft; P = 200 lb; and the cylinder weighs 100 lb. A, B, and E are pin joints. The inclination of the plane at F is 30° to the horizontal. Determine the forces acting on the cylinder and those acting on the bell-crank ACD.

7. A light bar 9 ft long is fastened to the floor at its lower end by a horizontal pin perpendicular to the bar. The angle between the bar and the floor is 50°; its upper end rests against a smooth vertical wall. Loads of 300 and 200 lb are hung at distances of 4 and 7 ft respectively from the lower end. Calculate the reactions at the ends of the bar.

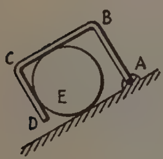

FIG. 63.

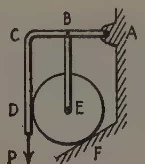

FIG. 64.

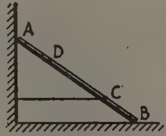

FIG. 65.

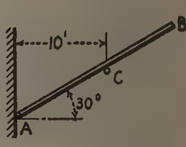

FIG. 66.

8. The bar AB, Fig. 65, weighs 100 lb; BC is one-fourth of AB. (*a*) Calculate the pull of the rope and the pressures at A and B. (*b*) If the rope (still horizontal) were fastened at D, would its pull be more or less than in (*a*)? Bar is inclined at 40°.

9. A and B are two horizontal pegs in a wall; they are 3 and 6 ft above the floor respectively, and the horizontal distance between them is 4 ft. A smooth straight bar CD, 15 ft long and weighing 200 lb, is placed under A and over B with its lower end on the floor, but is not sprung into that position. Determine all the pressures on the bar due to its own weight. *Ans. Reaction of floor* = 200 *lb.*

10. (*a*) AB, Fig. 66, is a bar 20 ft long, weighing 10 lb. It rests on a peg C and against a smooth wall at A, as shown. What vertical force applied at B will preserve the equilibrium of the bar? (*b*) If the weight of the bar is 12 lb and a load weighing 4 lb is suspended at B, at what angle must the bar be placed to insure equilibrium? *Ans.* (*a*) 11.7 *lb.*

11. Figure 67 represents a slender bar, not uniform in weight per unit length, which is to rest on a smooth floor and against a smooth wall, and to be prevented from slipping down by a rope attached to the floor at O and to the bar at some point. Determine by graphical construction the limiting positions of this point of attachment.

12. Calculate the pressures at A, B, and C, Fig. 68, due to loads: (a) $W_1 = 100$ lb and $W_2 = 0$. (b) $W_1 = 0$ and $W_2 = 200$; (c) $W_1 = 100$ lb and $W_2 = 200$ lb. See whether the answers you get for (c) are the sum of the corresponding answers for (a) and (b).

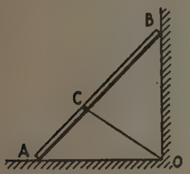

FIG. 67.

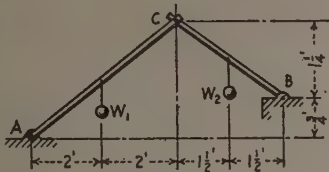

FIG. 68.

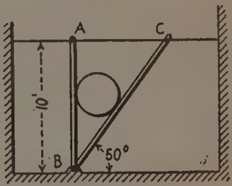

FIG. 69.

13. Two rods AB and CD of equal length are pinned together at their middle points and then laid upon a table so that AEC is 60°, E being the pin joint. A cord joins A and C, and a second cord joins B and C. If the cord AC is under a tension of 10 lb, determine the tension in cord BC and the magnitude and direction of the force on AB at E. *Ans. Tension in BC* = 17.3 *lb.*

14. In Fig. 69, AB and BC are planks pinned at B to the floor. A cord extends horizontally from wall to wall, securing the upper ends of the planks; this cord may be regarded as straight but without any initial tension. A cylinder weighing 100 lb and 3 ft in diameter is placed between the planks as shown. Determine the resulting tension in each segment of the string. (The planks may be regarded as weightless.)

15. The bar AB, Fig. 70, weighs 50 lb. Will it hang in the horizontal position as shown when suspended by the two cords? If not, what is the least weight which, suspended from the bar, will keep it in that position?

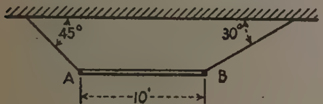

FIG. 70.

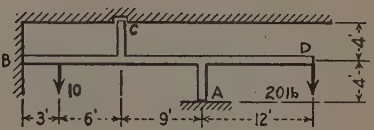

FIG. 71.

16. The bar $ABCD$, Fig. 71, rests at A on a horizontal support on a floor, at B against a wall, and at C against a side of a hole in a ceiling. Determine the supporting forces due to the two loads indicated.

Art. 41. Noncoplanar Concurrent System in Equilibrium

1. The tripod shown in plan and elevation in Fig. 72 sustains a load of 1000 lb. Determine the reaction on the lower end of each leg of the tripod. The legs are hinged at O and rest in smooth sockets at their lower ends.

2. A heavy body W, weighing 600 lb, is suspended beneath a square table by three chains fastened on the under side of the top at A, B, and C, Fig. 73. The chains are of such length that the ring D is 6 ft directly below the center of the top. Determine the pull of each chain. *Ans. Tension in chain at A* = 233 *lb.*

3. The x and y axes, Fig. 74, are in the front face of a building, and the z axis is perpendicular to it. OC is a pole, pinned at O to the face and supported at C by two chains CA and CB fastened to the face at A and B. $OC = 10$ ft; A and B are 8 ft from the x axis and 10 ft from the y axis. Calculate the three forces at C due to a load $W = 0.8$ ton. *Ans. Force from pole* $= 1.0$ *ton.*

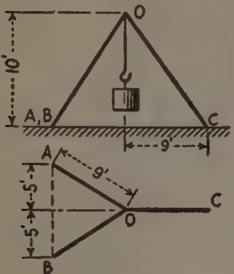

FIG. 72.

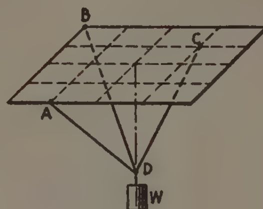

FIG. 73.

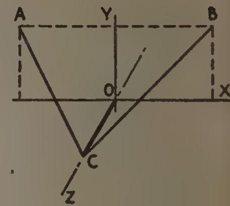

FIG. 74.

4. Change the location of A of the preceding problem so that its distances from the x and y axes respectively are 6 and 12 ft. Then solve.

5. Four spheres of diameter D and weight W are placed in a box whose bottom is $2D$ by $2D$. Then a similar sphere is laid upon the four. Calculate the pressures of the spheres upon the sides and bottom of the box.

Art. 42. Noncoplanar Parallel System in Equilibrium

1. Having been given necessary dimensions and weight of loading on a table with three legs, how would you proceed to find the reactions or pressures on the lower ends of the legs? Assume simple data, and try out your plan. What is your plan for a table with four legs?

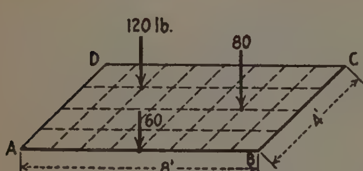

FIG. 75.

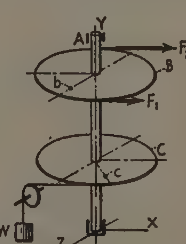

FIG. 76.

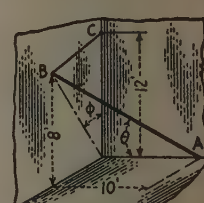

FIG. 77.

2. A horizontal platform carries three loads as shown in Fig. 75. It weighs 50 lb and is uniform in thickness. (*a*) It is to be supported by two vertical ropes, one attached at A, the other to an edge of the platform wherever necessary. Find the point of attachment of this second rope and the tension in each rope. (*b*) Suppose that the platform is supported by three vertical ropes at A, B, and D. Determine the pull of each. (*c*) Suppose that the platform is supported by vertical ropes at the four corners. What can you say about the pull of each?

3. Suppose that the tripod of Prob. 1, Art. 41, instead of being made and supported as

described, has the three legs rigidly welded together at their junction and is supported on a smooth, level floor. Determine the reaction on each leg under these circumstances, and compare with the vertical components of the reactions found before.

Art. 43. Noncoplanar Noncurrent Nonparallel System in Equilibrium

1. The vertical shaft in Fig. 76 carries a pulley at B weighing 150 lb and one at C weighing 80 lb. The radius of pulley B is 20 in., of pulley C, 15 in. $AB = 2$ ft; $BC = 5$ ft; the shaft is 11 ft long. The center of gravity b of B is 2 in. from the axis of the shaft; c of C is $\frac{1}{2}$ in. $F_1 = 200$ lb; $F_2 = 50$ lb; and $W = 200$ lb. Find the reactions on the bearings at the ends when the angle between the radius through c and the x axis is 30°.

2. Determine the reactions on the bearings of the shaft of Prob. 3 of Art. 32 in the circumstances there described.

3. The uniform bar AB, Fig. 77, shown in a corner of a rectangular room is pinned at A and rests against a smooth wall at B. It is held by a rope BC. The bar weighs 80 lb and is 15 ft long. Determine all forces that act on the bar.

Art. 45. Anchorage; Degrees of Freedom

1. Figure 78 represents a solid cubical block partially anchored by the three links **1, 2, 3.** The block is to be loaded in any way. Describe three lines with respect to which supporting forces exerted by these links cannot satisfy the equation $\Sigma M = 0$. Add three links, parallel to the x, y, and z axes respectively, that will anchor the block. Then determine the stress in each of the six links due to a force $F = 100$ lb applied as shown.

General

1. Figure 79 shows a frame consisting of three rigid bars pinned together at A and B and to the floor at C and D. A weight W is to be suspended as shown, and to preserve equilibrium a rope is attached at E, the midpoint of AB, and to some outside support. Determine, by graphical construction, the limits of inclination of this rope. At what angle should it be arranged in order that the tension in it will be least? At what point of AB should it be attached in order to hold the frame with the least tension?

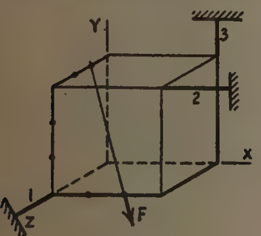

FIG. 78.

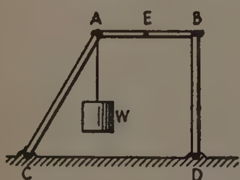

FIG. 79.

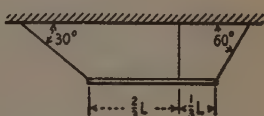

FIG. 80.

2. Figure 80 represents a rigid bar of negligible weight supported by three cords. What are the limits as to point of application and direction of a force which, applied to the bar, will be balanced by the tensions in the three cords?

3. Figure 81 represents a riveting machine operated by compressed air. It consists of a rigid frame on which the air cylinder C is mounted; the piston rod is pinned to the piston so that the rod can be rotated somewhat about the pin inside of the (hollow) piston; the toggle link BD is pinned to the frame at D; the toggle link BE is pinned to the plunger (movable in a vertical guide on the frame) at E; HH are the rivet dies between which the rivet is squeezed. The piston rod is 19 in. long; $BD = 13$ in.; $BE = 10$ in.; the diameter of the cylinder is 10 in. Assume the air pressure to be 100 lb/in.², and then determine the pressure

at the pins D and E, the pressure against the guide, and the pressure on the rivet. (To " lay out " this mechanism begin at D, then fix the piston pin, then B, and then E.) Solve the problem when the piston is advanced 2 in. from the position shown.

4. A drier drum used in a paper factory consists of a hollow, cast-iron cylinder 10 ft in diameter and weighing 15 tons. This cylinder is mounted on horizontal axial trunnions so that it can be rotated about its longitudinal axis. Two smaller cylinders of equal length press against the drum as it rotates. One of these, the " pressure roll," is 17 in. in diameter, is mounted so

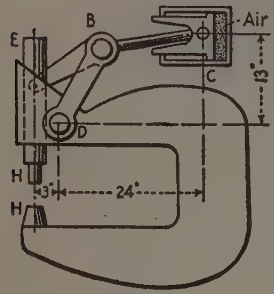

Fig. 81.

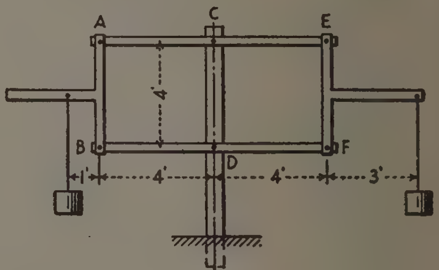

Fig. 82.

that its center is 27 in. from the vertical plane containing the axis of the drier, and exerts against the drier a pressure of 15 tons. The other, the " print roll," is 20 in. in diameter and is mounted so that its center is 8 in. below the axis of the drier, and on the same side of the drier as the pressure roll, and exerts against the drier a pressure of 12 tons. (*a*) Determine the reactions on the trunnions of the drier. (*b*) Assuming the position of the pressure roll to be fixed, where should the print roll be located in order that the reactions on the drier trunnions may be as small as possible?

5. The frame shown in Fig. 82 consists of rigid parts pinned at A, B, C, D, E, and F. With equal weights placed as shown, would this frame be in equilibrium? Is it possible to determine all forces on all members?

CHAPTER IV

Note: To determine the stress in a member of a truss means to determine both the amount and the kind of stress. The results obtained in a solution should always be recorded on a sketch of the truss, the kind of stress being indicated by the appropriate letter *t* or *c*, or by the use of arrowheads as explained in (*a*) of Art. 51 and in Art. 53.

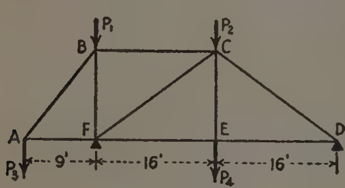

Fig. 83.

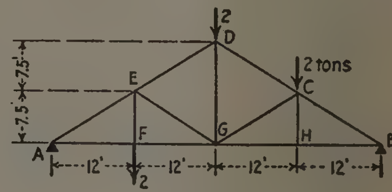

Fig. 84.

Art. 50. Determination of Stress, General Method

1. The truss represented in Fig. 83 is supported at F and D; $BF = CE = 12$ ft, $P_1 = P_2 = 2000$ lb, and $P_3 = P_4 = 1000$ lb. Determine the stress in each of the members BC, CF, and FE. *Ans. Stress in BC = 750 lb tension.*

2. The truss of Fig. 84 is supported at each end; it sustains three loads as shown. Find the stress in EG without first finding any other stress. Do the same for ED; for FG.

3. For the truss shown in Fig. 85, determine the stress in the members AB, BC, and AD. *Ans. Stress in BC* = *1667 lb tension.*

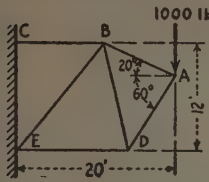

FIG. 85.

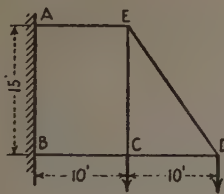

FIG. 86.

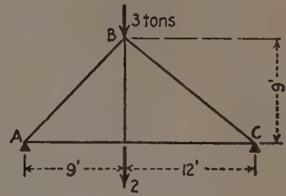

FIG. 87.

4. Figure 86 represents an incomplete truss which is to carry two 1000 lb loads. (*a*) Complete the truss with a tension member, and determine the stress in that member. (*b*) Complete the truss with a compression member, and determine the stress in that member.

Art. 51. Stress Analysis of a Truss

1. Determine the stress in each member of the truss of Fig. 87. *Ans. Stress in BC* = 3.55 *tons comp.*

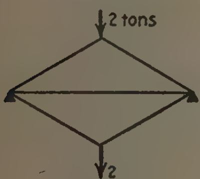

FIG. 88.

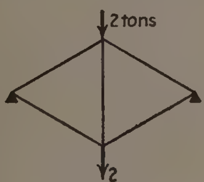

FIG. 89.

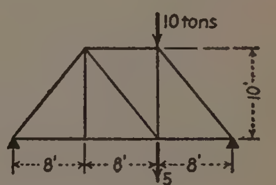

FIG. 90.

2. The slope of each inclined member of the truss of Fig. 88 is 30°. (*a*) Determine the stress in each member. (*b*) Make the lower load 4 tons, then solve. (*c*) Make the upper load 4 tons, and then solve.

3. Solve as in Prob. 2 but for the truss in Fig. 89.

4. The truss in Fig. 90 is supported at its ends; it has two loads. (*a*) Make a stress analysis for the truss. (*b*) What stress changes would result from interchange of the two loads?

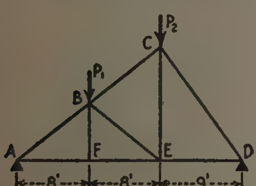

FIG. 91.

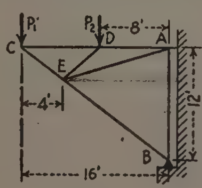

FIG. 92.

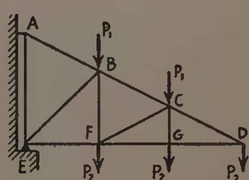

FIG. 93.

5. The truss represented in Fig. 91 is supported at A and D; CE = 12 ft, P_1 = 1000 lb, and P_2 = 2000 lb. Determine the stress in each member. *Ans. Stress in BC* = *1500 lb comp.*

6. The truss represented in Fig. 92 is held by a pin at joint B and by a horizontal tie at joint A. Each load = 1000 lb. Solve for the stress in each member.

7. The truss represented in Fig. 93 is supported by a pin at E and by a horizontal tie at A. $AE = 9$ ft; $EF = FG = GD = 6$ ft. Each load $P_1 = 1000$ lb; each load $P_2 = 500$ lb. Determine the stress in each member. *Ans.* *Stress in AB = 4460 lb tension.*

8. The tower represented in Fig. 94 is supported at E and I. Determine the stress in each member due to $P = 1$ ton. Suppose that P acts toward the left; then solve.

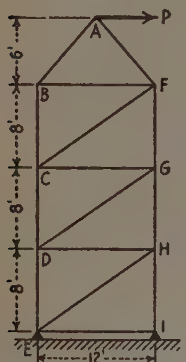

FIG. 94.

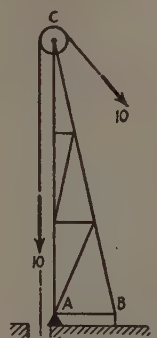

FIG. 95.

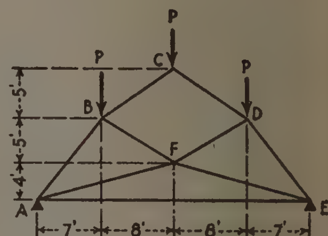

FIG. 96.

9. The structure represented in Fig. 95 is a steel head-frame for hoisting ore from a mine. The frame is pinned at A and is anchored at B so that either an upward or a downward reaction can occur at that point. The load is 10 tons. The distance $AC = 90$ ft; the distance $AB = 20$ ft; the inclined portion of the rope makes an angle of 50° with the horizontal. Determine the stress in each member of the frame. *Ans.* *Stress in AB = 6.43 tons tension.*

10. The truss represented in Fig. 96 is supported at A and E; each load $P = 1000$ lb. Determine the stress in each member. *Ans.* *Stress in AB = 1700 lb comp.; stress in AE = 1625 lb tension.*

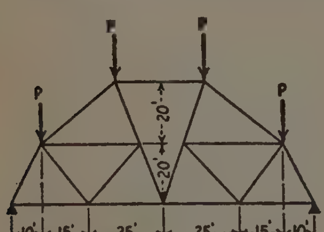

FIG. 97.

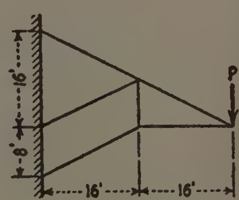

FIG. 98.

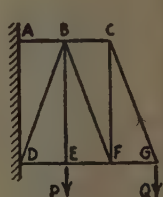

FIG. 99.

11. The truss of Fig. 97 is supported at each end. Number the members (1, 2, 3, etc.) in the order in which you could determine their stresses due to the four loads P indicated, and explain how you would solve for each.

Art. 52. Solution by Inspection

1. Without any calculations, ascertain the kind of stress due to the load P in each member of the truss of Fig. 98.

2. See Fig. 99. Compare the following: (*a*) stresses in *DE* and *EF*; (*b*) stress in *BE* and load *P*; (*c*) stresses in *BC* and *FG*; (*d*) stress in *CF* and load *Q*; (*e*) stress in *CG* and load *Q*. (*f*) What kind of stress is in *BD*; *AB*; *DE*? (*g*) Which is the larger stress, *AB* or *DE*?

3. Make a copy of Fig. 100. Indicate by *t* and *c* the kind of stress in each member, which you should ascertain by inspection. Also by inspection, compare the magnitudes of S_1 (the stress in member 1) and S_6; S_3 and S_4; S_1 and S_7; S_2 and S_5; S_2 and S_3; S_8 and S_9; S_8 and the reaction at A; S_8 and S_7.

Art. 54. The Stress Diagram

1. The truss represented in Fig. 101 is supported at each end. The total width of the truss is 32 ft; the total height is 20 ft; the horizontal members are 8 ft long; the vertical member is 10 ft long. Each load P_1 = 1000 lb, and P_2 = 2000 lb. Determine the stress in each member graphically.

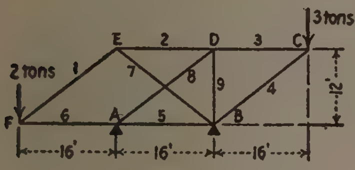

FIG. 100.

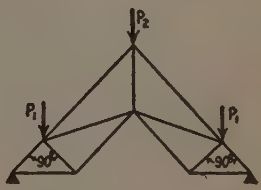

FIG. 101.

2. The truss represented in Fig. 102 is supported at each end. The points 1, 2, 3, 6 and the points 3, 4, 5, 7 are at the vertices of parallelograms. Determine the stress in each member graphically. *Ans. Stress in* 1 – 2 = 10.6 *tons comp.*

3. Each load *P* in Fig. 103 is 5000 lb. Determine graphically the amount and kind of stress in each member of the truss.

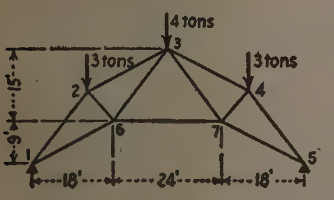

FIG. 102.

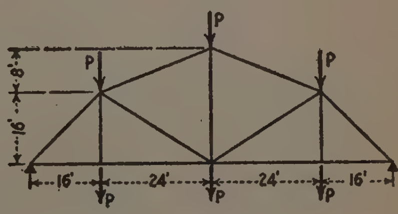

FIG. 103.

4. The truss represented in Fig. 104 is held by a pin at the right end and rests upon a roller at the left end. Each load P_1 = 1000 lb; P_2 = 2000 lb. Solve graphically for the stress in each member.

Art. 56. Force Analysis of a Crane

1. Figure 105 represents a crane consisting of three members, post *AB*, boom *BC*, and brace *CD*. The post extends through a hole at *E* in an upper floor and rests in a socket *A* in a lower floor. The supporting force at *E* is horizontal; at *A* there is a vertical supporting force on the bottom and a horizontal supporting force on the side of the post. Take a = 8 ft and b = 6 ft; then calculate all forces acting on each member. How do changes in a and in b affect the required forces?

2. Determine all forces acting on each member of the crane shown in Fig. 106. *Ans.* $B_x = 2.86$ *tons right;* $B_y = 4.0$ *tons up.*

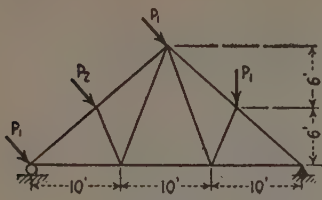

FIG. 104.

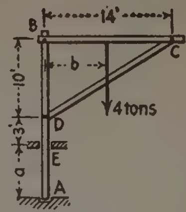

FIG. 105.

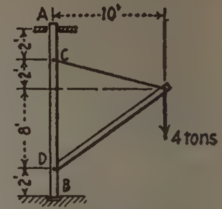

FIG. 106.

3. Determine all forces acting on each member of the crane shown in Fig. 107.

4. Figure 108 represents a crane consisting of three members, a boom AC, a brace AD, and a post BF. The crane is supported at E and F by two floors. The load $W = 5$ tons. Determine all forces acting on each member. *Ans. Pressure of upper floor* $= 6.88$ *tons left.*

5. Solve Prob. 4 but take into account the weights of the members, which are as follows: post $BF = 0.5$ ton, brace $AD = 0.2$ ton, and boom $AC = 0.7$ ton. The boom is 18 ft long; its center of gravity is 2 ft 6 in. from B. The center of gravity of each of the other members is at its center.

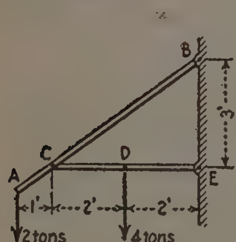

FIG. 107.

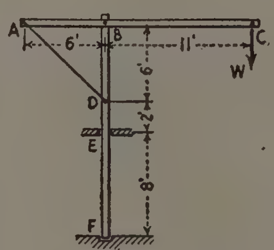

FIG. 108.

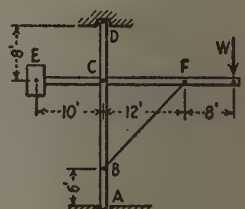

FIG. 109.

6. The crane represented in Fig. 109 rests in a socket at A and bears against the smooth side of the hole in the floor at D. There are pins at B, C, and F. The load W is 4000 lb; the counterweight E weighs 5000 lb. $BC = 14$ ft. Determine all the forces which act upon each member of the crane. *Ans. Stress in BF* $= 3295$ *lb comp.*

7. The crane represented in Fig. 110 consists of a post AB, a boom CD, and braces DE and FG. The crane is supported by sockets at A and B as shown. The boom passes freely through a smooth slot in the post at H so that any reaction existing there will be vertical. The counterweight at D is $\frac{1}{2}$ ton, and the load W is also $\frac{1}{2}$ ton. Determine all the forces which act upon each member. *Ans. Force at H on boom* $= 1.1$ *tons down.*

8. In Fig. 111 the three members are pinned to the wall at A and C and to each other at E. At B the member DB simply bears against the end of member AEB. Determine all forces that act on each member. (Could this problem be solved if the members were pinned together at B?)

Art. 57. Force Analysis (Continued)

1. The wall crane in Fig. 112, is pinned at A, B and C. The pulley diameter is 3 ft. The hoisting rope extends through the wall to a hoist. Determine all forces on each member due to the load $W = 1$ ton. (For the first fbd choose the entire crane, with pulley and all

ropes shown; then select other fbd's and complete the solution. Make another solution choosing the two members together for the first fbd.) *Ans.* $A_x = \frac{1}{6}$ *ton left;* $A_y = \frac{3}{8}$ *ton up.*

2. Suppose that the rope in Fig. 112 extends not horizontally to the right but upward at an angle of 30° to the horizontal. Then solve.

3. Suppose that the rope of Fig. 112, instead of running to the wall, runs vertically from the pulley to a winch mounted on member BC. Then solve.

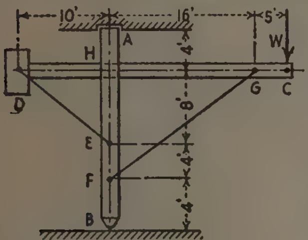

FIG. 110.

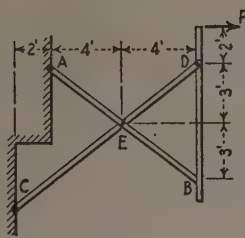

FIG. 111.

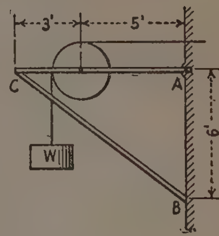

FIG. 112.

4. The diameter of each pulley in Fig. 113 is 2 ft. Load $W = 2$ tons. Determine all forces on each member of the crane. *Ans.* $B_x = 1.81$ *tons left;* $B_y = 0.17$ *ton down.*

5. The diameter of the pulley in Fig. 114 is 4 ft. Determine all forces acting on each member.

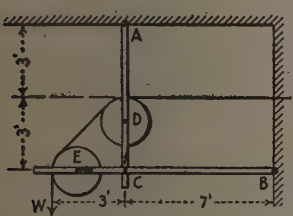

FIG. 113.

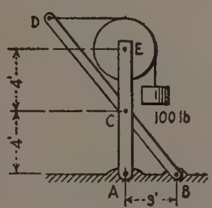

FIG. 114.

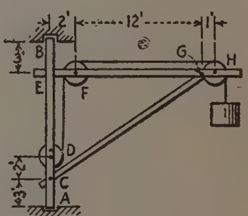

FIG. 115.

6. The crane shown in Fig. 115 rests in a socket at A and projects into a hole in the ceiling at B, the sides of the hole affording horizontal support. The diameter of each pulley is 2 ft. $ED = 8$ ft. The rope is vertical from F to D and is fastened to a pulley on the post at D. The load W is 8 tons. Determine all forces acting on each member.

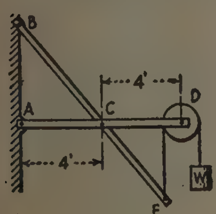

FIG. 116.

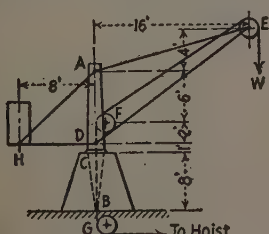

FIG. 117.

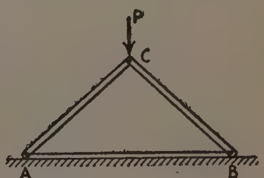

FIG. 118.

7. In Fig. 116 the members are pinned to the wall at A and B and to each other at C. The diameter of the pulley is 2 ft; the load W is 1000 lb; $AB = 5$ ft. Determine the forces which act upon each member of the structure.

8. Figure 117 represents a crane supported by a foot-step bearing at B and a collar bearing at C. B can furnish horizontal and vertical support, and C can furnish horizontal support only. The pulleys E and F are 1 ft in diameter; the hoisting cable enters the post at F, descends through the post, over pulley G, and to the hoist as shown. The pulley at F is attached to the post. The counterweight H is 2 tons and the load 4 tons. Determine all the forces which act upon each member. *Ans. Force on post at C* = 6.25 *tons.*

General

1. Figure 118 represents a simple truss, consisting of two timbers with the lower ends connected by a cable, supported at its ends. The timbers are each 15 ft long; they will safely stand a compressive stress of 36,000 lb, and the cable will safely stand a tensile stress of 28,000 lb. What is the maximum span that can safely be employed if the load P is 40,000 lb? For what span would the timbers and cable both be stressed to their safe limit?

2. The diagonal members of the truss in Fig. 119 are wires and can take tension only. The other members can take tension or compression. G is a joint. Each of the four loads is 2 tons. Analyze the truss. (Suggestion: First find stresses due to loads at C and F only; then find stresses due to loads at B and E only; then add results algebraically.)

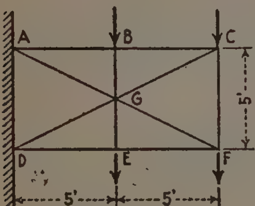

FIG. 119.

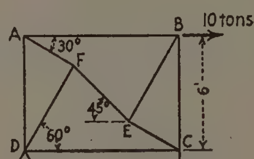

FIG. 120.

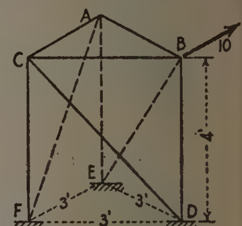

FIG. 121.

3. The framework shown in Fig. 120 is not a simple truss, since the members are not arranged to form a series of triangles, but it is rigid and determinable. It is pinned at D and supported by a vertical link at C. Determine the stress in each member due to a horizontal load applied at B as shown. Suppose that the triangle BEC is inverted so that member EF is horizontal. Then calculate the stress in FE. Is the structure now rigid?

4. (All the trusses so far considered are plane frameworks; that is, all members and external forces lie in the same plane. Trusses of which this is not true are called *space frameworks;* like ordinary trusses, they consist of straight members pinned together at the ends, and they are assumed to be loaded only at the joints. Since all the members are two-force bodies, the forces that act on any joint constitute a noncoplanar concurrent system, and such a system can be solved for as many as three unknowns by the equations $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$.) Figure 121 represents a tower, the cross section of which is an equilateral triangle. It is pinned to the foundation at D, E, and F and is loaded at B by a horizontal force parallel to member AC. Determine the stress in each member of the tower and the reactions at D, E, and F.

5. The structure shown in Fig. 122 is pinned to the ground at A, B, C, and D. Determine the stress in each member due to the loads shown.

6. Figure 123 represents a certain type of hydraulic crane. It consists of a post AB, an hydraulic cylinder C mounted on the post, a large sleeve which can be slipped along the post, two rollers D and E mounted on the sleeve, a boom EF, and a tie rod FG. When water, under pressure, is admitted to the cylinder, the pistons are pushed upward; the upper one bears against the sleeve, and rolls the entire part $DEFG$ up along the post. Let the load $W = 2$ tons, and suppose that it is 10 ft out from the axis of the post; then determine all the forces that act upon each pin (D, E, and G).

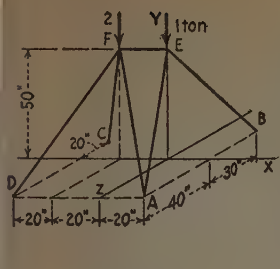

FIG. 122.

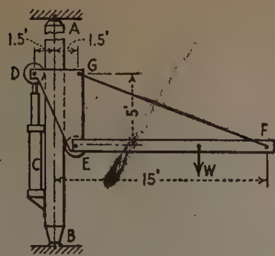

FIG. 123.

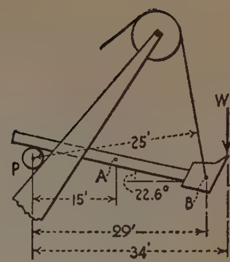

FIG. 124.

7. Figure 124 represents part of a steam shovel, comprising the boom, the handle and dipper, the cable, and the pinion. The handle is moved back and forth by the pinion, which engages with a rack on the handle, and the dipper is raised and lowered by the cable. The weight of the handle is 20,000 lb, and its center of gravity is at A; the dipper (empty) weighs 20,000 lb; and its center of gravity is at B. The cable from B slopes at 26.5°. It is required to determine the digging force W and the normal and tangential components of the pinion pressure when the handle is in the position shown and the cable tension is 120,000 lb.

CHAPTER V

Art. 59. Preliminary

1. A rectangular block A weighing 100 lb rests on another rectangular block B weighing 200 lb; B rests on a horizontal floor. All surfaces are rough. A horizontal force P of 20 lb acts on A to the right; it is not large enough to cause slipping. Make fbd's of A and B, and determine all forces on each.

2. Assume that the force P of Prob. 1 acts upward and to the right at 20° to the horizontal; then solve.

3. Assume that the force P of Prob. 1 acts downward and to the right at 20° to the horizontal; then solve.

4. A block weighing 100 lb rests on a plane inclined at an angle α to the horizontal, and a force P of 40 lb, inclined at an angle β to the plane (β is positive when P slopes upward more steeply than the plane), is applied to the block. The surfaces in contact are rough enough to prevent slipping. Determine the normal pressure and the friction force on the block for each of the following conditions:

(a) $\alpha = 20°$, $\beta = 0°$
(b) $\alpha = 20°$, $\beta = +30°$
(c) $\alpha = 20°$, $\beta = -30°$
(d) $\alpha = 30°$, $\beta = -30°$

5. Blocks A and B rest on the inclined plane, in the position shown in Fig. 125, without slipping. A weighs 100 lb. B weighs 50 lb. Determine normal pressure and friction on the top and bottom surfaces of A and on the lower surface of B. *Ans. Normal pressure of plane = 130 lb.*

Art. 60. Impending Slip

1. A body rests on a plane inclined at 35° to the horizontal. (a) What can you say about the coefficient of friction for the body and plane? (b) Suppose that the coefficient of friction for body and plane is 0.85; what is their angle of repose?

2. A large number of shingles are laid on top of one another, their thick ends all at the same end of the wedge-shaped pile thus formed. If the shingles are 14 in. long, $\frac{1}{2}$ in. thick at one end and $\frac{1}{32}$ in. thick at the other, and the coefficient of friction between them is 0.2, how many can be thus stacked up before slipping occurs?

Art. 62. Equilibrium When Slip Impends

1. A block A weighing 100 lb rests on a horizontal floor for which $\mu = 0.6$. What force is required to make the block slip if applied to the block: (a) horizontally; (b) upward at 20° to the horizontal; (c) downward at 20° to the horizontal? What is the friction force in each case? *Ans.* (a) 60 *lb.*

2. In Prob. 1, what are the direction and magnitude of the least force that will produce slip?

3. A rectangular block A weighing 100 lb rests on a second rectangular block B weighing 200 lb; B rests on a horizontal floor C. For A and B, $\mu = \frac{1}{2}$; for B and C, $\mu = \frac{1}{3}$. A is held by a horizontal cord running to a wall to the left. What horizontal force P, acting to the right, is required to make B slip? *Ans.* 150 *lb.*

4. Suppose that in Prob. 3 the cord that holds A slopes upward at 30° to the horizontal. Then solve.

5. Suppose that in Prob. 3 the cord that holds A slopes downward at 30° to the horizontal. Then solve.

6. A block weighs 100 lb and rests on a plane inclined at 20°. The angle of repose for A and plane is 30°. Calculate the least force parallel to the plane required to start the block: (a) down; (b) up. *Ans.* (b) 88.4 *lb.*

7. Referring to the preceding problem, suppose that the inclination of the plane is 40°. Calculate the least force (parallel to the plane) (a) that would prevent A from slipping; (b) that would start A up.

8. A block A weighing 200 lb rests on a plane inclined at 35° for which $\mu = 0.3$. (a) What horizontal force will just prevent A from slipping down the plane? (b) What are the direction and magnitude of the least force that will prevent slipping? (c) What horizontal force will just start A up the plane? (d) What are the direction and magnitude of the least force that will start A up the plane?

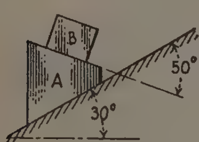

FIG. 125.

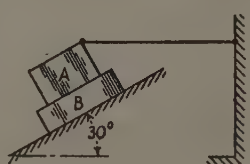

FIG. 126.

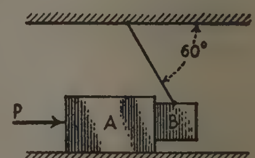

FIG. 127.

9. A, Fig. 126, is tied to a wall and rests on B, which in turn rests on the inclined plane. The coefficient of friction for all contacts is 0.3. B weighs 100 lb. What minimum weight of A would prevent slipping? *Ans.* 28.8 *lb.*

10. Block A, Fig. 127, rests on a horizontal floor; block B is suspended by an inclined cord and rests against A. A weighs 80 lb, and B weighs 60 lb. The coefficient of friction for all surfaces of contact is 0.2. Determine the horizontal force P, applied as shown on A, which will (a) just prevent slipping to the left; (b) just cause slipping to the right.

11. A cylinder weighing 200 lb rests in a trough formed by two surfaces each inclined at 40° to the horizontal. The coefficient of friction is 0.2 between the cylinder and each surface. What horizontal force P, applied to the cylinder at its topmost point, will make the cylinder rotate?

12. A straight bar rests in a vertical plane with one end on a horizontal floor and the other against a vertical wall. The coefficient of friction for floor and bar is 0.4; for wall and bar, 0.3. At what minimum angle between bar and floor would the bar rest? *Ans.* 47.7°.

13. Suppose that the bar of the preceding problem weighs 100 lb, and is set at an angle of 60°. Determine the necessary downward force applied at the upper end to cause slip of the bar. At what angle would the bar have to be placed in order that slipping could not be made to occur in this way?

14. AB, Fig. 128, is a bar on two inclined planes. R denotes the resultant of the weight of the bar and the pulls due to several bodies suspended from the bar. Evidently, if R is between two certain points C and D, the bar will not slip. Locate these limiting points for these data: $AB = 10$ ft and $\mu = 0.1$ at A and at B. (Graphical solution suggested.)

15. A ladder rests against a vertical wall, its lower end on a horizontal floor. The coefficient of friction between ladder and floor is 0.3; between ladder and wall, 0.1. What part of the total length of the ladder could a man ascend when the ladder is inclined at 60° to the horizontal? (Neglect weight of ladder.) *Ans.* 0.533.

16. A car wheel is 29 in. in diameter (outside); the brake drum is 14 in. in diameter. The coefficient of friction between tire and pavement is 0.6; between brake and drum, 0.3. If the weight on the wheel is 900 lb, what must be the brake pressure to make the wheels slip on the pavement? *Ans.* 3730 *lb.*

17. A rod weighing 100 lb rests in a horizontal position on two posts, Fig. 129. A vertical wire, straight but without initial tension, and practically inextensible, connects the rod to an anchorage in the ground. The coefficient of friction between rod and posts is 0.2. Determine all forces on the rod when a horizontal pull P of 50 lb is applied. Derive a formula for the wire tension in terms of P.

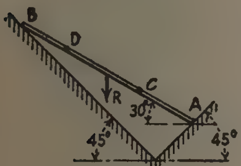

FIG. 128.

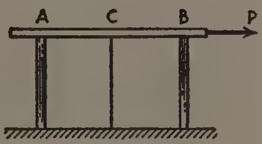

FIG. 129.

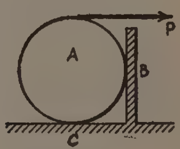

FIG. 130.

18. It is desired to roll a cylinder 4 ft in diameter over a curb 1 ft high by a horizontal force P. If the coefficient of friction between the cylinder and the edge of the curb is 0.2, what is the highest point on the cylinder at which P can be applied? What is the lowest point? What happens if P is applied above the upper point? Below the lower point?

19. A large tripod, similar to that for a transit, is sometimes used in construction work to support a length of pipe or other load. If the coefficient of friction between the tripod legs and ground is 0.6 and the legs are 14 ft long, what is the maximum spread (distance between lower ends of legs) that can be employed, assuming that the tripod legs are symmetrically arranged with respect to the load?

20. A brick weighing 4 lb rests on a roof that slopes at 20°. The coefficient of friction between brick and roof is 0.6. What horizontal force applied to the brick in a direction parallel to the roof ridge would just cause slipping? *Ans.* 1.79 *lb.*

Art. 63. Test for Rest or Motion

1. A block weighing 120 lb rests on a horizontal floor. For block and floor, $\mu = 0.4$. A pull of 60 lb is applied to the block, upward at 20° to the horizontal. Will the block slip? *Ans.* *Yes.*

2. Suppose that the block of Prob. 1 has two horizontal forces applied to it, one of 20 lb acting northward and one of 30 lb acting eastward. Will the block slip?

3. Referring to Prob. 4 of Art. 59, suppose that the coefficient of friction is 0.5. Determine for each case whether there is slipping.

4. The cylinder in Fig. 130 weighs 100 lb; $\mu = 0.2$ for each contact. Will A slip if $P =$ 20 lb?

5. Can a cylinder be rolled over a cleat, the height of which is 0.2 times the diameter of the cylinder, by a force applied at the top of the cylinder, if the coefficient of friction between cleat and cylinder is 0.4? *Ans. No.*

6. A straight bar is placed with its lower end on a horizontal floor and its upper end against a vertical wall. At each point of contact $\mu = 0.1$. The center of gravity of the bar is 0.4 of its length from the lower end. If the angle of inclination is 60° to the horizontal, will the bar slip?

7. A and B, Fig. 131, respectively weigh 400 and 150 lb. Their coefficients of friction respectively are 0.5 and 0.8. Do A and B move or rest?

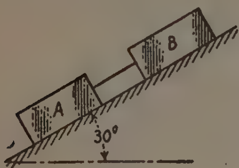

FIG. 131.

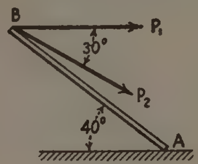

FIG. 132.

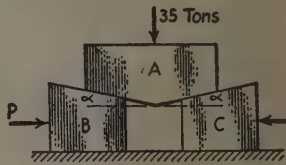

FIG. 133.

8. AB, Fig 132, is a bar supported at A by a floor. Could the position of the bar shown be maintained by a horizontal force P_1? by an inclined force P_2? The coefficient of friction under the bar is 0.7. (Graphical solution suggested.)

9. Two horizontal pegs A and B project from a wall; B is 2 ft above and 3 ft to the right of A. Can a uniform slender bar 10 ft long be placed over B and under A in such a way as not to slip down, if the coefficient of friction between the bar and the pegs is 0.3?

Art. 66. Wedges

1. The wedge angles α, Fig. 133, are 12°. Body A weighs 35 tons; weights of wedges B and C are negligible. The coefficient of friction for all contacts is 0.1. (*a*) Calculate P for raising the load. (*b*) Are any forces P required to prevent the wedges from slipping out? If so, how large must each be? If not, pulls of what value are required to start them out? *Ans.* (*a*) $P = 7.34$ *tons.*

2. Assuming that the weight of A in Fig. 133 is W and the angle of friction at each point of contact is ϕ, show that the forces Q_1 required to raise the load W are given by $Q_1 = \frac{1}{2}W[\tan(\phi + \alpha) + \tan\phi]$ and that the forces Q_2 required to lower the load are given by $Q_2 = \frac{1}{2}W[\tan(\phi - \alpha) + \tan\phi]$.

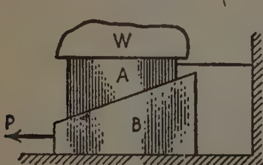

FIG. 134.

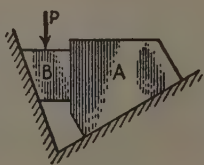

FIG. 135.

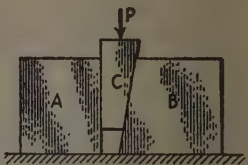

FIG. 136.

3. The coefficients of friction between A and B and B and the floor, Fig. 134, are $\frac{1}{4}$. The top of B is inclined at 20° to the horizontal. W weighs 10 tons. How great must P be to start the wedge B?

4. A, Fig. 135, weighs 10,000 lb; the coefficient of friction for all contacts is $\frac{1}{4}$. The surface against which B rests is inclined at 70° to the horizontal; that against which A rests, at 30°. What value of P is required for starting A up the plane? *Ans.* 97,300 *lb.*

5. A and B, Fig. 136, are equally heavy, and the coefficients under them are equal. The weight of C is negligible. (*a*) Suppose that C is smooth and that P is increased to start A or B. Ascertain which moves. (*b*) Suppose that the left face of C is rough, the coefficient there being 0.4, and that the wedge angle is 10°. Which moves, A or B?

6. Wedge A, Fig. 137, is used to raise the weight; the vertical wall is smooth, and the coefficient of friction on other surfaces is 0.2. What is the maximum wedge angle θ which will still give the wedge a "mechanical advantage," i. e., make P less than W?

Art. 67. Screw Jack

1. Referring to Prob. 1 of Art. 66, suppose that the forces P are to be applied by a right and left screw through tapped holes in the wedges B and C. Take screw data as follows: mean diameter $= 1\frac{1}{2}$ in., pitch $\frac{1}{4}$ in., $\mu = 0.15$, and calculate the torque applied at one end required for raising the load.

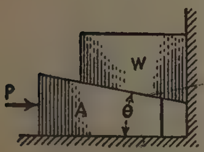

Fig. 137.

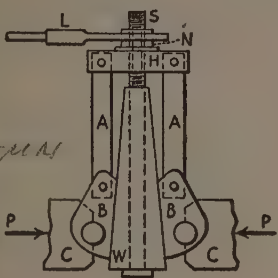

Fig. 138.

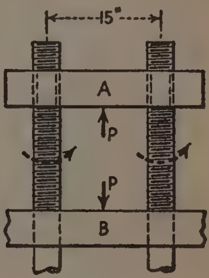

Fig. 139.

2. Figure 138 represents, somewhat conventionalized, an adjusting device used in making the closure (insertion of the last few members) of a large cantilever bridge (Beaver River). The mechanical elements are a double wedge W, a screw S, and a lever L. The accessories are a head piece H, two struts A, and two wedge blocks B; they are pin-connected as shown. C and C are two portions of the bridge member to be connected; they are under compression P and pin-bear against the wedge blocks B. The nut, which bears against the head piece, can be turned by means of the lever, and the screw and wedge raised or lowered. Raising the wedge separates the wedge blocks and parts C and C. Determine the necessary moment of the force on the lever for raising the wedge against pressures $P = 1{,}235{,}000$ lb, assuming that the struts A are vertical, and given the following data: mean diameter of screw $= 4\frac{1}{2}$ in.; pitch of screw $= \frac{1}{4}$ in.; bevel of wedge (each side) $= 1$ in 10; mean radius of nut where it bears on the head piece $= 9$ in.; coefficient of friction for all rubbing surfaces $= \frac{1}{4}$. (Consider first a wedge block, and determine all the forces which act upon it.) *Ans.* 2,530,000 *in.-lb.*

3. A, Fig. 139, is the cross head and B the weighing base of a screw testing machine. When turned from below, in the direction indicated, the screws depress A and compress any specimen, such as a concrete cylinder or wooden block, between A and B. (*a*) Calculate the torque T on each of the two screws for the following data: mean radius $= r = 1.025$ in., pitch $h = \frac{1}{4}$ in., $P = 100{,}000$, $\mu = 0.2$. (*b*) Of course the screws exert equal torques but in the opposite sense on the head A. Consider a fbd of A, and decide how these torques are resisted. Are the screws subjected to any bending action?

4. If made with a very large pitch angle a screw jack would not be self-locking; it would "overrun" or turn when loaded, unless prevented by an applied force or couple. A "Yankee screw driver" is an example of such a screw; when an axial thrust is applied, the screw turns.

Suppose that the pitch angle for such a screw driver is 70°, its mean rádius is 0.15 in., and the coefficient of friction is 0.06. What moment or torque is exerted by the screw driver when a push of 20 lb is applied to it?

Art. 68. Tackle

1. A man wishes to raise a load of 1200 lb by means of block and tackle similar to that shown in Fig. 133 of Art. 68. The rope is 1 in. in diameter, and the constants given in the table of Art. 68 can be considered applicable. How many pulleys must there be in each block if the man is to lift the load by exerting a pull of 120 lb? How much pull would the man have to exert in lowering the load?

Art. 69. Coil and Belt Friction

1. Figure 140 represents a band brake. The diameter of the wheel is 1 ft 8 in., the angle of lap = 255°, $P = 60$ lb, and the coefficient of friction is $\frac{1}{5}$; the wheel is turning clockwise. Compute the frictional moment and the pull on the pins A and B. Solve for the case when the wheel is turning in the other direction. *Ans. Moment = 1720 ft-lb when turning clockwise.*

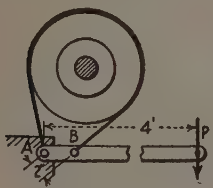

FIG. 140.

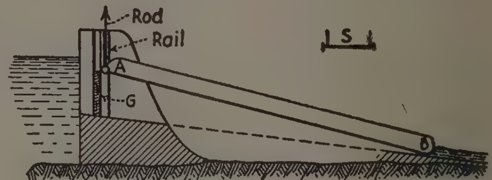

FIG. 141.

2. By means of a rope wrapped around a snubbing post, a man, by exerting a pull of 60 lb, resists a pull of 1500 lb at the other end of the rope. If the coefficient of friction between rope and post is 0.1, what must be the angle of wrap?

Art. 70. Rollers

1. Some men wish to move a heavy load weighing 2 tons by means of rollers. The load is to be moved some distance along a hardwood floor, and the rollers available are of hardwood, 4 in. in diameter. What horizontal push must the men exert to move the load?

2. To allow for temperature expansion, one end of a bridge is supported on a group or "nest" of segmental rollers, that is, diametral segments of cylinders, which function exactly like cylinders as long as the angular displacement is small. The rollers are of steel, 8 in. in diameter, and rest on steel plates. Using the data and formulas given in Art. 70, calculate the horizontal force required to produce motion if the load on the nest of rollers is 100 tons.

General

1. Figure 141 represents the cross section of a dam, a sluice gate, and a log sluice or trough AB (shown in section at S). Water is shown passing over the gate and down the sluice permitting the passage of logs. The sluice is made adjustable to the water level. The front wheels at A rest against vertical rails, and the wheels at B on rails inclined at 10° to the horizontal. The end A is raised or lowered by means of a vertical rod operated from above by a suitable winch. The log sluice weighs 10 tons, $AB = 80$ ft. Determine the pull at the rod required to overcome the weight of the sluice and the friction at A and B when the sluice is inclined at 25° to the horizontal. (As the diameter of the wheels A and B is small compared to AB, regard the sluice as slipping on the two ends, and take the equivalent coefficient of friction as $\frac{1}{10}$.)

2. Refer to Fig. 77. Suppose that instead of being supported by a rope the upper end of the bar is prevented from sliding down by friction. Derive a formula for the maximum value ϕ can have, in terms of θ and μ, the coefficient of friction between bar and wall.

3. A uniform bar rests at an inclination of 60° to the horizontal, with its lower end on a horizontal floor and its upper end against a vertical wall. At the upper end, $\mu = 0.6$; at the lower end, $\mu = 0.25$. What horizontal force, parallel to the wall, applied at the upper end of the bar, would be required to cause slipping? Would the bar slip simultaneously at both ends, or at the upper end only? Solve also taking μ at the lower end of the rod as 0.4.

CHAPTER VI

Art. 72. Center of Gravity by Experiment

1. It was desired to locate partially the center of gravity of a heavy irregular body on a truck. The loaded truck was run up to a street scales and the load on the front pair of wheels weighed and then that on the rear; they were 4000 and 8000 lb respectively. The wheel loads on the unloaded truck were found likewise to be 2500 and 3500 lb respectively. The wheel base was 132 in. What information about the center of gravity of the heavy body is afforded by these data?

2. In order to locate partially the center of gravity of an elongated irregular body, it was supported at two points A and B at the same level, resting on a scale first at A and then at B. The weights recorded were 1450 lb and 785 lb respectively; the distance $AB = 3$ ft. What information about the position of the center of gravity of the body is afforded by these data?

3. A certain box, 30 in. wide, 24 in. high, and 50 in. long, weighs 280 lb. The box is supported on a tranverse knife edge 1 in. from one end, and the supporting reaction at the bottom edge at the other end is determined, by weighing, to be 150 lb when the box is level and 120 lb when it is tipped up at 30° to the horizontal. What do these data enable you to say concerning the center of gravity of the box?

4. A heavy body has three feet like a three-legged stool. The triangle determined by these feet is equilateral. The location of the projection of the center of gravity of the body on the plane of this triangle is desired. Can this location be determined by weighing if only one of the feet can be put onto a scales at one time? If so, determine just how many weighings you would make, and how to calculate the location.

Art. 73. Center of Gravity by Calculation

1. Figure 142 represents a bent wire; the length of the part extending to the left of the y axis is 12 in. Determine the coordinates of its center of gravity. *Ans.* $\bar{x} = 5.76$, $\bar{y} = 5.83$, $\bar{z} = 7.07$ *in.*

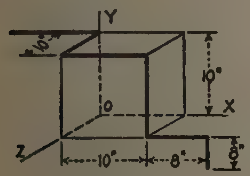

FIG. 142.

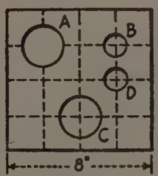

FIG. 143.

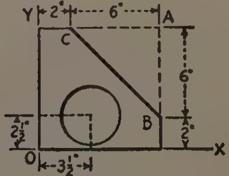

FIG. 144.

2. Figure 143 represents a thin plate into which holes were punched at A and B, and the pieces glued on at C and D, respectively. Area of hole $A = 4$ in.^{2}; that of $B = 2$ in.^{2} Locate the center of gravity of the modified plate. *Ans.* $\bar{x} = 4.13$, $\bar{y} = 3.69$ *in.*

3. One corner ABC, Fig. 144, of a square piece of uniform sheet metal is folded forward on the line BC so that it stands at right angles to the rest of the piece, and a 4-in. hole is cut as shown. Locate the center of gravity of the piece when thus altered.

4. Figure 145 represents an irregular piece of uniform sheet metal. $OABC$ is a rectangular part in the xz plane, ODE is a triangular part in the yz plane, and $OEFC$ is a square part with a 4-in. (diameter) hole in the xy plane. Determine the x and z coordinates of the piece of sheet metal.

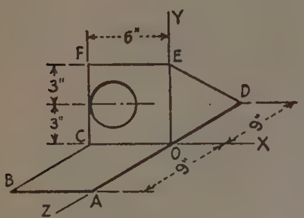

FIG. 145.

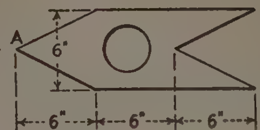

FIG. 146.

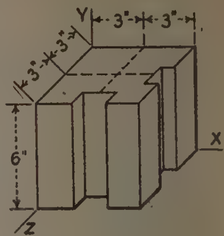

FIG. 147.

5. Figure 146 represents a piece of uniform sheet metal whose center of gravity is at the center of the hole. How far from A is the center of the hole?

6. Locate the center of gravity of the grooved cube, Fig. 147. The grooves are 2 in. wide and 1 in. deep.

7. Figure 148 represents an axial section of a right-circular cylinder (diameter 10 in. and height 7 in.) having two cylindrical holes with axes A and B (diameters 4 and 2 in.). The axes A, B, and Y are coplanar. Find the x and y coordinates of the center of gravity.

8. The center of gravity of a fairly homogeneous disk 10 in. in diameter is 0.015 in. from the geometrical center. Where should one drill a $\frac{3}{4}$-in. hole in order to put the center of gravity at the center of the disk?

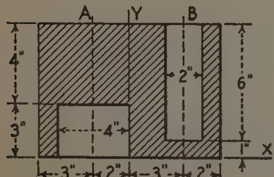

FIG. 148.

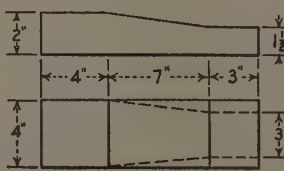

FIG. 149.

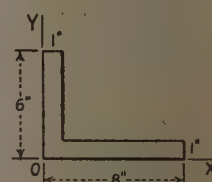

FIG. 150.

9. Figure 149 represents a crank arm for a shaft, by plan and elevation — dotted lines to be disregarded. Locate the center of gravity of the arm. *Ans.* $\bar{x} = 6.55$ *in.*

10. Solve Prob. 9, but change the width at the thin end as shown by dotted lines.

11. Into one end of a cast-iron cylinder 4 in. in diameter and 10 in. long a hole 2 in. in diameter and 6 in. deep is drilled, and this hole is filled with lead. The axis of the hole is collinear with the axis of the cylinder. Locate the center of gravity of the composite cylinder.

Art. 74. Centroid of a Line, Surface, or Solid

1. Calculate the x and y coordinates of the L-shaped area, Fig. 150, first regarded as consisting of two narrow rectangles, then as consisting of a 6-by-8-in. rectangle less a 5-by-7-in. rectangle. Which is the better method?

2. Figure 151 is the cross section of a steel beam built-up of two angles 5 by 4 by $\frac{1}{2}$ in. and a plate 8 by $\frac{3}{4}$ in. The centroid of each angle is 1.57 in. from the back of the shorter leg. Determine the position of the centroid of the entire section.

3. Determine the coordinates of the centroid of the area in Fig. 152. The width of each rectangle is 1 in. *Ans.* $\bar{x} = 3.75$ *in.*

4. Determine the centroid of the area in Fig. 153. The height of the rectangle is 10 in.; of the triangle, 9 in. The area of the hole is 8 in.², and the coordinates of its centroid are $x = 3$ and $y = 4$ in.

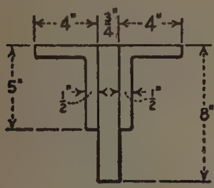

FIG. 151.

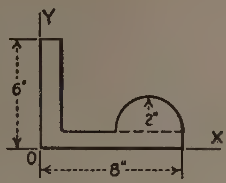

FIG. 152.

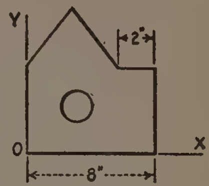

FIG. 153.

5. Locate the centroid of the area in Fig. 154. The curved lines are circular arcs. Check your results as directed under the following problem.

6. Figure 155 represents an area bounded by two circular arcs and two common tangents. The line AB joining the centers of the arcs is 10 in. long, and the radii of the arcs are 4 and 6 in. Locate the centroid of the area with respect to A or B. Check your solution experi-

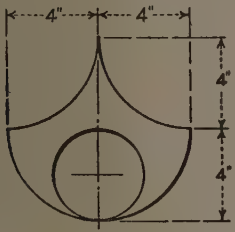

FIG. 154.

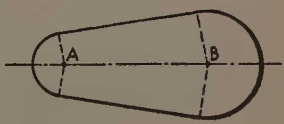

FIG. 155.

mentally; cut the figure out of stiff cardboard, and see whether it will balance when supported at the center of gravity as determined. Compare the time required to do this with the time required for the algebraic solution. *Ans.* 6.67 *in. to right of A.*

Art. 75. Centroid Determined by Integration

1. Prove that the distance of the centroid of a triangle from its base equals one-third the altitude.

2. Prove that the centroid of a semicircular area is distant $0.4242r$ from the diameter.

3. Imagine an ellipsoid separated into halves by one of its planes of symmetry. Determine the position of the centroid of one of these halves.

4. Prove that the distance from the centroid to the base of a paraboloid of revolution formed by revolving a parabola about its axis equals one-third the altitude.

General

1. For stability and maneuverability in flight, it is necessary that the center of gravity of a loaded airplane should lie between certain limiting points, especially with reference to its position along the longitudinal axis of the plane. In describing the position of any weight item, it is customary to give its "station," which is simply its distance in inches from the nose of the airplane, measured parallel to the longitudinal axis of the plane.

An empty DC-3 type cargo airplane is weighed in the horizontal position with each landing wheel and the tail wheel on a separate scale, and the weights recorded are as follows: left main wheel, 7925 lb; right main wheel, 7930 lb; tail wheel, 820 lb. The main wheels are at station 219.5, and the tail wheel at station 666.7.

(a) At what station is the center of gravity of the empty plane?

(b) To the empty airplane the following loads are added at the stations indicated: 2 pilots at station 70; 1 radio operator at station 154; 420 gal of gasoline in the front wing tanks at station 233.0; 402 gal of gasoline in the rear wing tanks at station 267.0; cargo of 500 lb at 115, 500 lb at 228, 475 lb at 328, 750 lb at 418, 920 lb at 508. (A crew member, with flying clothes and parachute, is assumed to weigh 200 lb, and gasoline weighs approximately 6 lb per gal.) At what station is the center of gravity of the loaded plane?

(c) The optimum position of the center of gravity for safety and operating efficiency is at station 256.2. If 1250 lb of cargo remains to be loaded after the load described in (b) is put in, where should the 1250 lb be placed to make the center of gravity of the fully loaded plane fall at the optimum location?

(d) For safety, the center of gravity must fall between stations 239.6 and 263.1. How far forward and how far aft could the 1250-lb load be put without causing the center of gravity of the fully loaded plane to fall outside these limits?

2. Owing to nonhomogeneity, the center of gravity of a cast-iron cylinder 20 in. long and 8 in. in diameter is 0.03 in. from midlength and 0.02 in. from the geometrical axis. It is desired to bring the center of gravity to the geometrical center by drilling a hole, 1 in. in diameter, into one end of the cylinder, this hole to be parallel to the axis. How deep must the hole be, and how far from the axis of the cylinder?

CHAPTER VII

Art. 77. Parabolic Cable

1. A cord is supported at two points on the same level 30 ft apart, and its lowest point is 8 ft below the level of the supports. If the load is 20 lb per horizontal foot, what are the tensions at the supports and at the lowest point? *Ans.* 411 *lb at supports*, 281 *lb at lowest point.*

2. A certain wire weighs 0.094 lb/ft and can sustain a pull of 1500 lb with safety. It is to be suspended between two points on the same level and 1000 ft apart. Assume that the suspended wire will be parabolic, and compute the shortest piece of wire that may be used.

3. A cable 100 ft long is laid symmetrically across a ditch 20 ft wide cut in level ground. The cable is held only by friction on the parts that lie on the ground, and the coefficient of friction between ground and cable is $\frac{1}{3}$. What is the sag?

Art. 78. Catenary Cable

1. Solve Prob. 2 of Art. 77 on the assumption that the suspended wire takes the catenary form.

2. A cable that weighs 0.5 lb/ft is to be suspended between two points at the same level 200 ft apart; the sag is to be 80 ft. Determine the length of the cable and the maximum tension in it. *Ans.* 270 *ft;* 76.5 *lb.*

3. Show that for minimum tension the sag of a catenary cable should be $f = 0.338a$, where a is the span.

CHAPTER VIII

Art. 81. Velocity

1. Reduce all the following speed records to miles per hour and to feet per second. Man running — 100 yd in 9.4 sec. Horse running — 9 furlongs in 1 min 53 sec. Homing pigeon — 1800 yd/min. Electric locomotive — 210 km/hr. Destroyer — 38 knots. Sculling — $4\frac{1}{4}$ miles in 22 min 48 sec.

Art. 82. Calculation of Velocity

1. A man drives 200 miles. If he has to stop for $1\frac{1}{3}$ hr to repair a tire, with what uniform speed must he drive the rest of the time in order to average 30 mi/hr for the entire run? *Ans.* 38.71 *mi/hr.*

2. A particle moves in a straight line so that $s = 2t^3 - 5t^2$, where s is in feet and t in minutes. Determine the velocity when $t = 1$ min; when $t = 2$ min. Interpret the signs. *Ans.* -4 *ft/min*, $+4$ *ft/min.*

3. A particle moves in a straight line in accordance with the law $s = t^3 - 40t$, where s is in feet and t in seconds. Find the velocity when $t = 5$ sec. What is the average velocity for the second preceding the instant named? for the second following the instant. *Ans.* 35, 21, 51 *ft/sec.*

4. A particle P moves in a straight path according to the equation $s = -t^3 + 7.5t^2 - 10t$, foot and second units. (*a*) Derive the general equation for velocity. (*b*) What are the velocities at $t = 0$, 2, 4, and 6? (*c*) Where is P and in which direction is P moving at those times? (*d*) Make graphs of the equations for s and v, up to $t = 6$, putting both on the same t axis. Use these scales: 1 in. for 2 sec, 5 ft, and 10 ft/sec. (*e*) Is the velocity zero at some time in the interval 0 to 6 sec? If so, when? *Ans.* *At* $t = 6$, $v = -28$ *ft/sec;* $s = -6$ ft. $v = 0$ *when* $t = 0.793$ *and* 4.207 *sec.*

5. A particle moves in accordance with the equation: $s = 32 + 24t - 2t^3$, where s is in feet and t is in seconds. (*a*) Derive the $v - t$ equation for this motion. (*b*) How far to the right of the origin does the point go? (*c*) What is the average velocity of the point while moving to the right? (*d*) When, if ever, does the point pass the origin? (*e*) What is its velocity when it passes the origin?

6. A point is 4 ft to the right of the origin when $t = 0$, and it moves so as to triple its distance every second for 5 sec. Derive the equation for the velocity, and determine the velocity at the end of 5 sec.

7. A certain point P of a mechanism is made to move in a straight line by means of a crank in such a way that $s = 3 \cos 2\theta$, where s = the (varying) distance in feet of P from a fixed origin in the path of P, and θ the (varying) angle which the crank makes with a fixed line of reference. The crank rotates uniformly at 100 rev/min. Determine the position and velocity of P when $\theta = 60°$. Interpret the signs of the results. *Ans.* *Velocity* $= -54.4$ *ft/sec.*

8. The space-time curve for a certain motion is plotted to the scales: 1 in. (horizontal) equals 2 min; 1 in. (vertical) equals $\frac{1}{2}$ mile. What speed, in feet per second, does a slope of the curve of 30° to the horizontal indicate?

9. In a certain gunnery experiment the shot was fired through screens placed 150 ft apart. The times (in seconds) of piercing were observed with the following results:

Screen	1	2	3	4	5	6	7
Time	0	0.0666	0.1343	0.2031	0.2729	0.3439	0.4161

Determine the velocity at the fourth screen.

10. Starting from rest, a railway test train moved the distance given after s in the time given after t:

$t =$ 5	10	15	20	25	30	35	40	45	50	55 sec
$s =$ 30	88	160	250	350	470	585	715	840	970	1130 ft

Plot the s-t graph for this motion, and determine the velocity of the train at $t = 20$ and the maximum velocity.

11. In order to study the braking of a passenger train, observers determine, by means of stop watches, the time intervals during which spaces between successive telegraph poles are passed over, observations being taken from the instant the brakes are applied till the train stops.

The telegraph poles are 160 ft apart, and the time intervals for successive spaces are as follows:

Space	First	Second	Third	Fourth	Fifth
Time (sec)	2.2	2.4	2.7	3.3	7.0

The train comes to rest at the end of the fifth space.

(*a*) Determine the approximate velocity of the train at the instant the brakes were applied. (*b*) Determine the approximate velocity of the train at the instant the fourth space has been traversed. (*c*) Determine the average velocity of the train while it is traversing the first four spaces. *Ans.* (*a*) $v = 74.8\,ft/sec$; (*b*) $v = 42.6\,ft/sec$; (*c*) $v_a = 60.4\,ft/sec$.

Art. 83. Acceleration

1. The value of g, acceleration of a falling body, is usually taken as 32.2 ft/sec². Express this acceleration in: (*a*) miles per hour per second; (*b*) yards per minute per second; (*c*) meters per second per second.

2. Compare the retardation of a train at 4 mi/hr/sec with the retardation of gravity on a ball thrown vertically upward. A certain electric car can get up a speed of 60 mi/hr in 20 sec. Compare its average starting acceleration with the acceleration of gravity.

Art. 84. Calculation of Acceleration

1. In a certain rectilinear motion $v = t^2 - 10t$, where v is velocity in feet per minute and t is time in minutes. Determine by calculus the acceleration when $t = 10$ min. What is the average acceleration for the minute preceding the instant named? for the minute following the instant? *Ans.* $+10, +9, +11\,ft/min^2$.

2. For the motion described in Prob. 2 of Art. 82 determine the acceleration when $t = 1$ min; when $t = 2$ min. Interpret the signs of the results.

3. Refer to Prob. 4 of Art. 82. (*a*) Derive the general equation for the acceleration. (*b*) What are the accelerations at times $t = 0, 2, 4,$ and 6? (*c*) Is the velocity increasing or decreasing at those times? (*d*) Is the speed increasing or decreasing then? (*e*) Make a graph of the equation for acceleration, using these scales: 1 in. for 2 sec and 20 ft/sec². *Ans.* (*b*) $15, 3, -9, -21\,ft/sec^2$.

4. For the motion described in Prob. 5 of Art. 82 ($s = 32 + 24t - 2t^3$): (*a*) Derive the a-t equation. (*b*) What is the value of a when $v = 0$? (*c*) What is the average acceleration while the point is moving to the right?

5. For the motion described in Prob. 6 of Art. 82 derive the equation for the acceleration, and determine the acceleration at the end of 5 sec.

6. A particle moves in a straight line according to the law $v^2 = k/s$, where k is a constant. Show that the acceleration is given by $a = -k/2s^2$.

7. For the motion described in Prob. 7 of Art. 82, determine the acceleration of P when $\theta = 60°$. Interpret the sign of the result. *Ans.* $658\,ft/sec^2$.

8. A point starts with an initial velocity of 5 ft/sec and moves so as to double its velocity every second for 4 sec. What is the a-t law, and what is the maximum acceleration? *Ans.* $Max\ a = 55.45\,ft/sec^2$.

9. In a test to determine the starting acceleration or "pick up" of a car at speeds in excess of 10 mi/hr the following data were obtained:

$t =$ 0	14	16	18	20	22	24	26	28	30	32	sec
$v =$ 10	49.0	54.5	59.5	63.6	67.5	71.0	74.0	76.6	79.0	81.2	mi/hr

(In the test the velocities were not measured to the nearest 0.1 mi/hr as these figures imply; the values given here were taken from a faired v-t curve based on observed data.) Draw the v-t curve, and determine the acceleration at 10, 60, and 80 mi/hr. (The v-t curve is straight from $t = 0$ to $t = 14$.)

10. In a certain run the velocity of an electric car changed in the following manner:

Time t	0	10	15	20	25	30	35	40	50	114	125 sec
Velocity v	0	15	21	24.5	26.5	28.5	29.5	31	32.5	21	0 mi/hr

The car coasted during the interval 50–114 and was braked during the interval 114–125. (Sheldon and Hausman, *Electric Traction and Transmission Engineering*, p. 63.) Find the acceleration when $t = 0$ sec; when $t = 25$ sec. *Ans.* 1.5 *and* 0.41 *mi/hr/sec.*

11. In a test of an electric locomotive the following data were obtained:

t=10	20	30	40	50	60	70	80	90	100	110	120	130	140	145 sec
v=10.4	19.2	24.5	28.0	32.0	34.2	36.8	38.0	39.5	41.0	40.2	37.0	29.0	16.0	0 mi/hr

Draw the v-t curve, and determine the maximum rate of speeding up and the maximum rate of slowing down.

Art. 85. Study of a Motion by Integration

1. A point P moves in a straight line so that $a = 4 - 2t$, where a is in feet per minute per minute and t in minutes. When $t = 0$, $v = 0$ and $s = 0$. Determine general formulas for v and s. What are v and s when $t = 4$? when $t = 5$? *Ans.* *When* $t = 5$, $v = -5$ ft/min, $s = 8\frac{1}{3}$ ft.

2 A particle starts from the origin and moves along the x axis in accordance with the equation $v = 20 - t^2$, the units being the foot and second. Determine the acceleration, velocity, and position of the particle at the end of 10 sec, the average acceleration for that period, and the actual distance traveled by the particle. *Ans.* *Distance* = 252.5 *ft.*

3. A particle moves in accordance with the equation $a = 24 - 6t^2$. When $t = 0$, $v = +20$ and $s = 0$. (*a*) Derive the v-t and s-t equations for the motion. (*b*) Determine the maximum positive velocity. (*c*) Determine the maximum distance to the right of the origin. (*d*) Determine the distance traveled during the interval $t = 2$ to $t = 5$. (*e*) Determine the average velocity and the average speed for that interval.

4. A particle starts from rest at the origin, moves with an acceleration of 8 ft/sec² until it reaches $s = 64$ ft, then moves with a constant acceleration of -4 ft/sec² until it stops. (*a*) What is its maximum speed? (*b*) How long does it take the complete motion described to occur? (*c*) How far does the particle travel? (*d*) What is the average velocity of the particle? *Ans.* (*a*) +32 *ft/sec;* (*b*) 12 *sec;* (*c*) 192 *ft;* (*d*) 16 *ft/sec.*

5. A particle starts from rest at the origin for s and moves with a velocity that increases uniformly with respect to time. It is to be 250 ft to the right of the origin 5 sec after starting; what are the a-t, v-t, and s-t laws?

6. A point starts from rest at the origin for s and moves with an acceleration that, initially zero, increases uniformly with respect to time. At the end of 5 sec, the point is 250 ft to right of the origin. What are the a-t, v-t, and s-t laws?

7. A body has an initial velocity (when $t = 0$) of 15 ft/sec and an acceleration expressed by $a = 90t - 24t^2$, the units being foot and second. (*a*) How far does the body move in the interval $t = 3$ to $t = 7$? (*b*) What are the velocities at the instants named? (*c*) What is the average velocity for the interval? *Ans.* (*a*) 772 *ft;* (*c*) 40.5 *ft/sec.*

8. A certain train can be retarded at the rate of 4 mi/hr/sec by braking. Determine the times (in seconds) and the distances (in feet) in which the train can be stopped from 10, 20, 30, and 40 mi/hr. (Assume that the retardation is the same at all speeds.) *Ans.* *From* 40 *mi/hr in* 10 *sec and in* 293.3 *ft.*

9. A certain electric train can get up full speed of 24 mi/hr in a distance of 150 ft and can stop from full speed in a distance of 100 ft. What is the shortest time in minutes in which the train can make a run between two stations 650 ft apart, the train starting from one station and coming to full stop at the other? (Assume that the starting and stopping are accomplished uniformly with respect to time.) *Ans.* 25.6 *sec.*

10. A train can get up a speed of 60 mi/hr in 5 min, and stop in 0.5 mi. About midway between two stations 10 mi apart a bad piece of track 1 mi long necessitates reduction of speed to 10 mi/hr. Assuming that acceleration and retardation can be applied uniformly with respect to time, determine the time between stations. (Sketch the velocity-time graph before calculating.) How much time was lost on account of defective track? *Ans. Time required* = 20.084 *min.*

11. (*a*) On a dry concrete pavement a car can, by proper braking, be stopped from 60 mi/hr in 142 ft. What acceleration does this represent, assuming the rate of retardation to be uniform? (*b*) For the average driver, $\frac{3}{4}$ sec elapses between the perception of danger and the application of the brakes. Assuming this "reaction time," and the stopping acceleration found in (*a*), calculate the distance that would be traveled between the instant danger was perceived and the instant the car could be brought to rest from a speed of 80 mi/hr.

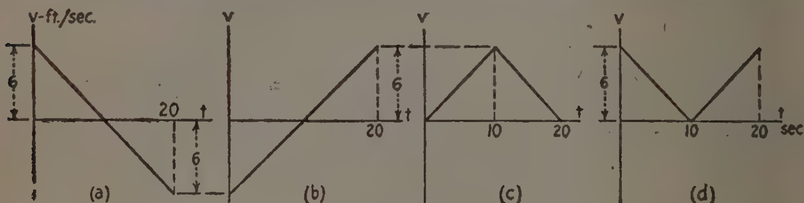

FIG. 156.

12. Figure 156 shows *v-t* graphs for four rectilinear motions in each of which *s* was zero when *t* was zero. Make the *a-t* and *s-t* graph for each without writing the equations. Then write the *a-t* and *s-t* equations and use them to verify the graphs.

13. Refer to Fig. 156 and change feet per second to feet per second², and regard the changed graphs as pertaining to four rectilinear motions in which *v* and *s* were zero when *t* was zero. Then make the *v-t* and the approximate *s-t* graphs for each motion without writing the equations. Then write the *v-t* and *s-t* equations and use them to verify the graphs.

14. A particle moves in accordance with the *v-t* graph shown in Fig. 157 until it returns to the origin, where it was at the instant $t = 0$. (*a*) Draw the *a-t* graph for this motion. (*b*) How far to the right of the origin does the point go? (*c*) When does it return to the origin? (*d*) Sketch the *s-t* graph, giving values of *s* at $t = 5, 10$, and 12.

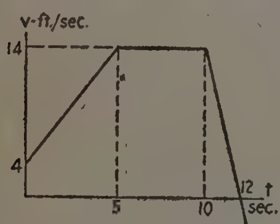

FIG. 157.

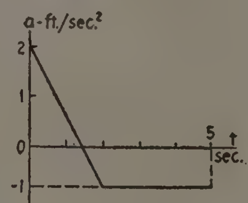

FIG. 158.

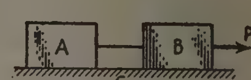

FIG. 159.

15. Figure 158 shows the acceleration-time graph for a certain rectilinear motion. When $t = 0$, v and $s = 0$. Construct the *v-t* and *s-t* graphs. Write the *a-t*, *v-t*, and *s-t* equations for the motion.

16. For the motion of the car described in Prob. 9 of Art. 84, determine the distance traveled in the entire period of 32 sec.

17. For the motion of the electric locomotive described in Prob. 11 of Art. 84, determine the distance required for attaining maximum speed and the distance required for stopping.

18. A particle starts at the origin with an initial velocity of +10 ft/sec and moves according to an a-t diagram that may be described as follows: A straight line from $t = 0$, $a = 0$, to $t = 4$, $a = 12$; a horizontal straight line $a = 12$ to $t = 10$; a straight line from $t = 10$, $a = 12$, to $t = 12$, $a = 0$, and on. (a) What is the maximum positive velocity attained? (b) How far does the point go to the right of the origin? (c) When, if ever, does it return to the origin? (d) Construct the approximate v-t and s-t diagrams up to $t = 12$. *Ans.* (a) *118 ft/sec;* (b) *1222 ft;* (c) $t = 26$ *(nearly).*

19. The following table gives the corresponding values of s, the distance traveled in feet along the bore of the gun, and a, the acceleration in thousands of feet per second per second of the projectile as it moved from breech to muzzle.

s	0.051	0.170	0.3315	0.542	0.817	0.988	1.195	1.803	2.88	5.125	6.21
a	246	398	473	493	478	445	391	273.5	131.5	83.8	73.0

(a) Determine the muzzle velocity of the projectile. (b) How much velocity would be lost by reducing the barrel length to 5 ft? *Ans.* (a) *1570 ft/sec;* (b) *62 ft/sec.*

Art. 89. Acceleration along Any Axis

1. Derive a formula for the acceleration with which a particle will slide down a smooth plane under the action of gravity, the plane being inclined at an angle θ to the horizontal. Do the same thing assuming a coefficient of friction μ between plane and particle. Does the acceleration, in either case, depend upon the weight of the particle?

2. At what angle should a smooth plane be inclined in order that a particle, allowed to slide down the plane, shall experience maximum horizontal displacement within a given time after starting?

3. A particle is projected up a plane inclined at 30° to the horizontal with an initial velocity of 20 ft/sec. The coefficient of friction between plane and particle is 0.15. Find the position of the particle after $\frac{1}{2}$ sec; after 5 sec.

Art. 91. Typical Problems; Examples

1. A, Fig. 159, weighs 200 lb, B weighs 100 lb; the coefficient of friction under A is $\frac{1}{6}$, that under B is $\frac{1}{4}$; $P = 300$ lb. Determine the acceleration of A and B, and the tension T in the rope connecting them. *Ans.* $T = 196.5$ *lb.*

2. Suppose that the supporting surface in the preceding problem is not horizontal but inclined at 30° to the horizontal. Then solve. *Ans.* $a = 10.07$ *ft/sec²; tension* = *197.2 lb.*

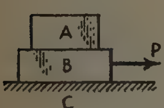

FIG. 160.

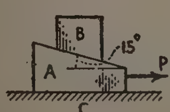

FIG. 161.

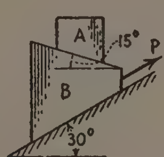

FIG. 162.

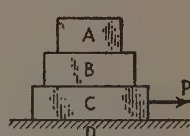

FIG. 163.

3. A, Fig. 160, weighs 100 lb, and B weighs 200 lb. The coefficient of kinetic friction under B is $\frac{1}{5}$; the coefficient of static friction under A is $\frac{1}{10}$. When $P = 75$ lb, will A slip? How great is the friction under A? How large a force P would just make A slip? *Ans. To cause slip* $P = 90$ *lb.*

4. A block A, Fig. 161, rests on a horizontal floor C, and a second block B rests on the sloping top of A. Between A and B the coefficient of friction (static and kinetic) is 0.4, and between A and C it is 0.1. A weighs 60 lb; B weighs 40 lb. (a) What horizontal force P, applied to A, would cause slipping as between the blocks? (b) What would P have to be if applied (in the same direction) to B instead of to A?

5. A, Fig. 162, weighs 50 lb, and B weighs 100 lb; the pull P gives A and B an acceleration of 12 ft/sec² up the plane. Determine the magnitude and direction (referred to the horizontal) of the pressure between A and B.

6. The weights of A, B, and C, Fig. 163, are 50, 100, and 200 lb respectively. Contacts between A, B, and C are very rough; between C and D, perfectly smooth; $P = 100$ lb. Determine the forces which the bodies exert upon each other. Sketch each body separately, showing the forces acting on it.

7. A box weighing 2000 lb is to be lowered from the deck of a ship and deposited on the floor of the hold, 40 ft below, without appreciable impact. This is to be done by means of a hoisting cable in which the tension must not exceed 6000 lb. What is the least time in which the load can be thus lowered?

8. A certain freight elevator weighs 3000 lb empty and 10,000 lb with load. It is raised and lowered between floors 50 ft apart by means of a single cable, the safe tension in which is 12,000 lb. Assuming that in each trip the car starts from rest at one floor and comes to rest at the other, and that no forces act on it except gravity and the pull of the cable, determine: (*a*) the least time in which the loaded elevator can be raised; (*b*) the least time in which the empty elevator can be lowered. *Ans.* (*a*) 4.328 *sec;* (*b*) 2.16 *sec.*

9. Two bodies are connected somewhat as two cars and are placed on a plane inclined at 30° to the horizontal. The lower one weighs 600 lb and is smooth; that is, there is no resistance to its sliding on the plane. The upper one weighs 1000 lb, and the coefficient of friction under it is $\frac{2}{10}$. With what acceleration will the bodies slide down when released? Will there be tension or pressure at the connection? What is its value?

10. Upon a plane inclined at 30° to the horizontal is placed a block A; on A is placed another block B. A weighs 120 lb; B weighs 50 lb; the top of A is parallel to the plane. Determine the acceleration of each block under each of the following conditions: (*a*) all surfaces of contact are smooth; (*b*) the coefficient of friction between A and the plane is 0.1, between A and B, 0.4; (*c*) the coefficient of friction between A and the plane is 0.4, between A and B, 0.1. *Ans.* (*a*) *Both blocks,* 16.1 *ft/sec²;* (*b*) *both blocks,* 13.3 *ft/sec²;* (*c*) *block A,* 1.46 *ft/sec².*

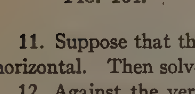

Fig. 164.

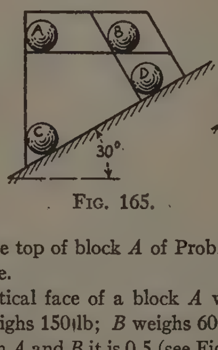

Fig. 165.

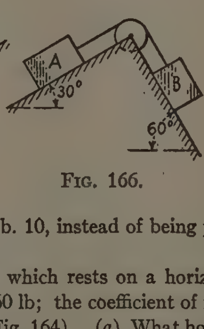

Fig. 166.

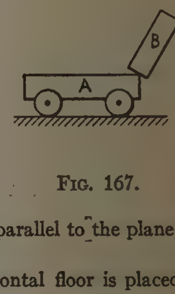

Fig. 167.

11. Suppose that the top of block A of Prob. 10, instead of being parallel to the plane, is horizontal. Then solve.

12. Against the vertical face of a block A which rests on a horizontal floor is placed a second block B. A weighs 150 lb; B weighs 60 lb; the coefficient of friction between A and the floor is 0.2; between A and B it is 0.5 (see Fig. 164). (*a*) What horizontal force P applied to A will prevent B from slipping down? (*b*) What horizontal force P applied to B will prevent B from slipping down? *Ans.* (*a*) 462 *lb;* (*b*) 151.2 *lb.*

13. Two spheres, A and B, are connected by a rubber band. A weighs 3 lb, B weighs 2 lb. The band is stretched until the tension in it is 10 lb; the system is then released in space, with the 3-lb sphere vertically above the 2-lb sphere. Determine the acceleration of each sphere at the instant of release. *Ans.* *Upper,* 139.5 *ft/sec² down; lower* 128.7 *ft/sec² up.*

14. The cylinders A, B, C, D, Fig. 165, rest in compartments of a box as shown. Each weighs 100 lb. Calculate all pressures on each: (*a*) when the box is dragged upwards with an

acceleration of 10 ft/sec²; (*b*) when it is dragged upward with constant speed of 5 ft/sec; (*c*) when it is at rest.

15. *A*, Fig. 166, weighs 160 lb, and *B* 250 lb. The coefficient of kinetic friction between *A* and the incline is 0.3, and between *B* and the incline 0.25. Determine the acceleration of *A* and *B*. (Consider the rope very flexible, and neglect the mass of rope and pulley.)

16. A light flexible rope passes over a frictionless pulley, the mass of which is negligible. To one end of the rope is attached a body *A* which weighs 100 lb; to the other end of the rope is attached a body *B* which weighs 160 lb. (*a*) Determine the acceleration of *A* and *B* if the system is allowed to move under the action of gravity. (*b*) What downward force applied to *B* would cause an acceleration of 15 ft/sec²? (*c*) What additional weight, attached to *B*, would cause an acceleration of 15 ft/sec²? *Ans.* (*c*) 114 *lb.*

Art. 92. Force Criterion for Rectilinear Translation

1. *B*, Fig. 167, is a straight post which rests upon the front edge of the car *A*, at an inclination of 60° to the horizontal. With what acceleration must *A* be made to move along a level track in order that *B* will remain in the position shown? *Ans.* $a = 18.6\ ft/sec^2$.

2. Assume that the car of the preceding problem moves up a 30° incline with a uniform acceleration of 12 ft/sec². At what angle with the vertical must the post be inclined in order that, as before, it may maintain its position? *Ans.* 15.5°.

3. Figure 168 represents a box *A* in which a straight uniform bar *B* is laid in the position shown. (*a*) If the box is smooth, what acceleration must be given it to prevent the bar from slipping? (*b*) If the bar weighs 12 lb, the box 30 lb, μ between bar and box = 0.1, and μ between box and floor = 0.3, determine the maximum and minimum values of the applied horizontal force *P* that will allow the bar to stay in the position shown relative to the box. *Ans.* (*a*) 55.8 *ft/sec²*; (*b*) 137.6 *lb and* 60.3 *lb.*

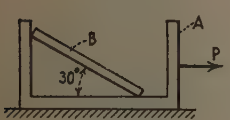

FIG. 168.

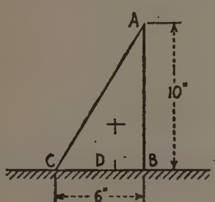

FIG. 169.

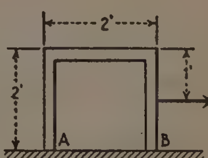

FIG. 170.

4. Figure 169 represents a wedge-shaped block (*ABC*) resting on a horizontal floor. Let *R* denote the resultant pressure of the floor on the base of the block. The intersection *D* of the line of action of *R* and the base is directly below the center of gravity of the block; hence *D* is 2 in. from *B*. If an increasing horizontal pull *P* is applied at *A* toward the right, then the intersection shifts toward *B* and eventually reaches *B*. Then the block is about to tip. What is the least value of *P* that would tip the block: (*a*) if there is no friction; (*b*) if there is friction and the coefficient is 0.1? *Ans.* (*a*) 0.3*W*; (*b*) 0.25*W*.

5. A homogeneous cylinder weighing 100 lb is drawn up an inclined plane by a force of 200 lb acting parallel to the plane. The cylinder is 4 ft long and 2 ft in diameter; it rests on end with its length normal to the plane. The coefficient of friction between cylinder and plane is 0.2; the angle of inclination of the plane is 30° to the horizontal. Determine the limits between which the point of application of the 200-lb force must lie in order that the cylinder may not tip over. *Ans.* 0.26 *ft above G*, 0.61 *ft below.*

6. Figure 170 represents an open box or trough made by bending a uniform plate twice at right angles. The trough rests as shown on a horizontal floor. At edge *A* the coefficient of kinetic friction is 0.1; edge *B* is roughened so that there the coefficient of friction is 0.6. A

horizontal force of 20 lb is applied as shown midway between the ends of the trough. If the trough weighs 30 lb what is the resulting acceleration?

7. A certain automobile has a wheel base of 100 in.; its center of gravity is 40 in. in front of the rear axle and 34 in. above the ground level. It has brakes on the rear wheels only. The engine can spin the wheels; the brakes can lock the wheels. If the coefficient of friction, static and kinetic alike, between tires and pavement is 0.2, and rolling resistance is negligible, determine: (*a*) the maximum possible starting acceleration, (*b*) the maximum possible stopping acceleration.

8. Figure 171 represents a box, to the bottom of which, at A, a uniform rod 3 ft long is pinned. The rod projects 1 ft through a smooth hole in the top of the box at B. Box and rod are allowed to slide down the inclined plane, the coefficient of friction between the plane and the box being 0.3. Determine all forces on the rod, which weighs 10 lb.

9. Figure 172 represents a block A, to the upper edge of which is pinned a straight uniform bar B. A weighs 120 lb; B weighs 80 lb; the coefficient of friction between A and the floor is 0.3. (*a*) What acceleration would be given the system by a force P of 100 lb if the coefficient of friction between B and the floor is 0.3? (*b*) What acceleration would be given the system by a force P of 100 lb if the coefficient of friction between B and the floor is 0.4? (*c*) Solve (*a*) and (*b*) assuming P to be a pull to the left of 100 lb. *Ans.* (*a*) 6.44; (*b*) 5.45; (*c*) 6.44 *and* 6.00 *ft/sec²*.

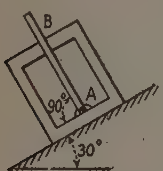

FIG. 171.

FIG. 172.

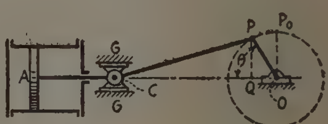

FIG. 173.

10. Determine the magnitude and point of application of a horizontal force that will make a uniform bar 6 ft long weighing 60 lb slide on end along a rough plane (coefficient of friction 0.3) with an acceleration of 80 ft/sec², while inclined at 40° to the horizontal: (*a*) when the rod slopes in the direction of the acceleration; (*b*) when the rod slopes in the opposite direction. *Ans.* (*a*) *Force of* 167 *lb*, 1.61 *ft from c.g. and below it.*

11. Consider the rod of Prob. 10. What force applied at the upper end of the bar will produce the specified motion? Answer for both (*a*) and (*b*) *Ans.* (*b*) *Force P having components* $P_x = 186$, $P_y = 63.4$ *lb.*

12. " A locomotive weighs 380,000 lb. Its cab is rigidly mounted on the truck side frames. It is designed to withstand a maximum end bump of 1,000,000 lb. A transformer mounted in this locomotive cab weighs 22,000 lb, and its center of gravity is 28 in. above and midway between its supporting feet. It is to be held down by a total of four ordinary clearance bolts at the lower corners of the transformer tank. The bolt centers will be 40 in. along the length of the locomotive. Assuming a working tensile stress of 9000 lb/in.² in the bolts, what size bolts should be used to hold the transformer?" (Westinghouse Electric and Manufacturing Co., Problem Service for Technical Schools.)

13. Answer and discuss the following questions: (*a*) How should a straight uniform rod be set upon a smooth inclined plane in order that it will slide down the plane on end without falling over? (Assume air resistance negligible.) (*b*) What difference, if any, would air resistance make? (*c*) What difference, if any, would friction make? (*d*) To what extent do the answers to (*a*), (*b*), and (*c*) apply to a skier sliding down hill, and what is the essential difference? (*e*) If a long slender block stands on a smooth horizontal floor, where should a horizontal force be applied to the block in order to slide it along the floor without overturning it?

(*f*) Answer (*e*) assuming the floor to be rough. (*g*) Why does a passenger standing in a bus tend to fall when the bus starts or stops? In which direction does he tend to fall in each case?

General

1. An old-time sporting event was a race between a mounted man and a man on foot; the quicker starting of the runner offset the greater speed of the horse and often enabled the runner to win a short race. Assume that an athlete can start with an acceleration of 30 ft/sec² which diminishes linearly with time, becoming zero when a velocity of 30 ft/sec is attained. Assume that the horse can start with an acceleration of 18 ft/sec² which diminishes linearly with time, becoming zero when a velocity of 45 ft/sec is attained. What is the longest race the man could win?

2. A mounted messenger at the rear of an infantry column 25 miles long rides to the head of the column to deliver a message, and returns immediately. When he gets back to the rear of the column it is at the spot where the head of the column was when the messenger started. Assuming the column to have marched continuously at 4 mi/hr and the messenger to have ridden at a uniform speed v, find v, the distance ridden by the messenger, and the time required for the whole trip.

3. Suppose that for a motion simultaneous values of s and v are known for a number of successive instants. If such data are plotted in the form of a v-s curve, with values of v as ordinates, the subnormal (projection on the s axis of the normal to the curve) for any value of s represents, to some scale, the acceleration at the corresponding instant. Prove that this is so, and ascertain the scale by which the length of the subnormal is interpreted in terms of acceleration.

4. In testing automatic safety cushions placed at the bottom of elevator shafts in the Woolworth Building, New York, the following data were obtained (*Engineering Record* for Sept. 5, 1914):

Distance from top of air cushion	0	20	40	60	80	100	120	130	135	137 ft
Pressure on bottom of car	4	4	7	10	12	9	9	9	8	0 lb/in²
Downward velocity of elevator	168	168	157	140	116	92	67	52	32	0 ft/sec

Plot the distance-velocity curve, and, by means of the relationship proved in the preceding problem, find the acceleration when the elevator had fallen 70 ft in the cushion. Check your result by computing the acceleration due to the forces acting on the elevator, taking the weight of the elevator as 7500 lb.

5. Data such as those given for the preceding problem can also be used to determine the time interval Δt corresponding to any given Δs. Since $v = ds/dt$, $dt = ds/v$ and $\Delta t = \int (1/v)\, ds$. Therefore, if a curve is plotted with values of s as abscissas and corresponding values of $1/v$ as ordinates, the area under this curve corresponding to any Δs represents the corresponding Δt. Making use of this relation, determine from the data of Prob. 4 the time required for the elevator to fall 70 ft into the air cushion.

6. For an ordinary automobile the wind resistance D is given by the equation $D = KAv^2$ and the rolling resistance R by the equation $R = 0.012W$, where D and R are in pounds, K is a coefficient that depends upon the shape of the car, A is the frontal area of the car in square feet, v is the velocity in miles per hour, and W is the weight of the car in pounds. For a certain sedan, $K = 0.0012$, $A = 29$, and $W = 3600$. How long will it take this car to come to rest from a velocity of 80 mi/hr if allowed to coast freely on a level road? (Hint: Since $a = dv/dt$, $dt = 1/a\, dv$. If a is expressed in terms of v, integration of the expression for dt between appropriate limits will give Δt.

7. Figure 173 represents a crank and connecting-rod mechanism such as is used in an

ordinary steam engine. OP is the crank, mounted on its shaft at O; PC is the connecting rod; C is the cross head; A is a piston; and AC, the piston rod. When the crank is rotated the cross head is constrained to move in a straight line by the guides G. Let r = the length of crank, L = length of connecting rod, $c = r/L$, n = number of revolutions of the crank per unit time (assumed constant), ω = angular velocity of the crank ($\omega = 2\pi n$), s = varying distance of the cross head from its leftmost position, θ = the "crank angle" QOP, and t = time required for the crank to describe the angle θ ($= \omega t = 2\pi nt$). Obviously, there is a definite relation between s and θ (or t). Express this relationship by an equation, and then by differentiation show that the velocity of the cross head is given by

$$v = r\omega\left(\sin\theta + \frac{c\sin 2\theta}{2(1 - c^2\sin^2\theta)^{\frac{1}{2}}}\right)$$

and its acceleration by

$$a = r\omega^2\left(\cos\theta + \frac{c\cos 2\theta + c^3\sin^4\theta}{(1 - c^2\sin^2\theta)^{\frac{3}{2}}}\right)$$

8. The apparatus described in Prob. 7 is being used to compress air. The piston is 10 in. in diameter and weighs 40 lb; the lengths of crank and connecting rod are 3 and 10 in. respectively; the crank rotates at 100 rev/min; the pressure on the left face of piston is 50 lb/in.^{2} Find the stress in the piston rod when $\theta = 40°$.

9. In Ex. 1 of Art. 85 it was assumed that the elevator car started upward with an acceleration that increased linearly with respect to time from 0 to its maximum value. The passengers suffer less discomfort if the acceleration is made to increase in accordance with the equation $da/dt = -6\sin(6t - \pi/2)$. If the acceleration is made to increase from 0 in this way, how long will it take for the velocity to become equal to $\frac{1}{3}$ ft/sec? What is the acceleration at the end of that time? How far will the car have moved by the end of that time?

10. A body weighing 10 lb falls a distance of 5 ft upon the top of a spiral spring. The spring is of such stiffness as to be compressed 1 in. by each additional 100 lb of load. (*a*) Determine the velocity of the body after it has compressed the spring 3 in. (*b*) Determine the total compression that the body will cause in the spring.

11. A box car stands on an east-west track. A straight uniform bar rests with its lower end on the bottom and 4 ft from the west end of the car, and its upper end against the west end of the car, 6 ft above the floor and 3 ft north of the lower end. The coefficient of friction is 0.2 for each point of contact. (*a*) Will the bar stay in place when the car is stationary? (*b*) If not, what is the least eastward acceleration (of the car) that will prevent slipping? (*c*) If so, what is the greatest eastward acceleration of the car that is possible without slip? (*d*) What is the greatest westward acceleration that is possible without slip?

CHAPTER IX

Art. 95. Position; Speed; Velocity

1. A particle moves in a circular path 12 ft in diameter in a vertical plane; it starts from an origin at the right end of the horizontal diameter and moves so that s (measured positive counterclockwise from the origin) is given by $s = 8t - \frac{1}{2}t^2$, the units being feet and seconds. (*a*) Determine the velocity of this point at the instant $t = 4$. (*b*) When does the point return to the origin for the first time? What is its velocity then? (*c*) When does the point return to the origin for the second time?

2. Suppose that the radius of the circle described in Prob. 1 is 20 ft, and that the particle starts at the right end of the horizontal diameter as before and moves counterclockwise in accordance with the equation $s = 2t^2$, the units being feet and seconds. Determine the velocity of the particle: (*a*) when $t = 3$, (*b*) when $s = 40$, (*c*) when the particle reaches the top of the circle for the first time. *Ans.* (*a*) $V = 12$ *ft/sec up and to the left at* 38.4° *to the horizontal.*

Art. 96. Axial Components of Velocity

1. For the motion described in Prob. 1 of Art. 95, take the horizontal and vertical diameters as x and y axes respectively and derive expressions for v_x and v_y: (*a*) by getting expressions for x and y in terms of time and differentiating, and (*b*) by resolving V, as expressed in terms of t, into x and y components. Check by computing V at the instant $t = 4$ and comparing with the value found in Prob. 1 of Art. 95.

2. For the motion described in Prob. 2 of Art. 95, derive expressions for v_x and v_y, and determine the velocity by means of its x and y components when $t = 3$.

3. A particle which is at the origin when $t = 0$ moves with a constant v_x and with $v_y = 6t^2$, the units being the foot and second. What must v_x be if the particle is to pass through the point $x = 20$, $y = 16$? What is the velocity of the particle when it passes through the point described? *Ans.* $v_x = 10$ ft/sec.

4. A particle moves according to the equations $x = 3t^2$, $v_y = 4t^2$, the units being the foot and second. When $t = 0$, $y = 0$. Where is the particle when $t = 4$? What is its velocity then? How far has it traveled then? What is the equation of its path?

5. The point Q, Fig. 174, on the rim of a wheel rolling in a straight line describes a curve known as cycloid. Let v' = velocity of the center of the wheel C, a' = the acceleration of C, and R = radius of the wheel. Find formulas for the x and y components of the velocity of Q when in the position shown. [Let s = the abscissa of C, and x and y = the coordinates of Q. Then $x = s - R\sin\theta$, and $y = R(1 - \cos\theta)$; also $s = R\theta$.]

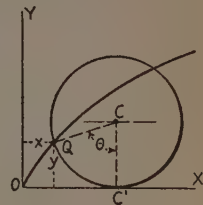

FIG. 174.

Art. 100. Axial Components of Acceleration

1. A particle P moves according to the equations $x = 4t^2 + 10$, $y = 2t^3$, foot and second units. (*a*) Derive general formulas for v_x, v_y, a_x, and a_y. (*b*) For $t = 2$ determine the position, velocity, and acceleration of P. (*c*) Plot the position, and represent V and A by means of arrows drawn through the point. *Ans.* (*b*) $V = 28.8$ *ft/sec at* 56° 20′; $A = 25.3$ *ft/sec*² *at* 71° 30′.

2. With reference to an origin O, a vertical axis OY, and a horizontal axis OX, a particle starts from rest on the x axis and moves according to the equations $x = 10 + 4t^2$ and $a_y = 3t$, the units being the foot and second. (*a*) Determine the position, velocity, and acceleration of the particle for the instant $t = 3$. Represent the results of your solution on a sketch. *Ans.* (*a*) $x = 46$; $y = 13.5$; $V = 27.6$ *ft/sec;* $A = 12$ *ft/sec.*²

3. A particle moves so that $v_x = 4t^3$ and $a_y = 6t$, foot and second units. At $t = 0$, P is at $x = 10$ and $y = 0$; and v_y is 4. (*a*) Derive general formulas for x, y, a_x, and v_y. (*b*) For $t = 2$, determine the position, velocity, and acceleration of P. (*c*) Plot the position, and represent V and A by arrows through the position. *Ans.* (*b*) $V = 35.75$ *ft/sec at* 26° 34′; $A = 49.45$ *ft/sec*² *at* 14°.

4. A particle starts from rest at the origin and moves so that $v_y = 4t^2$. (*a*) What constant x acceleration must it have to pass through the point $y = 36$, $x = 45$? (*b*) When will it pass through that point? (*c*) What will its velocity be at that instant? (*d*) What will its acceleration be at that instant? *Ans.* (*d*) $A = 26$ *ft/sec*² *at* 67° 23′.

5. A particle moves with a constant v_x and a constant a_y. When $t = 0$, $x = 0$, $y = 0$, $v_y = 0$, and $v_x = 30$ ft/sec. What must a_y be if, when $t = 4$ sec, $V = 50$ ft/sec? What is the direction of V then? Where is the particle then?

6. For the motion discussed in Prob. 1 of Art. 95 and Prob. 1 of Art. 96, derive expressions for a_x and a_y. Then determine the acceleration for the instant $t = 4$.

7. For the motion discussed in Prob. 2 of Art. 95 and Prob. 2 of Art. 96, derive expressions for a_x and a_y, and determine the acceleration by means of its x and y components when $t = 3$.

8. For the motion of Q, Prob. 5 of Art. 96, derive expressions for a_x and a_y.

9. A particle moves along the curve $y = 0.5x^2$ with a constant speed of 10 ft/sec. Determine the x and y components of the acceleration when $x = 2$. *Ans.* $a_x = -8$ *ft/sec²*; $a_y = +4$ *ft/sec²*.

10. A particle starts from rest at the origin and moves in the xy plane along the path $x = \frac{1}{6}y^2$ in such a way that $a_x = 18t^2$, the units being the foot and the second. Completely determine the position, velocity, and acceleration of the point for the instant $t = 2$ sec. Represent your results by means of a sketch showing the axes, path, and appropriate vectors.

11. A particle moves with a constant speed of 10 ft/sec along a path whose equation is $y = 0.5x^2$ Determine a_x and a_y when $x = 2$ ft. Then determine A from a_x and a_y.

Art. 101. Tangential and Normal Components of Acceleration

1. For the motion described in Prob. 1 of Art. 95, determine the acceleration for the instant $t = 4$. Compare with the result obtained in Prob. 6 of Art. 100, where A was found by means of its x and y components.

2. For the motion described in Prob. 2 of Art. 95, derive expressions for a_t and a_n, and determine the acceleration by means of its tangential and normal components when $t = 3$.

3. A point starts at the right end of the horizontal diameter of a 60-ft circle and moves according to the equation $s = 4t^3 + 3t^2$, foot and second units. Determine a_t and a_n for $t = 2$ sec. *Ans.* $a_t = 54$ *ft/sec²*; $a_n = 120$ *ft/sec²*.

4. A point starts from rest at the right end of the horizontal diameter and travels counterclockwise around a circle of 5-ft radius in accordance with the equation $a_t = 2t$, foot and second units. (*a*) Determine V and A for $t = 3$. (*b*) Derive an expression for a_x in terms of t. *Ans.* (*a*) $V = 9$ *ft/sec down and left at* 13.2°; $A = 17.27$ *ft/sec²*.

5. A particle starts from rest at the right end of the horizontal diameter of a circle with radius = 8 ft and moves counterclockwise around the circle according to the law $a_n = 2t^2$ (units are the foot and the second). Determine the velocity and the acceleration of the particle when it first reaches the left end of the horizontal diameter.

6. A point starts from rest and travels once around a circle of 20-ft diameter with uniformly increasing speed. What is the shortest time in which the point can go around the circle without having at any time an acceleration of more than 100 ft/sec². *Ans.* 3.98 *sec.*

7. A particle moves in the xy plane in accordance with the equations $v_x = 20$, $a_y = 2$. When $t = 0$, $x = 0$, $y = 0$, and $v_y = 10$. Determine, for the instant $t = 2$, the acceleration and velocity of the particle. Then determine the radius of curvature of the path at the then position of the particle and the rate at which the speed of the particle is then changing. Show the approximate form of the path at the then position of the particle. Also determine the distance traveled by the particle during the interval $t = 0$ to $t = 2$.

8. A point starts from rest at $x = 4$, $y = 0$, and moves according to the equations $a_x = \frac{1}{2}t$, $y = 2t^2$, units being the foot and the second. Determine (*a*) the position of the point when $t = 4$; (*b*) the velocity of the point then; (*c*) the acceleration of the point then; (*d*) the rate at which the speed of the point is changing then; (*e*) the radius of curvature of the path at the place where the moving particle is. Represent the answers to (*a*), (*b*), and (*c*) on a sketch. *Ans.* (*a*) $x = 9\frac{1}{3}$, $y = 32$; (*b*) $V = 16.5$ *ft/sec up and to right at* 76° *to horiz.* (*c*) $A = 4.48$ *ft/sec² up and to right at* 63.4° *to horiz.* (*d*) 4.37 *ft/sec²*.

9. A particle is moving at a constant speed of 2 in./sec along an ellipse whose axes are 6 and 2 in. long. Determine the amount and direction of the acceleration of the particle at the instant it is passing one end of the longer axis. *Ans.* $A = a_n = 12$ *in./sec²*.

10. Does the acceleration of a point moving with uniform speed along a curved path depend on the direction in which the point is moving? Upon what does it depend? Upon what does it depend if the point is moving with varying speed? Could a point move along a curved path in such a way as to have zero acceleration? What are the limits as to direction of acceleration in curvilinear motion? Could a point on a curved path have, at a given instant, acceleration but no velocity?

Art. 102. Average Speed; Average Velocity; Average Acceleration

1. At 10 A.M. a boat is traveling northeast at 20 mi/hr. At 10:30 A.M. the boat is 12 miles southeast of its former position and is traveling due west at 28 mi/hr. For the half-hour interval, what are: (a) the displacement, (b) the average velocity, (c) the increment in velocity, (d) the average acceleration? Can you tell from the data given anything about distance traveled, average speed, or maximum speed? *Ans.* (c) $\Delta V = 44.4$ *mi/hr south* $71\frac{1}{2}°$ *west.*

2. In Art. 82 it is explained how the acceleration of a particle in rectilinear motion can be found by estimating the limit of the average acceleration. Acceleration in curvilinear motion can be estimated in the same way. For the motion described in Prob. 2 of Art. 95, determine the velocities at the instants $t = 0, 1, 1.5, 2, 2.5, 3$, and 3.5, and draw the hodograph for the interval 0 to 3.5. Then determine the value of the average accelerations for the intervals 1 to 3, 1.5 to 3, 2 to 3, and 2.5 to 3. Then from these average accelerations estimate the magnitude of the acceleration for the instant $t = 3$, and from the tangent to the hodograph determine its direction. Compare the results thus arrived at with the results obtained in previous solutions of this problem.

3. For the motion described in Prob. 1 of Art. 95, determine the velocity at $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}$, and 2. Construct the hodograph, and ascertain as nearly as possible the acceleration of the point when $t = 2$.

4. A particle starts from the right end of the horizontal diameter and travels counterclockwise around a circle of 25-ft radius in accordance with the equation $s = 0.1t^3$, s being the distance in feet measured along the path from the right end of the horizontal diameter. (a) Determine the position and velocity of the particle for each of the instants $t = 0, 1, 2, 3, 4, 5, 6, 7, 8$. (b) Construct the hodograph of the motion for the interval $t = 0$ to $t = 8$. (c) Determine the acceleration of the particle for the instant $t = 6$ by the same method as used in the preceding problem, and then check your results by computing A from its calculated normal and tangential components. *Ans.* (c) $A = 5.88$ *ft/sec² down and left at* 12° *to horiz.*

Art. 103. Simple Harmonic Motion

1. A particle Q describes a simple harmonic motion; the frequency = 100 (to-and-fro) oscillations per minute, and the amplitude = 3 ft. Determine the average accelerations of the particle for the following distances traversed: first 6 in. from one end of its path; second 6 in.; third 6 in.; and first 18 in. *Ans. For first* 6 *in.*, 310 *ft/sec.²*

2. The period of a certain simple harmonic motion is 8 sec, and the amplitude is 6 in. What is the maximum velocity? the maximum acceleration? For the motion from one extreme point in the path to the center, what is the average velocity? the average acceleration?

3. Four particles, Q_1, Q_2, Q_3, and Q_4, are describing simple harmonic motions in AB, Fig. 175; the period of each motion is 8 sec. At a certain instant the four particles are at points 1, 2, 3, and 4, respectively; Q_1 and Q_3 are moving toward the right, and Q_2 and Q_4 are moving toward the left. Write the expressions for the x coordinates of the moving points t sec after the instant mentioned. (AB = 12 in. and is divided into sixths by the points.)

Art. 104. Force-Acceleration Relationship for a Particle in Curvilinear Motion

1. A box weighing 1000 lb rests on the floor of a truck which slows down at the rate of 100 mi/hr/min while rounding a (flat) curve of 600-ft radius. Determine the reaction of the truck on the box at the instant the truck is going 40 mi/hr. *Ans. Components of R are* 75.8 *lb;* 178 *lb;* 1000 lb.

2. A highway curve has a radius of 600 ft and a transverse slope (upward and outward) of 1 in./ft. If $\mu = 0.12$, what is the maximum speed in miles per hour at which a car can round the curve without side slipping?

3. At the crest of a short steep hill a concrete highway is, in longitudinal section, a circular arc. What is the least radius of curvature of this arc if cars traveling 70 mi/hr are not to lose

contact with the pavement? (Consider the car as a rigid body; i.e., neglect effect of springs. Then explain what difference the springs might make.) *Ans.* 327 *ft.*

4. Figure 176 represents a rigid arm, mounted on a horizontal axle so as to permit rotation in the vertical plane, and having at its end a flat ring. In this ring is placed a smooth sphere weighing 12 lb and of such diameter as to just fit loosely in the ring. Determine all forces acting on the sphere when the system is made to rotate uniformly at 100 rev/min, for each of the extreme positions at top, left, bottom, and right of path. Explain how the pressure of the ring against the sphere varies in magnitude and direction during one complete revolution of the system.

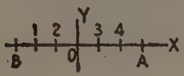

FIG. 175.

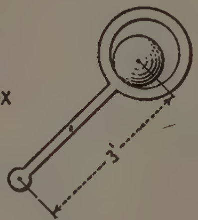

FIG. 176.

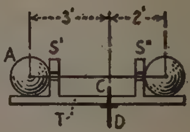

FIG. 177.

5. T, Fig. 177, is a horizontal whirling table. A and B are spheres connected by an elastic cord, the tension in which is 30 lb when the table is at rest. A weighs 10 lb and B weighs 40 lb. What are the pressures of the stops S' and S'' against the spheres when the table is rotated about CD at 20 rev/min? *Ans.* $S' = 25.9$ *lb;* $S'' = 19.1$ *lb.*

6. A shot putter releases the shot at a height of 7 ft above the ground; when released the shot has a velocity of 34 ft/sec directed 40° above the horizontal. How far from the point of release will it strike the ground?

7. A disk is so mounted that it can be made to rotate in a plane inclined at 20° to the horizontal. If a small body is placed on the disk, at a distance of 15 in. from the axis of rotation, what is the maximum uniform speed at which the disk can rotate without causing the body to slip? The coefficient of friction between disk and body is 0.6.

8. Figure 178 represents a hollow cone that can be made to rotate about its own axis. A small object, A, weighing 3 lb, is placed in the cone where its radius is 2 ft and the cone is made to rotate uniformly at 100 rev/min; it is assumed that the friction is sufficient to prevent A from slipping. Determine the normal pressure (normal to the surface of the cone) and the friction on A for each of the positions 1, 2, and 3 indicated. *Ans. Position* 1, $N = 16.82$ *lb;* $F = 5.03$ *lb.*

9. Determine how much the apparent weight of a body (as measured with a spring balance) is diminished by the rotation of the earth at (*a*) the equator and (*b*) latitude 45°. For the purposes of this problem the earth may be assumed to be a sphere with a radius of 4000 miles. *Ans.* (*a*) 0.347%; (*b*) 0.17%.

10. Imagine a wedge-shaped block laid upon a sloping plane, the inclination of which is such as to make the top of the block horizontal. Describe the motion of a particle (of negligible weight) placed on top of the block when the block is allowed to slide down the plane. (The plane and the top of the block are both smooth.)

11. Referring to the preceding problem, assume that the particle weighs 20 lb, the block 40 lb; the coefficient of friction between the particle and the block (static and kinetic) is 0.1, and between the block and the plane 0.2. The plane slopes at 30°. Determine the acceleration of the block and of the particle when the system is released.

Art. 105. Curvilinear Translation

1. A straight uniform bar 12 ft long weighing 60 lb is suspended in a horizontal position by two vertical ropes each 10 ft long. One rope is attached to the bar at 4 ft from the left end; the other is attached to the right end. The bar is raised to such a height that when released and allowed to swing freely it will have, when in its lowest position, a velocity of 16 ft/sec. Determine the tension in each rope when the bar is in this position. *Ans.* 80.7; 26.9 *lb.*

2. Figure 179 represents a uniform bar bent at right angles at midlength. The bar weighs 20 lb. It is suspended from a horizontal ceiling by two cords A and B of equal length and made to swing back and forth in the plane of the figure. The angle of swing is such that when the bar is in its lowest position its speed is 8 ft/sec. Determine the tension in each of the cords at that instant. For what position of the bar (while swinging) is the tension in cord A zero? cord B? Could the cords be attached to the bar at such points as to make the tensions in the two cords equal?

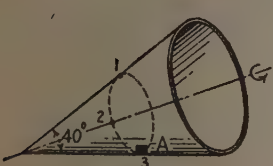

FIG. 178.

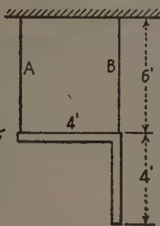

FIG. 179.

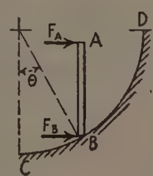

FIG. 180.

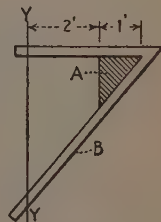

FIG. 181.

3. The uniform bar AB, Fig. 180, of length L and weight W is made to slide with uniform speed v up the curved track CD (quadrant of a circle) by two horizontal forces F_A and F_B. Derive expressions for F_A and F_B in terms of W, L, v, R, and the angle θ. Can the bar be made to slide all the way to D by horizontal forces F_A, F_B?

4. (*a*) A man throws a uniform straight rod, as though it were a javelin, but in throwing it he exerts only a propelling force along the axis of the rod. How does the rod move through the air, i.e., what are its successive positions, and in what position will it strike the ground? How can the thrower modify this behavior? (*b*) A skier slides down a hill, his knees and body straight. In passing onto a flatter slope, which way does he tend to fall, and how does he prevent it? What if he passes onto a steeper slope? (*c*) A uniform bar, suspended by a cord of about the same length attached to one end, is raised somewhat and released, so as to swing. When released, bar and cord are in a straight line. Will they remain so while swinging?

Art. 107. Typical Problems; Examples

1. A wedge-shaped block A is placed in the bend of a rod B, as shown in Fig. 181, and the system is made to rotate uniformly about the vertical axis Y–Y at 90 rev/min. A weighs 12 lb; the surfaces of contact are smooth. Angle of A is 60°. Determine the pressures exerted on A by B. *Ans.* 32.6 *lb down on top;* 89.1 *inward on sloping side.*

2. Suppose that the horizontal part of the rod in Fig. 181 is removed and that μ between the block and the rod is 0.3. Determine the maximum and minimum angular velocity with which the system can rotate without the block's slipping.

3. A cylinder C, Fig. 182, is suspended by a cord and rests against a smooth inclined plane P as shown. The cylinder weighs 20 lb; its diameter is 1 ft. The plane is rotated at 30 rev/min about the vertical axis AB. Determine the tension in the cord and the pressure against the plane. *Ans.* *Tension* = 21.2 *lb.*

4. CD, Fig. 183, is a vertical axis about which E can be rotated. A is a body resting on E, and B is suspended by means of a cord fastened to A as shown. A weighs 10 lb and B weighs 20 lb. Suppose that E makes 30 rev/min; then compute the pressure at the stop S. The centers of A and B are 5 and 3 ft from CD respectively. (Neglect friction under A, at B, and at the pulley axle.)

5. Suppose that A and B in Prob. 4 are rough, the coefficient of static friction being $\frac{1}{4}$ for each. What rate of rotation would lift B?

6. AB, Fig. 184, is a board lying upon a table. C is a vertical peg in the table top projecting upward through a suitable hole in the board. The board weighs 20 lb. $AC = 8$ ft, $CB = 3$ ft, $DC = 2$ ft. The table top (and board) are spun about C at 400 rev/min. Determine the stress at the smallest section of the board. *Ans.* 3160 *lb.*

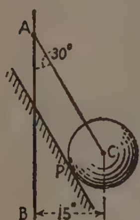

Fig. 182.

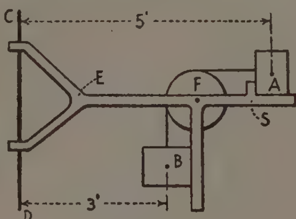

Fig. 183.

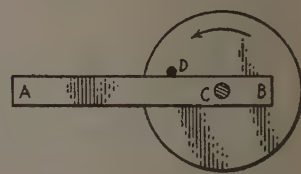

Fig. 184.

7. Two spheres, A and B, are connected by a rubber band. A weighs 3 lb; B weighs 2 lb. The band is stretched until the tension in it is 10 lb. The system is then released in space, with the 3-lb sphere vertically above the 2-lb sphere. Determine the acceleration of each sphere at the instant of release, and the acceleration of their center of gravity. *Ans.* -139.5, $+128.8$, -32.2 *ft/sec*^{2}.

8. A circular hoop 5 ft in diameter is made of uniform wire weighing 0.5 lb/ft. This hoop, supported on a smooth horizontal table top, is made to rotate about a vertical axis through its center at 2000 rev/min. Determine the resulting tension in the hoop. (Do this in three ways, first by taking as the body under consideration half the hoop, then by taking an elementary length of the hoop dL then by taking any finite length ΔL.)

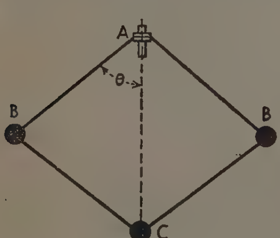

Fig. 185.

Fig. 186.

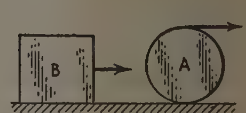

Fig. 187.

9. A slender uniform bar weighing 12 lb is bent into the form of a semicircle of 3-ft radius and is mounted on a horizontal axle through one end so that it can be made to rotate in its own (vertical) plane. It is made to rotate uniformly at 40 rev/min. In what position is the bar when it exerts the maximum force on the axle, and what is this force? *Ans. Force* = 35.25 *lb.*

10. Figure 185 represents a device consisting of four links hinged to the collar A and to the weights B, B, and C. The system rotates about the vertical axis indicated at a uniform rate

of 60 rev/min. The links are each 4 ft long and of comparatively negligible weight; B and B each weigh 60 lb, C weighs 40 lb. Determine the angle θ and the tension in each of the links. *Ans.* $\theta = 70° 9'$.

11. Suppose that the surface of contact between the two blocks of Fig. 161 is quite smooth, and that the floor is also smooth. What will happen if the blocks are simply arranged as shown and released? What would be the acceleration of each block?

12. The two unconnected blocks A and B, Fig. 186, may be collectively considered a nonrigid body. By applying the principle of motion of the mass-center to this nonrigid body, determine the acceleration of the body A due to the force P shown. The floor is horizontal and smooth; A weighs 100 lb, B weighs 200 lb, and $P = 80$ lb.

13. In Fig. 187, A is a cylinder and B a block; they have exactly the same weight, and both rest on a perfectly smooth horizontal floor. Suppose equal horizontal forces P_1 and P_2 to be applied to these bodies as shown. Which center of gravity will have the greater acceleration?

14. A large, light, airtight box, in which a bird is confined, is supported by a sensitive spring balance. Does the reading of the balance change when the bird flies about in the box? If so, explain in what manner, under what circumstances, and why?

15. (*a*) If the center of gravity of a locomotive driver is not on the axis of the axle, does the pressure which that driver exerts on the track vary during a revolution? (*b*) Does the reaction of the driver on the axle vary?

16. State precisely what answer the "principle of motion of the mass-center" enables you to give to each of the following questions: (*a*) A uniform straight slender rod is set endwise upon a perfectly smooth sloping plane. In what manner will the rod fall, and where will it be, relative to its initial position, when it strikes the floor? (*b*) The rod lies upon a smooth horizontal floor, its axis pointing north and south. A horizontal force acting due east is applied to the north end of the bar. What can you say as to the resulting motion of the bar? (*c*) What if a clockwise couple, consisting of two horizontal forces very close together, is applied near the north end of the bar? (*d*) What if the rod is replaced by a flexible rope or chain?

17. (*a*) Is it theoretically possible to give a symmetrical projectile so high a velocity that it will travel, for some distance, horizontally? (*b*) Would it be possible for a baseball pitcher to throw a curve through a tunnel, or chamber, from which all air had been exhausted? (*c*) Why is it that when dynamite is exploded in contact with a body it can, even though unconfined, exert a great force against that body? Could it exert such a force if exploded in a vacuum? (*d*) It has been proposed to fire a rocket to some other planet, or to the moon. Is this theoretically possible? Could the rocket be made to move with increasing velocity after it had passed beyond the earth's atmosphere? Could it be retarded as it approached the other planet?

General

1. At velocities considerably less than the velocity of sound, the resistance offered by the air to a moving body varies very nearly as the square of the velocity. It has been stated that because of this fact a parachutist who jumps from a rapidly moving plane will fall less rapidly (lose less altitude in a given time) than one who jumps from a slowly moving plane or from a stationary balloon. (The motion under consideration is that which occurs before the parachute opens.) Do you think that this statement is true? If so, explain why.

2. The following terms, used in aerodynamics, have the meanings given: *Relative wind* is the motion of the air relative to the plane; for example, if a plane is gliding downward and westward at 60 mi/hr, the relative wind is upward and eastward at 60 mi/hr. *Lift* is the force exerted by the air on the plane normal to the direction of the relative wind; thus for a plane in level flight the lift is vertically upward and equals the weight of the plane. *Drag* is the force exerted by the air on the plane in the direction of the relative wind. *Angle of attack* is the angle between the plane of the lower wing surface and the relative wind, usually very small.

Lift coefficient is the coefficient C_L in the formula: Lift in pounds $= C_L \times \frac{1}{2}\rho V^2 S$, where $\rho =$ mass density of air (0.002378 slug/ft³), V is the velocity of the relative wind in feet per second, and S is the area of the wing in square feet. Drag coefficient is the coefficient C_D in the formula: Drag in pounds $= C_D \times \frac{1}{2}\rho V^2 S$.

When a plane in level flight encounters an upward-moving column of air, called "sharp-edged vertical gust," the angle of attack changes and C_L and C_D are suddenly increased. As a result both the lift and the drag are increased, and, since the other forces — the weight of the plane and the pull of the propeller — do not change, the airplane is accelerated.

Assume that a plane weighing 5000 lb and having S = 300 ft² is flying 200 mi/hr when it runs into a vertical gust having a velocity of 30 ft/sec. The value of C_L changes from 0.165 to 0.560, and the value of C_D changes from 0.01 to 0.027. Determine the resulting horizontal and vertical acceleration of the airplane. How many times greater does the total pressure of the seat on the pilot's body become than it was before?

3. The following is an extract from a description of the Sheepshead Bay Motor Racetrack (*Engineering News* for Aug. 19, 1915): "There are two parallel straightaway stretches connected by two turns of 180° each. Each turn consists of a circular arc of about 135° connected by 'spirals' to the straightaway stretches; the radius to the inner edge of the circular track is 850 ft. The outer edges of the circular turns are given a maximum superelevation of 25 ft 6 in., computed for a speed of 96 mi/hr by the common railway formula $C = dv^2/gR$, where $C =$ superelevation for a radial width d, and $R =$ radius of curvature. The width d was taken in 14-ft strips commencing at the inside of the track, and the superelevation was computed for speeds of 40, $52\frac{1}{2}$, 65, $77\frac{1}{2}$, and 96 mi/hr. This gives a cross section theoretically of a parabolic curve." Verify the correctness of the formula, prove the last statement, and show how the formula gives 25 ft 6 in. for the maximum superelevation.

4. In determining the proper superelevation of highways at curves, the following formula is used: $E + F = \dfrac{0.067V^2}{R}$, where $E =$ superelevation, $F =$ coefficient of friction of tires against side slipping, $R =$ radius of curve in feet, and $V =$ speed of automobile in miles per hour. Show that this formula errs slightly on the side of safety. Determine the error if $E = 0.10$ and $F = 0.16$.

5. If a car is moving straight north with a speed of 100 ft/sec, what is the least time in which it can be made to move straight east with the same speed, without giving it at any time an acceleration of more than 12 ft/sec²? (Note that the change could be effected by turning on a circular path of minimum radius without reducing speed, or by slowing down more or less, making a sharper turn, and then speeding up again. The problem is to determine the quickest way, and to calculate the time required when that method is followed.) Calculate the maximum rate of speeding up and the maximum rate of slowing down required by the adopted method, the form of the path followed, and the least radius of turn involved. *Ans. Minimum time,* 11.8 *sec.*

CHAPTER X

Art. 110. Angular Velocity

1. Figure 188 represents a straight bar AB that leans against a vertical wall or partition that can be moved to right or left as viewed. The bar is pinned at A, so that when the wall is moved the bar rotates about that point. Derive a formula for the angular velocity of the bar in terms of the length of the bar L, the angle of inclination θ, and the velocity of the wall v. Evaluate the angular velocity for L = 10 ft, θ = 30° and 60°, and for v = 20 ft/sec to the right and also for v = 20 ft/sec to the left. Pay particular attention to the interpretation of signs. *Ans.* $\omega = 4$ *rad/sec clockwise for* $\theta = 30°$ *and* $v = 20$ *ft/sec right.*

2. Suppose that the wall of Prob. 1, instead of being made to slide, is hinged so that it can

be turned about its lower edge C. Suppose that the wall is made to turn clockwise with uniform angular velocity ω'. Show that the angular velocity ω of the bar is given by $\omega = \omega'[1 - (a \sin \phi/\sqrt{L^2 - a^2 \cos^2 \phi})]$, where a is the distance from A to C and ϕ is the angle the wall makes with the vertical.

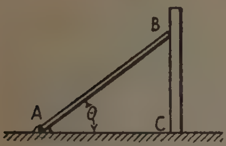

FIG. 188.

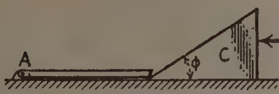

FIG. 189.

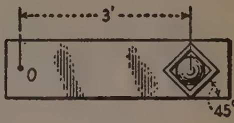

FIG. 190.

3. The slender bar, Fig. 189, of length L is pinned at A and lies on the horizontal floor. It is raised by sliding the wedge C at uniform velocity v under the right end. Date t from the instant the point of the wedge is at the right end of the bar, and derive an expression for the angular velocity the bar would have at any instant after motion started until the bar sloped at the wedge angle ϕ. Determine the numerical value of ω for $L = 10$ ft, $\phi = 30°$, $v = 2$ ft/sec, $t = 3$ sec. *Ans.* $\omega = 0.103$ *rad/sec.*

Art. 111. Angular Acceleration

1. A flywheel starting from rest is given a constant counterclockwise angular acceleration of 3 rad/sec². (*a*) How long will it take for the wheel to make 2 revolutions? (*b*) How fast will it then be rotating? (*c*) How long will it take to stop it if it is retarded at the uniform rate of 5 rad/sec²? *Ans.* (*a*) 2.89 *sec.*

2. A flywheel making 300 rev/min is subjected to a constant retardation of 1 rad/sec². How many revolutions will it make before stopping?

3. A rotating disk starting from rest is subjected to a constant angular acceleration of $\frac{1}{2}$ rad/sec² for the first minute and $\frac{1}{4}$ rad/sec² for the second minute. Find the number of revolutions made at the end of the second minute and the angular velocity at that time.

4. A flywheel starting from rest and subject to a constant acceleration for 3 min makes 6000 revolutions. Find the acceleration, also the velocity at 3 min.

5. A motor acquires its maximum speed, starting from rest, in accordance with the equation $n = 10t - \frac{1}{2}t^2$, where n is in revolutions per second and t is in seconds. Determine the maximum angular acceleration, the maximum angular velocity, and the number of revolutions made before this maximum velocity is attained.

6. (*a*) For the motion of the bar of Prob. 1, Art. 110, derive an expression for the angular acceleration and determine the value of α for the data given. (*b*) Suppose that the wall has an acceleration a. Then derive an expression for α and determine its value for $a = 10$ ft/sec² to the right, the other data being as previously given.

7. (*a*) For the motion of the bar of Prob. 3, Art. 110, derive an expression for the angular acceleration and determine the value of α for the data given. (*b*) Suppose that the block has an acceleration a. Then derive an expression for α and determine its value for $a = 4$ ft/sec² to the left, the other data being as previously given. *Ans.* (*a*) $\alpha = 0.00212$ *rad/sec² clockwise.*

Art. 112. Motion of Any Particle of a Body in Rotation

1. Assume that the wheel of Prob. 1 of Art. 111 has a diameter of 6 ft. Determine the velocity and acceleration of a point on the perimeter when it has completed the first revolution. (Consider the point that was at the upper end of a vertical diameter when motion started.)

2. Determine the velocity and acceleration of the center of gravity of the rod in Prob. 3 of Art. 110 assuming the rod to be uniform and the numerical data to be as given in Prob. 3.

3. A disk is mounted on a vertical shaft so that it can be made to rotate in a horizontal plane. A small block is placed on the disk 18 in. from the axis of rotation, and the disk, starting from rest, is given an angular acceleration of 2 rad/sec². If the coefficient of friction between the block and the table is 0.3, how long will it be before the block slips on the table? *Ans.* *1.24 sec.*

4. Figure 190 shows a top view of a horizontal plank, mounted so that it can rotate about a vertical axis through O, and carrying near its outer end a square box in which a smooth sphere fits loosely. The sphere weighs 20 lb. Determine the forces exerted on the sphere by the sides and bottom of the box when the angular velocity of the plank is 30 rev/min and is increasing at the rate of 20 rev/min/sec.

Art. 114. Typical Problems; Examples

1. A cylinder is mounted on a horizontal axle so that it can rotate freely. The cylinder weighs 200 lb; it has a diameter of 3 ft and a radius of gyration of 15 in. (*a*) A couple of 300 in.-lb, in a vertical plane perpendicular to the axle, acts on the cylinder for 1 min. How many revolutions will the cylinder make during that minute, and what angular velocity will it have at the end of that time? (*b*) A rope is wound around the cylinder, and a force of 50 lb is applied to it. How long will it take to unwind 100 ft of the rope? *Ans.* (*a*) 738 *rev;* 154.5 *rad/sec.* (*b*) 4.16 *sec.*

2. (*a*) What pull applied to the rope of Prob. 1 will give the cylinder an angular acceleration of 3 rad/sec²? (*b*) What weight, attached to the rope and allowed to descend under the action of gravity, would give the cylinder this acceleration? (*c*) What is the greatest acceleration that could possibly be given the cylinder by the latter method? *Ans.* (*a*) 19.4 *lb;* (*b*) 22.5 *lb;* (*c*) 21.5 *rad/sec².*

3. A certain flywheel weighs 1200 lb and is 3 ft in diameter. (*a*) A torque of 100 ft-lb applied to the wheel produces how much angular acceleration? (*b*) How much torque is required to produce a speed of 75 rev/min in 10 sec? (*c*) in 10 revolutions? *Ans.* (*a*) 2.385 *rad/sec²;* (*b*) 33 *ft-lb;* (*c*) 20.6 *ft-lb.*

4. Consider a cylinder of a given moment of inertia, mounted as described in Prob. 1, but with the rope wrapped around the axle instead of around the body of the cylinder. A weight W is attached to the rope and allowed to descend as in Prob. 2. What should be the radius r of the axle in order that the cylinder may be given the maximum angular acceleration? *Ans.* $r = I(\omega/g)$.

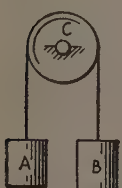

FIG. 191.

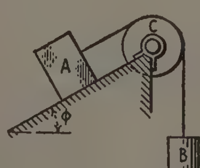

FIG. 192.

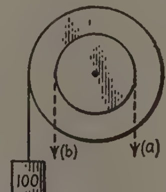

FIG. 193.

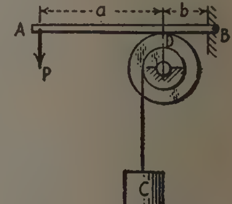

FIG. 194.

5. A homogeneous solid cone is mounted so that it can rotate about its own axis. The cone is 10 in in diameter at the base and 20 in. long; it weighs 140 lb. What constant couple, applied to the cone in a plane at right angles to the axis, would give it an angular velocity of 100 rev/min in 2 min?

6. A, B, and C, Fig. 191, weigh 50 lb, 100 lb, and 150 lb, respectively. C is a solid disk of cast iron 16 in. in diameter. Determine the acceleration of A, B, and C, and also the pulls of the cord on A and B. (Neglect axle friction.) *Ans.* *Acceleration of A* = 7.18 *ft/sec².*

7. A, B, and C, Fig. 192, weigh 100 lb, 30 lb, and 34.4 lb, respectively. The diameter of C = 2 ft 2 in., and the radius of gyration of C about the axis of rotation = 1 ft; $\phi = 30°$. Friction under A, when the system is moving, is 10 lb. Determine the acceleration of A, B, and C, and the tensions in the rope, the system being in motion. (Neglect axle friction.) *Ans.* $\alpha = 1.86$ *rad/sec*^{2}.

8. Figure 193 represents a pulley and attached drum mounted on a horizontal axle. A rope is wound around the pulley, and a block weighing 100 lb is suspended therefrom as shown. The pulley is 6 ft in diameter; the drum is $2\frac{1}{2}$ ft in diameter; the weight of the drum and pulley combined is 200 lb, and the radius of gyration is 18 in.; the axle is frictionless. Determine the angular acceleration of the drum and pulley: when a force (a) of 400 lb is applied; (b) of 400 lb is applied. *Ans.* (a) 4.79; (b) 35.8 *rad/sec*^{2}.

9. Two pulleys, A and B, are mounted on separate axles (horizontal and parallel) so that each can rotate in a vertical plane. A weighs 100 lb, is 3 ft in diameter, and has a radius of gyration of 15 in. B weighs 200 lb, is 4 ft in diameter, and has a radius of gyration of 18 in. A rope is wound about B and then passes over A, so that when this rope is pulled both pulleys are made to turn. If a pull of 80 lb is applied to the rope, what is the angular acceleration of each pulley? *Ans.* 9.45 *and* 7.08 *rad/sec*^{2}.

10. AB, Fig. 194, is a brake for regulating the descent of the suspended body C. C weighs 1000 lb, the drum 2000 lb; the diameter of the drum = 12 ft, that of the brake wheel = 14 ft; a = 4 ft; b = 6 in.; and the radius of gyration of the entire rotating system about the axis of rotation = 4 ft. When P = 100 lb and the coefficient of brake friction is $\frac{1}{4}$, what is the acceleration of C? (Neglect axle friction.)

11. The wheel A, Fig. 195, is a solid cylinder weighing 1000 lb, and its diameter is 8 ft. It is desired to arrange a brake BC as shown, by means of which the speed of the wheel may be reduced from 100 rev/min to zero in 10 sec. The coefficient of friction is $\frac{1}{4}$; the available pull P is 100 lb. Determine the ratio a/b. (Neglect axle friction.)

12. The drum shown in Fig. 196 rests upon a floor and against a low vertical wall as shown. A rope is wound around the axle and passes off horizontally over the wall. The diameter of the drum is 4 ft; the diameter of the axle is 3 ft; the radius of gyration of the drum and axle is 20 in., and their combined weight is 180 lb. The coefficient of friction between drum and floor is 0.3 and between drum and wall 0.2. The pull P = 200 lb. Determine the angular acceleration of the drum and all forces acting on it. *Ans.* $\alpha = 8$ *rad/sec*^{2}.

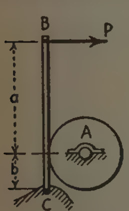

FIG. 195.

FIG. 196.

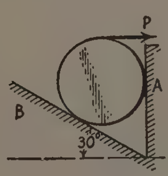

FIG. 197.

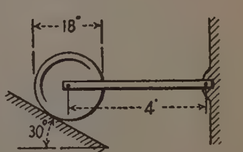

FIG. 198.

13. The homogeneous cylinder of Fig. 197 weighs 100 lb and has a diameter of 3 ft. It lies in a trough formed by a vertical wall A and a sloping floor B. For all surfaces of contact $\mu = 0.2$. A horizontal force P of 150 lb is applied to the cylinder by means of a cord. Determine the resulting angular acceleration of the cylinder.

14. Figure 198 represents a disk mounted on the end of a (weightless) horizontal arm which is pinned at its other end to a fixed support. The disk is 18 in. in diameter, weighs 120 lb, and has a radius of gyration of 6 in. Suppose that the arm is lifted slightly and that the disk is given an angular velocity of 400 rev/min and then lowered and allowed to rest upon the

inclined plane. If the coefficient of friction between disk and plane is 0.3, determine the time required for the disk to stop rotating: (*a*) when its initial angular velocity is clockwise, and (*b*) when its initial angular velocity is counterclockwise. *Ans.* (*a*) 1.032 *sec.*

15. A table with a circular top has a smooth border along the edge and is mounted so that it can be rotated about the vertical through its center. Suppose that a straight rod lies on the top. When the table is rotated the rod rolls to, and its ends bear against, the border. Take the diameter of the top = 9 ft, length of rod = 6 ft, weight of rod = 10 lb, and speed of rotation = 200 rev/min. Compute the pressures at the ends of the rod. *Ans.* 304 *lb.*

16. A uniform slender rod is placed as shown in Fig. 199 on the top of a square table which is provided with a vertical rim. The table, with the bar in place, is made to rotate uniformly at 90 rev/min about a vertical central axis O. Determine the force exerted on the rod by each of the three sides of the rim that touch it. (The rod weighs 12 lb, the table is 5 ft square, and all surfaces of contact are smooth.) *Ans.* *Each force* = 41.4 *lb.*

17. Suppose the table and bar of Prob. 16 to be given a counterclockwise angular acceleration of 4 rad/sec², starting from rest. Determine the magnitude of all forces on the rod 2 sec after motion commences. *Ans.* 29.8; 32.3; 30.4 *lb.*

18. AB, Fig. 200, is a stiff slender rod pinned to a round table at B and restrained at A by nails driven alongside the rod into the table. The rod is 4 ft long, weighs 40 lb, and lies 2 ft from the center of the (circular) top. The top can be rotated about a vertical axis through its center. Determine the pressures at A and B for the instant when the angular velocity and acceleration respectively are 30 rev/min and 40 rev/min/sec clockwise. *Ans.* 14.05 *lb at A.*

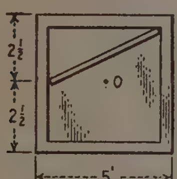

FIG. 199.

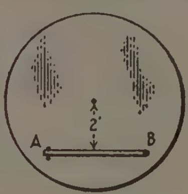

FIG. 200.

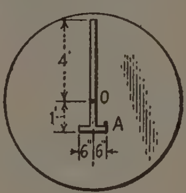

FIG. 201.

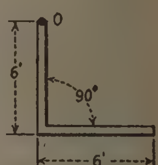

FIG. 202.

19. Figure 201 represents a top view of a slender uniform T-shaped bar, which is pinned at O to the center of a horizontal table top. The bar bears against another pin at A. The table is made to rotate about a vertical axis through O with a clockwise angular acceleration of 30 rev/min/sec. Determine the force exerted by each pin on the bar at the instant ω becomes 80 rev/min.

20. Figure 202 represents a top view of a uniform bent bar weighing 40 lb which rests on a smooth horizontal floor and is made to rotate about a frictionless pin at O by a constant clockwise couple (not shown) of 20 ft-lb acting in the horizontal plane. (*a*) Determine the reaction R of the pin at the instant the couple is first applied. (*b*) Determine the reaction of the pin on the bar when the couple has acted for 5 sec and is still acting. *Ans.* (*b*) R_t = 3.17 *lb;* R_n = 42.7 *lb.*

21. A uniform straight bar 10 ft long weighing 50 lb is supported in a horizontal position by two vertical cords attached at 2 ft from the ends of the bar. One of the cords breaks; what is the tension in the other cord just after this occurs?

22. A uniform slender rod 4 ft long and weighing 20 lb is pinned at its upper end so that it can rotate in the vertical plane. The rod is raised to the vertical position and given a violent downward throw, so that when it reaches the horizontal position it has an angular velocity of 100 rev/min. The starting force now no longer acts on it. Determine the pin reaction R at this instant. *Ans.* R_n = 134 *lb,* R_t = 5 *lb.*

23. A slender uniform bar is pinned at its upper end O so that it can swing in a vertical

plane. To the outer end of the bar is attached a small open box of negligible weight, the bottom of which is at right angles to the bar. The bar and box are raised to the horizontal position, a small block B of negligible weight compared with the bar is placed on the bottom of the box, and the system is released and allowed to swing down. What is the direction of the friction force exerted by the box on the particle? Does it change during the swing? What is its magnitude when the bar makes an angle θ with the vertical?

Art. 115. Center of Percussion

1. Consider a slender uniform rod, suspended by a pin through its upper end, hanging freely. (*a*) Determine the pin reactions at the upper end of the rod at the instant a horizontal force P is applied at the lower end. (*b*) Determine the pin reactions at the instant a horizontal force P is applied at the center of the rod.

2. A hoop is made by bending a slender uniform rod into a circle and welding the ends together. This hoop is pinned at a point on its circumference so that it can swing in its own plane. Locate the center of percussion of the hoop with respect to the point of suspension.

Art. 116. Gravity Pendulum

1. Compute, for your locality, the length, to the nearest hundredth inch, of the simple seconds pendulum.

2. The length of a simple seconds pendulum at a certain place is 3.56 ft. Find the length of a pendulum which at the same place swings from one side to the other in 5 sec. (U. S. Civil Service examination.)

3. In order to compute the counterweight required to balance (as nearly as possible) the connecting rod of a locomotive, it is necessary to know the distance from the axis of the cross-head pin to the center of percussion of the rod. This is determined by suspending the rod by the cross-head pin and letting it swing as a pendulum. The distance L_p from the pin axis to the center of percussion is computed by the formula $L_p = 9.775/N^2$, where L_p is in inches and N is the number of to-and-fro or double swings per second. Can you verify this formula? (Formula from publication of the Baldwin Locomotive Works.)

General

1. Consider the cylinder described in Prob. 1 of Art. 114, and suppose this cylinder to be rotating at 100 rev/min. It is stopped by braking, the coefficient of friction between brake and cylinder being 0.2 and the brake pressure P increasing uniformly with time according to the formula $P = 10t$, where P is in pounds and t is in seconds. How long will it take for the cylinder to stop?

2. Consider the wedge-bar combination described in Prob. 3 of Art 110. Suppose that all surfaces of contact are smooth, that the wedge weighs 10 lb and the bar 20 lb, and that the wedge is placed so that the bar is inclined at 20° to the horizontal and the system then released. At the instant of release, what is the acceleration of the wedge?

3. Consider the cone described in Prob. 5 of Art. 114. Suppose that the cone has wrapped around it, smoothly and in one layer, a flexible rope $\frac{1}{4}$ in. in diameter which leads off at the large end of the cone. A constant pull of 5 lb is applied to this rope. What is the angular velocity of the cone at the end of 4 sec?

4. Refer to Prob. 6 of Art. 111. Assume the bar to be uniform and to weigh 20 lb, and the surface of the wall to be smooth. Take other data as given in part (*b*) of the problem referred to, and determine the pressure of the wall on the rod and the pin reaction.

CHAPTER XI

Art. 120. Motion of Any Particle of the Body

1. A wheel 6 in. in diameter rolls on a straight track. At a certain instant its center has a velocity of 3 ft/sec and an acceleration in the same direction of 2 ft/sec². Determine the velocity and acceleration of the highest point of the wheel at that instant; of the lowest point.

2. Suppose the wheel of Prob. 1 to be rolling on the circumference of a cylinder 2 ft in diameter. At a certain instant, when the wheel is at the top of its path, its center has a velocity (clockwise) of 3 ft/sec and a tangential acceleration (clockwise) of 2 ft/sec². Determine the velocity and acceleration of the highest point of the wheel at that instant; of the lowest point. Compare with the results obtained in Prob. 1. *Ans. Highest point,* $a_x = 4$, $a_y = 43.2$ *ft/sec² down.*

3. A bar AB, which is 8 ft long, slides along a horizontal floor on its lower end A while inclined forward. The motion is in the plane of the bar. At a certain instant A has a velocity of 2 ft/sec which is increasing at the rate of 10 ft/sec²; the bar an angular velocity rearward of 3 rad/sec which is increasing at the rate of 2.5 rad/sec². At that instant the bar slopes at 60° to the horizontal. Determine the velocity and acceleration of the upper end B then.

4. The spool of Fig. 203 has an outer radius r_2 and inner radius r_1; it rolls on the track AB without slipping. At a certain instant the center of the spool has a velocity v to the right and an acceleration a to the right. Determine, for that instant, the velocity and acceleration of each of the points 1, 2, 3, and 4. Solve also for the case when v is to the right and a is to the left.

5. Suppose that the lower end B of the rod BC of Ex. 2, Art. 120, has a velocity v and an acceleration a. Assume v and a positive when to the right. Derive formulas for ω and α, and for the velocity and acceleration of C. Determine ω, α, and the velocity and acceleration of C for the numerical data: $L = 10$ ft, $v = +12$ ft/sec, $a = -6$ ft/sec², $\theta = 40°$.

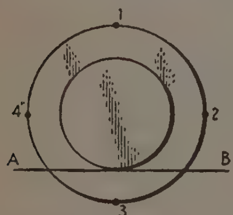

FIG. 203.

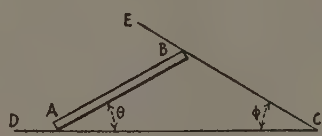

FIG. 204.

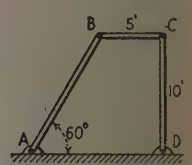

FIG. 205.

6. The rod AB, Fig. 204, is made to move so that the upper end B follows the line EBC while the lower end A follows the horizontal line DAC. At a certain instant the rod slopes at the angle θ and the end A is moving with a velocity v and an acceleration a, both toward the right. At that instant, what is the angular velocity ω and what is the angular acceleration α of the rod? Evaluate ω and α for $L = 10$ ft, $\theta = 30°$, $\phi = 30°$, $v = 10$ ft/sec, $a = 5$ ft/sec². *Ans.* $\omega = 1$ *rad/sec clockwise;* $\alpha = 1.23$ *rad/sec² counterclockwise.*

7. For the crank and connecting-rod mechanism described in Prob. 7, General, Chapter VIII, derive formulas for the angular velocity ω and angular acceleration α of the connecting rod and for the velocity v and acceleration a of C in terms of the crank radius r, the length of the rod L, the angular velocity of the crank ω', and the crank angle θ. Evaluate ω, α, v, and a for the numerical data: $r = 1.5$ ft, $L = 5$ ft, $\omega' = 100$ rev/min, $\theta = 145°$.

Art. 122. Forces on a Body in Plane Motion

1. A uniform circular disk, 4 ft in diameter and weighing 100 lb, rests on a horizontal floor. A horizontal force of 50 lb is applied at the top of the disk, making it roll. The floor is rough enough to prevent any slipping. Determine the acceleration of the center of the disk and the friction component of the reaction of the floor. *Ans.* $a = 21.5$ *ft/sec²; friction* $= 16.6$ *lb.*

2. A heavy spool, consisting of a pair of wheels 3 ft in diameter attached to either end of a drum 1 ft in diameter, weighs 200 lb and has a radius of gyration with respect to its central axis of 18 in. A rope wound about the drum passes off horizontally from the under side and

has applied to it a pull of 60 lb. If the floor on which the spool rests is so rough as to prevent slipping, determine the acceleration of the drum and the reaction of the floor upon it.

3. A 4-ft circular disk is counterweighted so that the center of gravity G is 6 in. from the geometrical center O. This disk is made to roll along a horizontal floor so that O has a uniform speed of 10 ft/sec. It is made so to roll by a horizontal force P applied at O, and there is no slipping at the point of contact with the floor. Determine the maximum and minimum values of P and the corresponding values and direction of the floor reaction on the disk, which weighs 100 lb. *Ans. Max* $P = 63.8$ *lb.*

4. The disk of Prob. 3 is made to roll in the manner described not by a force applied at O but by a couple applied to the disk. Determine the maximum and minimum values of this couple.

5. A uniform bar of length L and mass M rests on a smooth horizontal floor, its axis pointing north-south. A force F, acting towards the east, is applied to the south end of the bar. (*a*) Determine the momentary accelerations this force gives the ends of the bar. (*b*) What point of the bar, if any, would not be given any acceleration by the force described?

6. Refer to Prob. 6 of Art. 120. Assume DC to be a smooth floor and EC a smooth inclined wall. Assume further that the rod is made to move as described by a horizontal force P at the lower end and a vertical force Q at the upper end. Determine P and Q, and the reactions of the floor and wall, for the numerical data given and for a bar weight $W = 100$ lb.

7. An earthquake shock imposes a horizontal acceleration of a ft/sec² on the base of a tall brick chimney. For simplicity, assume the chimney to be a uniform slender column of length L ft, weighing W lb, and free to rock. Determine the horizontal force at the base, the acceleration of the mass-center, and the angular acceleration of the chimney at the instant the acceleration is imposed.

8. A slender uniform rod of length L and weight W is placed in an inclined position with its upper end against a smooth vertical wall and its lower end on a smooth horizontal floor. The bar is then released. Show that at the instant of release the angular acceleration taken on by the bar is $\alpha = \frac{3}{2} \cos \theta g/L$, that the floor reaction is equal to $(1 - \frac{3}{4} \cos^2 \theta)W$, and that the wall reaction is equal to $\frac{3}{4} \sin \theta \cos \theta W$, where θ = inclination of the bar to the horizontal.

Art. 123. Plane Motion as Rotation about Successive Axes; Instantaneous Center

1. Bars AB, BC, and CD, Fig. 205, are pinned together at B and C and to the floor at A and D. At the instant they are in the position shown (AB at 60°, BC horizontal, CD vertical), AB has an angular velocity of 2 rad/sec clockwise. Determine, for that instant, the velocity of the center of bar BC and the angular velocity of bar CD.

2. Refer to Prob. 4 of Art. 120. Determine the velocity of each of the points 1, 2, 3, 4 by the instantaneous-axis method, and compare with the results previously obtained.

3. Refer to Prob. 6 of Art. 120. Determine the angular velocity of the rod and the velocity of B by the instantaneous-axis method.

Art. 126. Absolute Velocity of the Moving Particle

1. The disk of Fig. 206 is 4 ft in diameter and is rotating uniformly about O at 1 rev/sec. A point P is moving along the diameter AB from A toward B with a uniform relative velocity of 4 ft/sec. Determine the absolute velocity of P when midway between A and O; when midway between O and B.

Suppose that P is moving from C toward A along the line CA; the angle $\phi = 150°$, and when P reaches A its relative velocity is 6 ft/sec (along CA). What is the absolute velocity of P then?

2. A certain square is 6 by 6 ft, and its corners are lettered A, B, C, and D in succession around the perimeter. The square is rotating uniformly about a line through A perpendicular to its plane at 1 rev/sec; a point P is moving along CD and in that direction with a relative

velocity of 6 ft/sec. Determine the absolute velocity of P when it reaches the mid position between C and D. *Ans.* 47.6 *ft/sec.*

3. A man who can swim 3 mi/hr undertakes to swim across a river 1 mi wide in which the current is 2 mi/hr southward. (*a*) If the man swims always toward the east, when and where will he reach the shore? (*b*) In what direction should he head in order to reach a point directly across from where he started? (*c*) Suppose that the current is 4 mi/hr. In what direction should the swimmer head in order to be carried downstream the shortest possible distance while making the crossing?

4. An airplane is flying toward the south at 100 mi/hr ground speed and climbing on a path inclined at 10° to the horizontal. The wind is blowing north at 30 mi/hr. What is the velocity of the airplane relative to the air?

Art. 127. Absolute Acceleration of the Moving Particle

1. For the motion described in Prob. 1 of Art. 126 determine the absolute acceleration of P.

2. For the motion described in Prob. 2 of Art. 126 determine the absolute acceleration of P. *Ans.* $A = 333$ *ft/sec*^{2}.

3. A flywheel 4 ft in diameter is mounted in the cab of a Diesel locomotive so that it can rotate in the vertical plane parallel to the axis of the locomotive. At a certain instant the locomotive is running north on a straight level track with a velocity of 60 mi/hr and is slowing at the rate of 6 mi/hr/sec, while the flywheel is rotating clockwise, viewed from the east, with an angular velocity of 2000 rev/min that is diminishing at the rate of 100 rev/min^2. Determine the absolute velocity and absolute acceleration at that instant of the highest point on the perimeter of the wheel.

General

1. Refer to Prob. 1 of Art. 123. Suppose that bar AB has a clockwise angular acceleration of 2 rad/sec^2, other data being as previously given. Determine the angular acceleration of each of the bars and the acceleration of point C.

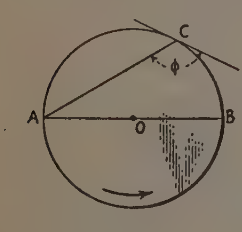

Fig. 206.

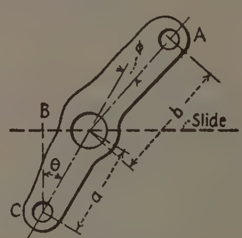

Fig. 207.

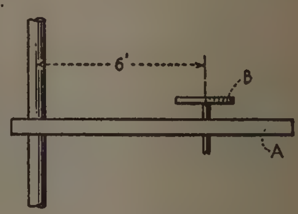

Fig. 208.

2. Figure 207 represents one of two similar elements in the "disappearing" carriage or mount of a coast defense gun. The gun is mounted on a trunnion at A and moves rearward and downward with A, while B moves horizontally on the slide and C moves vertically, raising a counterweight. Take $a = 10$ ft, $b = 12$ ft, $\phi = 20°$, and determine the velocity and acceleration of A and of B when $\theta = 30°$ and C has an upward velocity of 20 ft/sec and a downward acceleration of 15 ft/sec^2.

3. Consider the counterweighted disk of Prob. 3, Art. 122. Suppose that the propelling force P is removed at an instant when G is on a level with O and behind O, the velocity of O at that instant being 10 ft/sec. Determine the angular acceleration of the disk and the floor reaction immediately after P ceases to act.

4. In Fig. 208, A represents an arm mounted on a vertical shaft about which it can rotate in a horizontal plane. B is a disk which spins about a vertical axle at its center, while the

vertical axle in turn can be made to move out along A. Take the radius of the disk as 6 in., and determine the absolute velocity and acceleration of the southernmost point on its perimeter at an instant when the arm A points west-east and is rotating with $\omega = 10$ rad/sec clockwise and $\alpha = 4$ rad/sec² counterclockwise, viewed from above, and the disk is 6 ft east of the axis of the vertical shaft, is spinning with a relative $\omega = 50$ rad/sec counterclockwise and $\alpha = 40$ rad/sec² clockwise viewed from above, and its center is moving with a relative velocity of 12 ft/sec eastward and a relative acceleration of 20 ft/sec² westward.

CHAPTER XII

Art. 133. Translation

1. Solve Prob. 1 of Art. 92 by d'Alembert's principle.

2. In Fig. 209, A represents a homogeneous rectangular block 2 ft by 2 ft in cross section and 6 ft high which stands on end on the horizontal top of block B. The coefficient of static friction between A and B is sufficient to preclude slipping. What is the least value the coefficient of kinetic friction between B and C can have if A is not to tip over when the blocks are allowed to slide down C together? For this limiting coefficient of friction, determine the reactions of B on A while the blocks are sliding, in terms of W, the weight of A.

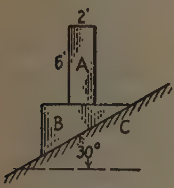

FIG. 209.

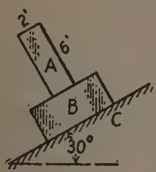

FIG. 210.

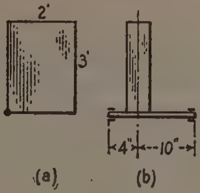

FIG. 211.

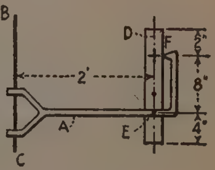

FIG. 212.

3. Refer to Fig. 210. Take the dimensions of A as in Prob. 2, and determine the maximum value the coefficient of kinetic friction between B and C can have if A is not to tip. As in Prob. 2, determine the reaction of B on A while the blocks are sliding.

4. Solve Prob. 10 of Art. 92 by d'Alembert's principle.

5. Solve Prob. 1 of Art. 105 by d'Alembert's principle.

6. Solve Prob. 3 of Art. 105 by d'Alembert's principle.

Art. 135. Rotation: Special Case I

1. Solve Prob. 2(a), of Art. 114, by d'Alembert's principle.

2. A homogeneous rectangular block, Fig. 211, is mounted so that it can be made to rotate on a light horizontal axle attached to one corner edge and supported in bearings as shown in (b). The block weighs 60 lb. Determine the bearing reaction due to the inertia forces (weight disregarded) when the block is rotating uniformly at 1000 rev/min.

Art. 136. Rotation: Special Case II

1. A, Fig. 212, is a rigid piece which can be rotated about the vertical axis BC. D is a slender uniform vertical bar pinned to A at E and resting against A at F; the bar is 14 in. long and weighs 20 lb. The speed of rotation is 100 rev/min. Determine the pressures on D.

Art. 137. Rotation: Special Case III

1. Figure 213 represents a vertical shaft having two horizontal arms, at the end of each of which there is a sphere. Sphere A weighs 10 lb, sphere B weighs 20 lb; the weight of the shaft is 15 lb, and the weights of the arms and their masses may be disregarded. Determine the

reactions of the bearings at the ends of the shaft C and D when the system is rotating uniformly at 100 rev/min. (The bearings are frictionless; the one at C exerts a horizontal force only, while the one at D exerts a horizontal and a vertical force.) *Ans.* $C_x = 5.0$; $D_x = 165$ *lb.*

2. Figure 214 represents a uniform slender bar bent at right angles; it is made to rotate about the vertical axis y with an angular acceleration of 5 rad/sec². At the instant the angular velocity becomes 300 rev/min, what are the moments of the applied forces about the x, y, and z axes? (Take the mass of the rod as m slugs per linear foot, and do not take into account the moment of the weight of the rod.)

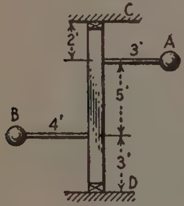

FIG. 213.

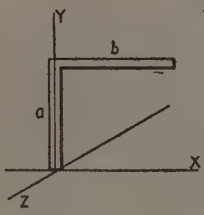

FIG. 214.

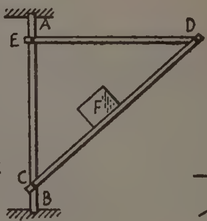

FIG. 215.

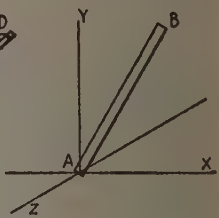

FIG. 216.

3. The frame in Fig. 215 consists of uniform slender bars; it can be rotated about the vertical shaft AB. The shaft is 12 ft long; $ED = 8$ ft, and $EA = BC = 2$ ft. The weights of AB, ED, and CD are respectively 500, 200, and 400 lb. F is a comparatively small body that may be regarded as a particle; it weighs 300 lb and is placed at midlength of CD. The entire system is rotated at 1800 rev/min. Determine all forces on each member.

4. The slender uniform rod AB, Fig. 216, is supported and made to rotate about the vertical axis Y by forces and couples applied at the lower end. Determine the necessary force and couples for the following data: Weight of bar, 20 lb; $\omega = 10$ rad/sec and $\alpha = 20$ rad/sec², both counterclockwise viewed from above; length of bar, 4 ft. Solve in two ways: first, by using the appropriate formulas; second, by representing graphically the reversed effective force for each elementary length of the bar and calculating their resultant directly from their graphical representation, without the use of the formulas.

Art. 138. Plane Motion

1. Consider the bar of Prob. 3 of Art. 120. Assume this bar to be uniform, to weigh 60 lb, and to be made to move as described along a smooth floor by a horizontal force F_A applied at the lower end and an inclined force F_B applied at the upper end. Determine F_A and F_B when the bar has the position and motion described in Prob. 3 of Art. 120.

2. Solve Prob. 6 of Art. 122 by d'Alembert's principle.

3. Consider the connecting rod whose motion was discussed in Prob. 7 of Art. 120. Take the combined weight of the reciprocating parts (piston, piston rod, and cross head) as 280 lb, the weight of the connecting rod as 360 lb, its mass-center G to be 2 ft from P, and its moment of inertia about an axis through G perpendicular to the plane of the motion to be 40 slug-ft². Assume that the total steam pressure is 8000 lb at the instant the rod has the position and motion described in Prob. 7 of Art. 120. Determine the pin reactions on the rod at P and C at that instant.

4. Solve Ex. 2 of Art. 127 by d'Alembert's principle.

Art. 139. Group of Moving Bodies

1. Solve Prob. 6 of Art. 114 by d'Alembert's principle.

2. Solve Prob. 7 of Art. 114 by d'Alembert's principle.

Art. 140. The Balancing of Rotors

1. A 5-lb disk is placed on a uniform horizontal shaft 20 in. long at 8 in. from the left bearing. The center of gravity of the disk is 3 in. from the center of the shaft. Show how to balance dynamically the shaft and disk in two planes normal to the shaft located at 3 and 16 in. from the left end.

2. A uniform horizontal shaft 20 in. long is loaded with two eccentric weights B of 2 lb at 6 in. from the left bearing and C of 2 lb at 12 in. from the left bearing. If the axis of the shaft is taken as the x axis, the center of gravity of B is at $z = +2$ in., $y = 0$, when the center of gravity of C is at $y = -1\frac{1}{2}$ in., $z = 0$. Show how to balance these weights in two planes normal to the shaft. Let the A plane be at 2 in. from the left end and the D plane at 16 in. from the left end. *Hint:* Balance B in each plane, then balance C in each plane. Finally combine vectorially the mr values in each plane, and indicate the value and direction of the resultant mr for each plane.

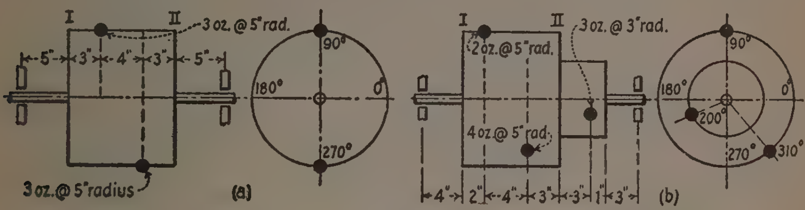

FIG. 217.

3. In Figs. 217a and b, the rotors shown are assumed to be in perfect balance except for the small masses shown attached. These rotors are to be balanced by placing additional material in planes I and II with only one addition in each plane at a radius of 4 in. in each case. Determine the weight in ounces of each corrective mass and the angular location at which each should be placed for each of these conditions: (a) unbalance due to the upper 3-oz mass only shown in Fig. 217a; (b) unbalance due to the two 3-oz masses shown in Fig. 217a; (c) unbalance due to the three masses shown in Fig. 217b.

Art. 141. Dynamic Stresses

1. A slender uniform rod of length L ft and weight W lb lies on a smooth horizontal floor. A couple of C ft-lb in the horizontal plane is applied to the bar. Determine the transverse shear V and the bending moment M at midlength of the bar. Do V and M change as the angular velocity of the bar increases with time? What is the tension P at midlength of the bar t sec after the couple starts to act? *Ans.* $V = \frac{3}{2}C/L$; $M = \frac{1}{4}C$.

2. Consider the slender bar of Prob. 1 subjected to a horizontal force of F lb, applied at one end, and normal to the axis of the bar. Determine V and M at midlength of the bar at the instant F is applied.

3. A steel bar of circular cross section 100 in. long tapers uniformly from a diameter of 2 in. at one end to a point at the other end. It rests on a smooth horizontal floor. An axial pull of 100 lb is applied at the larger end. Derive an expression for the unit tensile stress (total tension divided by cross-sectional area) at any section of the bar distant x in. from the larger end.

General

1. Consider Fig. 217b of Prob. 3 in Art. 140. Balance this rotor by adding or subtracting the necessary mass at each of the following points, described by distance x from plane I, distance r from the axis of rotation, and angle θ measured as indicated in the figure. Point 1: $x = 0$ in., $r = 4$ in., $\theta = 0°$; point 2: $x = 2$ in., $r = 5$ in., $\theta = 90°$; point 3: $x = 6$ in., $r = 3$ in., $\theta = 180°$; point 4: $x = 9$ in., $r = 4$ in., $\theta = 270°$.

2. Show that any rigid body having a plane of symmetry and having plane motion in that plane can be replaced by a "dynamically equivalent" body or system consisting of two concentrated masses m_1 and m_2 located on a straight line through the mass-center G of the body. What conditions must the masses m_1, m_2 and their respective distances d_1, d_2 from G satisfy? (By "dynamically equivalent" is meant having the same effective force system.)

CHAPTER XIII

Art. 143. Calculation of Work Done by a Force

1. A block rests on a horizontal floor; the block weighs 100 lb, and the coefficient of friction between the block and the floor is 0.2. Determine the work done on the block by each of the forces that act on it while it is being dragged a distance of 10 ft by a force of 60 lb that acts: (*a*) horizontally; (*b*) up at an angle of 30° to the horizontal; (*c*) down at an angle of 30° to the horizontal. *Ans.* (*c*) *By 60-lb force*, 519.6 *ft-lb; by friction*, 260 *ft-lb.*

2. A small pierced bead or ball weighing 0.5 lb slides on a smooth rod, Fig. 218, from A to B under the action of a constant horizontal force P of 6 lb, a vertical force $Q = 4s$, and a force parallel to the rod $F = 10 + s^2$ (here s is the distance of the bead from A, in feet, and Q and F are in pounds). Determine the work done on the bead by each of the forces acting on it, and the work done by all the forces collectively.

3. A horizontal force of 50 lb is applied to a block, weighing 100 lb, that rests on a smooth horizontal floor. (*a*) How much work is done by the 50-lb force during the first 5 sec after it is applied? (*b*) how much during the first second? (*c*) how much during the fifth second? *Ans.* (*a*) 10,060 *ft-lb.*

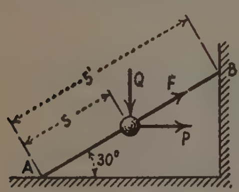

FIG. 218.

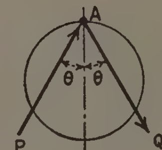

FIG. 219.

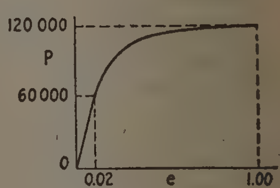

FIG. 220.

4. A 300-ft length of chain, weighing 10 lb per linear ft, is laid out straight on a horizontal floor. The coefficient of friction between chain and floor is 0.2. How much work is done against friction in hauling this chain in and coiling it in a pile at the point where one end was originally?

5. A steam cylinder is 20 in. in diameter; when the piston is within 2 in. of one end the steam in the space between piston and cylinder end is under a pressure of 200 lb/in.^{2} If the pressure changes according to the law $pv = k$, how much work is done by the steam on the piston while the piston moves 20 in.? *Ans.* 301,000 *in.-lb.*

6. The circle, Fig. 219, represents a wheel, 3 ft in diameter, that is made to rotate about its axis by several forces applied to it, two of them being P and Q which are equal, constant in direction, applied at the same point A, and equally inclined to the vertical as shown. Calculate the work done by each force for one revolution of the wheel.

7. The earth's attraction for a small body below its surface varies directly as the distance from the center of the earth to the body; the attraction for a body above its surface varies inversely as the distance squared. Let W = weight of the body at the surface, and P and Q respectively = the earth's attraction when the body is below and above the surface; then

$$P = \frac{Wx}{r} \quad \text{and} \quad Q = \frac{Wr^2}{y^2}$$

where r = earth's radius and x and y the distance from the earth's center to the body. (*a*) Show that the work done by gravity on the body in a fall from the surface down a (vertical) shaft of depth a is given by

$$Wa\left(1 - \frac{a}{2r}\right)$$

(*b*) Show that the work done by gravity on the body in a fall from a height h above the surface to the surface is given by $\dfrac{Wr}{1 + r/h}$.

8. Suppose that three horizontal forces P, Q, and F act on a block that rests on a horizontal floor, moving it from A to B, a distance of 8 ft. All three forces are efforts. P is constantly 20 lb; Q varies uniformly from 10 to 20 lb in the displacement from A to B, and F varies as the square of the displacement from A, reaching a value of 20 lb at B. Make a work diagram for each force, and from the diagrams calculate the amount of work done by each force. *Ans.* (*P*) 160 *ft-lb;* (*Q*) 120 *ft-lb;* (*F*) 53.3 *ft-lb.*

9. Figure 220 is a graph obtained from a tension test of a steel bar; the pull P (in pounds) applied to the bar is plotted against the corresponding stretch e (in inches). Calculate approximately the amount of work done on the bar in breaking it.

10. In order to retard the motion of a launching ship, ropes were fastened to it and to points on the shore, so that the ship broke many of the ropes as it progressed. In order to estimate the retarding effect of each rope broken, tension tests were made on samples of the rope (7-in. manila). Figure 221 shows the average tension-stretch curve for these tests. The average strength of the samples was about 32,500 lb. It was assumed that the efficiency of the knots used would be about 80 per cent, and, therefore, that the ropes would fail at about 26,000 lb. On the basis of this assumption and the curve, it was estimated that each rope (20 ft long) would do 60,000 ft-lb of work on the ship before breaking. Is this estimate correct? If not, make a correct estimate. (Data taken from *Trans. Soc. Naval Architects and Marine Engineers,* 1903, p. 295.)

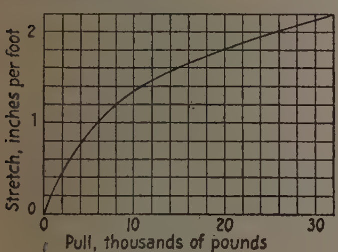

FIG. 221.

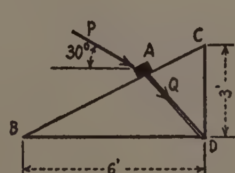

FIG. 222.

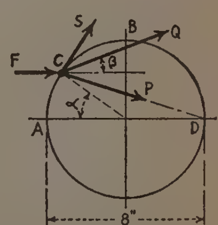

FIG. 223.

11. Consider a small block drawn across a rough horizontal platform by means of an attached cord. Is any work done on the small block by friction? Consider that the small block is held stationary (as by a string attached to some outside fixed support) and that the platform is drawn out from under it. Is any work done on the block in this case? Suppose that both block and platform are moved in the same direction at different speeds? In opposite directions?

12. (*a*) Does work have to be done in order to exert a force? (*b*) Does work have to be done in order to maintain a force? (*c*) Does work have to be done in order to give a body acceleration? (*d*) Does work have to be done in order to maintain a constant acceleration? (*e*) In what way does the amount of work done by a team in starting a heavy wagon depend on the harness?

13. Suppose that a man has to drag a small heavy object some distance along a rough horizontal floor. In what direction should he pull in order to move the object with the least amount of *work*? In what direction should he pull in order to move the object with the least *force*? In what direction should he pull in order that, with a given force, the object will *most quickly* be moved a given horizontal distance?

Art. 144. Calculation of Work Done by a Force: Special Cases

1. The small block A, Fig. 222, weighs 10 lb and moves from B to C along the smooth inclined plane under the action of its own weight, the constant force $P = 30$ lb, and the pull Q of a rubber cord AD which stretches 1 ft for each 10 lb of tension and which has a normal length of 2 ft. Determine the work done on A by each of the forces that acts on it. *Ans. By gravity,* -30 *ft-lb; by* P, $+110.9$ *ft-lb; by* Q, $+75$ *ft-lb.*

2. C, Fig. 223, is a bead on a circular wire ABD; it is subjected to four forces, F, P, Q, and S. $F = 10$ lb and is always horizontal; $P = 40$ lb and is always directed toward D; Q varies in magnitude and direction so that the simultaneous values of α, β, and Q are as follows:

α =	0°	15°	30°	45°	60°	75°	90°
β =	0°	5°	10°	15°	20°	25°	30°
Q =	10	15	18	20	15	12	10 lb

S is a tangential pull, and its value (in pounds) is $40s^2$, where s is the arc AC in feet. Compute the amount of work done by each force for the displacement of C from A to B.

Art. 145. Work Done by a System of Forces

1. A heavy block is dragged up a 30° incline for a distance of 40 ft by three constant forces, F, P, and Q, all applied to the block at the same point. $F = 50$ lb and acts vertically upward; $P = 100$ lb and acts upward at an angle of 45° to the horizontal; $Q = 80$ lb and acts horizontally. All three forces are efforts — that is, they tend to drag the block up the plane. Determine the work done by the three forces in each of these ways: (*a*) Find the work done by each force, and add these works. (*b*) Find the work done by the horizontal and vertical components of the system, and then add these works. (*c*) Find the work done by the components of the system normal and parallel to the incline, and add these works.

2. A block of stone 4 by 4 by 10 ft, weighing 150 lb/ft^3, lies on one side. It is tipped up so as to stand on end. What is the net work done on the block by gravity during this displacement? How much positive work must be done in raising the block to its final position?

3. A circular reservoir 10 ft in diameter and 20 ft deep, dug in the ground, is full of water. How much work is done against gravity in pumping this reservoir dry, the water being discharged at ground level? Suppose that the water is pumped from the reservoir into a conical water tank, the apex angle of the cone being 60° and the lowest point of the tank being 10 ft above ground level. How much work is done on the water by gravity during the operation?

4. When a spring is being stretched moderately, the amount of stretch varies directly with the pull, and when allowed to recover against a decreasing pull then again the stretch is directly proportional to the pull. (This behavior is an example of Hooke's law. But if the pull is changed rapidly the spring does not obey the law precisely.) (*a*) Show that the work required to stretch a spring that obeys Hooke's law is given by $\frac{1}{2}ke^2$, where e is the stretch and k the spring factor or modulus (the required force per unit stretch). (*b*) Show that, when such a spring is stretched an amount e, its strain energy or potential energy is $\frac{1}{2}ke^2$.

5. A steel bar 1 in. square stretches 1/30,000 of its length for every 1000 lb of pull applied to it. How much work will be done in applying a pull of 20,000 lb to such a bar if its length is 10 ft? *Ans.* 800 *in.-lb.*

6. The force between two electrically charged bodies varies inversely as the square of the distance between them. If the force is 0.01 lb when the distance is 2 ft, how much work is done in changing the distance between the bodies from 0.1 ft to 10 ft? *Ans.* 0.396 *ft-lb.*

7. A horizontal rod is so mounted that it can be made to rotate about a vertical axis through

its center. Two spheres, each weighing 10 lb, are mounted so that they can be moved inward or outward along this rotating rod. Suppose that the rod is rotating about the vertical axis at the rate of 90 rev/min and the centers of the spheres are each 4 ft from the axis of rotation. How much work would be done in drawing them in to a distance of 1 ft from the axis, the rod meanwhile being made to rotate at 90 rev/min? *Ans.* 414 *ft-lb.*

8. A variable couple C whose magnitude in foot-pounds is given by the equation $C = 2\theta^2$, where θ is the angle in radians turned through by the body on which C acts, acts on a flywheel during 10 revolutions. The plane of C is normal to the axis of rotation and $\theta = 0$ when C starts to act. Determine the work done by C.

Art. 146. Virtual Work

1. Refer to Fig. 82 of Prob. 5, Chapter III, General, and, by using the method of virtual work, ascertain whether the frame is in equilibrium.

2. Show, by the method of virtual work, that the horizontal force required to push a body of weight W up a smooth plane inclined at an angle θ to the horizontal is equal to $W \tan \theta$. Show that if applied parallel to the plane the force must be equal to $W \sin \theta$.

3. Consider a screw-jack such as that described in Art. 67. Assume the moment M to be obtained by equal and opposite forces P, P acting at the end of a lever of length L, and determine the common magnitude of these forces required to raise a load W, assuming friction to be negligible.

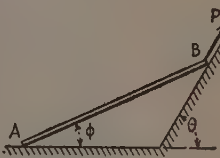

FIG. 224.

4. In Fig. 224, AB represents a slender uniform bar of length L; its lower end rests on a smooth horizontal floor, its upper end against a smooth sloping wall. Determine the force P, applied at the upper end and parallel to the wall, required to hold the bar in place. Show that P is independent of ϕ, and that when acting in the specified direction it is the least force which, applied at B, would maintain equilibrium.

Art. 148. Calculation of Power

1. (*a*) A man drags a block weighing 100 lb along a horizontal floor at the rate of 2 mi/hr. If the coefficient of friction between the block and the floor is 0.3, what horsepower is required to thus drag the block? (*b*) Suppose that the man picks up the block and carries it instead of dragging it; what horsepower is then required? *Ans.* (*a*) 0.16 *hp.*

2. What power is required to move a block weighing 1200 lb up a 30° incline at a uniform speed of 100 ft/sec if the coefficient of friction between block and plane is 0.2?

3. What horsepower is required to give an automobile weighing 3000 lb an acceleration of 5 mi/hr/sec at a speed of 15 mi/hr? *Ans.* 27.3 *hp.*

4. About how long would it take a 10-hp pump, working continuously at full capacity, to empty a well 40 ft deep and 10 ft in diameter, filled with water? Why is your result only approximate? What additional data would be required for an exact solution? *Ans.* 11.9 *min.*

5. At a certain instant during the travel of a 15-lb projectile along the bore of the gun, its acceleration is 41,000 ft/sec² and its velocity is 800 ft/sec. What horsepower is being developed by the powder pressure at that instant? *Ans.* 27,800 *hp.*

6. An airplane weighing 5000 lb is climbing, flying upward at an angle of 15° to the horizontal and at a speed of 120 mi/hr. The total drag (resistance of the air to the plane's motion) is 400 lb. What thrust horsepower is required? What forces are doing work on the airplane, and what kind of work is each of these forces doing?

7. Imagine that a body, falling freely (without air resistance) "from rest," weighs 100 lb. Calculate the work done by gravity on that body during the first 3 sec of the fall. Discuss, for the period stated, the power of gravity. *Ans.* *Work* = 14,490 *ft-lb.*

8. (*a*) A shaft is made to rotate steadily at 1000 rev/min against a resisting couple of

200 ft-lb. What horsepower is required? (*b*) A constant driving torque or couple of 1000 in.-lb is applied to a perfectly balanced flywheel mounted on a frictionless axle. What power is being developed 10 sec after motion starts, if the weight of the flywheel is 120 lb and its radius of gyration is 15 in.?

9. A block weighing 100 lb is dragged along a smooth horizontal floor by a horizontal force, which varies so that the power developed by it is equal to $1000t^2$ ft-lb/min, t being in seconds. (*a*) What is the velocity of the body when $t = 10$? (*b*) What is the acceleration then? (*c*) How far does it move in the first 10 sec? (*d*) What power is the force developing when $t = 10$? (*e*) What is the average power developed for the first 10 sec? *Ans.* (*a*) 59.8 *ft/sec;* (*b*) 8.99 *ft/sec*^{2}*;* (*c*) 239 *ft;* (*d*) 3.03 *hp;* (*e*) 1.01 *hp.*

10. S and S', Fig. 225, are two portions of a shaft. Arms A and A' are rigidly attached to the adjacent ends of the shaft as shown. The ends of the arms are furnished with hooks which are connected by two like coil springs as shown. Thus it is possible to transmit energy from one portion of the shaft to the other; indeed, the device illustrates, in principle, a "transmission dynamometer." Let the length of each arm $= CB = 24$ in., the natural length of each spring $= 8$ in., the stiffness of each spring $= 40$ lb/in. (40-lb pull required for each inch of stretch). When the shaft is rotating at 200 rev/min, the angle between the arms is 60°. What horsepower is being transmitted?

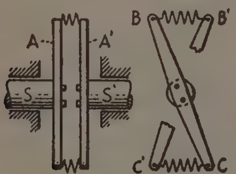

FIG. 225.

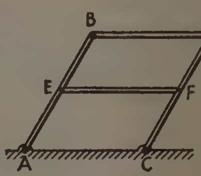

FIG. 226.

11. Put your solution of the preceding problem into general terms, using the following notation: a = length of each arm in inches, b = natural length of each spring in inches, p = stiffness of spring in pounds per inch, n = speed in revolutions per minute, θ = angle between arms in degrees.

12. A formula for the horsepower required to overcome the rolling and wind resistance of an automobile on a level road is: hp $= (32WV + 3.33AV^3)10^{-6}$, where W is the weight of the car in pounds, A is its frontal area in square feet, and V is its velocity in miles per hour. The rolling resistance of a car is commonly taken as $R = 0.012W$ and the wind resistance as $D = KAV^2$, where R and D are in pounds, W, A, and V are as defined above, and K is a constant which may be taken as 0.00125. Using the formula, calculate the power required by a truck weighing 24,000 lb, with a frontal area of 50 sq ft, when going up a 3 per cent grade at 30 mi/hr.

Art. 151. Kinetic Energy of a Rigid Body

1. The four slender bars are pinned to each other and to a base at the lettered points as shown in Fig. 226. Each bar is 4 ft long and weighs 40 lb. E and F are midpoints. The whole system is moved by rotation of AB and CD about the pins A and C respectively, at a constant speed of 30 rev/min. Determine the kinetic energy of each bar, and state the units of your answer. *Ans.* *AB and CD,* 32.8 *ft-lb;* *BD,* 97.6 *ft-lb;* *EF,* 24.4 *ft-lb.*

2. Two wheels A and B are connected by an endless chain. A weighs 10 lb, is 18 in. in diameter, and has a radius of gyration of 8 in. B weighs 8 lb, is 12 in. in diameter, and has a radius of gyration of 4 in. The chain, whose thickness in the plane of the wheels is very small, weighs 0.5 lb/ft and is 10 ft long. What is the kinetic energy of the system when A is rotating at 300 rev/min?

3. When a solid (circular) cylinder is rolling on a straight roadway, what portion of its total kinetic energy is "translational"? *Ans.* $\frac{2}{3}$.

4. Show that the rotational part of the kinetic energy of a rolling sphere is two-sevenths of its total kinetic energy.

5. When a thin hoop is rolling on a straight roadway, what part of its total kinetic energy is due to rotation?

6. Compare the kinetic energy of a truck when running 30 mi/hr on a straight road and when rounding a curve of 400-ft radius at the same speed. The truck weighs 12,000 lb, and it may be regarded as a homogeneous rectangular prism 20 ft long, 7 ft tall, and 6 ft wide.

7. A cast-iron cylinder 3 in. in diameter and 10 in. long moves with motion of translation and a velocity of 20 ft/sec. (*a*) What is the kinetic energy of the cylinder? (*b*) How fast would it have to rotate on its longitudinal axis to have this kinetic energy? (*c*) How fast would it have to rotate about an axis through its middle perpendicular to its length in order to have this kinetic energy? (*d*) How fast would it have to roll to have this kinetic energy? *Ans.* (*a*) 114 *ft-lb;* (*b*) 225.6 *rad/sec;* (*c*) 80.7 *rad/sec;* (*d*) 16.3 *ft/sec.*

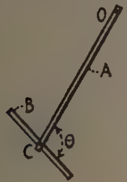

FIG. 227.

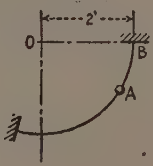

FIG. 228.

8. The arm A, Fig. 227, rotates uniformly counterclockwise about the fixed axis O, while the uniform bar B whirls about a pin C which connects it at midlength to A. Both motions are in the plane of the figure. A is 6 ft long and makes 30 rev/min. B weighs 10 lb, is 3 ft long, and whirls so that the angle θ between it and A is decreasing at the uniform rate of 60°/sec. Determine the kinetic energy of B.

Art. 154. Typical Problems; Examples

1. Consider a particle that is dropped from rest and allowed to fall freely. Determine, by the principle of work and energy, its velocity after falling a distance s feet.

2. A particle starts from rest and slides down a plane inclined at 25° to the horizontal. If the coefficient of friction is 0.3, how fast will the particle be moving when it has traveled 20 ft? *Ans.* 13.91 *ft/sec.*

3. Suppose that the particle of Prob. 2 is projected up the plane with an initial velocity of 30 ft/sec. How far will it go before it stops? *Ans.* 20.1 *ft.*

4. A man driving 50 mi/hr along a level road throws out the clutch and coasts just as he reaches the foot of an 8° slope. How far up the slope will the car go? (Air resistance and friction are neglected.) *Ans.* 599 *ft.*

5. A certain body weighing 400 lb is dragged along a rough horizontal plane by a force of 80 lb. The force is inclined 20° upward from the horizontal; the coefficient of friction between the body and plane is $\frac{1}{10}$. At a certain point in the motion, the velocity of A is 5 ft/sec. What is the velocity of A 10 ft beyond the point? *Ans.* 9.28 *ft/sec.*

6. Suppose that the wind resistance, road resistance, and internal friction that affect the motion of an automobile are equivalent to a single retarding force of 50 lb/ton of weight of car. How fast must a car be going in order that it will coast to the top of a 30-ft hill on a 15° slope? *Ans.* 46 *ft/sec.*

7. A, Fig. 228, is a bead on a smooth wire bent into a circular arc 4 ft in diameter. A

weighs 3 oz. It is allowed to fall from B. Calculate the pressure between A and the wire as A passes through its lowest position. *Ans.* $\frac{9}{16}$ *lb.*

8. A particle, under the action of gravity alone, moves from rest from the highest point on the outer surface of a smooth sphere whose diameter is 10 ft. Neglecting friction and the small force necessary to start the particle, find: (*a*) at what point the particle leaves the sphere; (*b*) what the velocity is at the instant of leaving. (U. S. Civil Service examination.) *Ans.* (*a*) 48° 10′ *from vertical.* (*b*) 10.4 *ft/sec.*

9. It has been remarked that the strain on an acrobat's grip when he swings on a high trapeze is so great that only a trained athlete can retain his hold. Show that if the acrobat's body is regarded as a particle the pull on his hands depends only on his weight and the angle of swing, and is entirely independent of the length of the trapeze ropes. Show also that, if he starts his swing from a point on the same level as the point of suspension, the maximum pull on his hands will be equal to three times his weight.

10. Several soldiers wish to swing across a 30-ft ditch by means of a rope suspended directly over the center of the ditch. If each man can grip the rope tightly enough to withstand a pull 50 per cent greater than the weight of himself and his equipment, what is the length of the shortest rope that could be used?

11. When an airplane pilot makes a vertical loop, it is desirable that he reach the top of the loop with just enough velocity so that the " lift " on the plane can be zero at that instant. Assume that a pilot flying horizontally with velocity V makes such a loop, in the form of a perfect circle, and that during the maneuver the propeller pull just balances the " drag " or air resistance. Prove that the radius of the loop is given by $R = V^2/5g$, and that the pilot is subjected to a pressure from the seat equal to six times his weight.

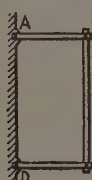

FIG. 229.

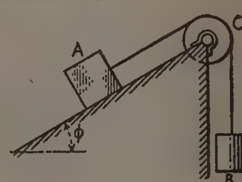

FIG. 230.

12. A 3-in. rifle has a barrel 6.2 ft long and shoots a shell weighing 15 lb; the cross-section area of the bore is 7.28 in². The powder pressure may be taken as varying linearly with the displacement of the projectile, from zero up to a maximum of 33,000 lb/in.² when the projectile has moved 0.5 ft, then to zero when the projectile is at the muzzle. (This does not closely approximate the actual way in which the pressure varies, but the assumption will serve the present purpose.) Neglect the effect of recoil and friction, and determine the muzzle velocity of the projectile and the maximum power developed during the discharge. *Ans.* $v =$ 1785 *ft/sec.*

13. In Fig. 229, AB, BC, and CD are slender uniform bars pinned at A, B, C, and D. AB and CD are each 3 ft long and weigh 8 lb apiece. BC is 5 ft long and weighs 14 lb. If the system is allowed to fall from the position shown (AB and CD horizontal and BC vertical) what is the speed of C just before the bars strike the wall?

14. The car of a mine hoist is being raised by means of a cable wound around a drum. When the car is 50 ft below the surface of the ground the power is shut off. What velocity must the car have at that instant if the kinetic energy of the car and drum is just sufficient to bring the car to the surface? The car weighs 1500 lb; the drum is 4 ft in diameter, weighs 500 lb, and has a radius of gyration of 18 in. The weight of the cable, stiffness of cable, and friction may be neglected.

15. A, B, and C, respectively, Fig. 230, weigh 100, 30, and 64.5 lb. The diameter of

$C = 30$ in., and its radius of gyration about the axis of rotation $= 1$ ft; $\phi = 30°$. The friction under $A = 10$ lb. Determine the velocity of the system when A has moved through 10 ft from rest. *Ans.* 6.14 *ft/sec.*

16. The suspended body C, Fig. 231, weighs 10 lb. The coefficient of friction under the brake is $\frac{1}{6}$; $r_1 = 4\frac{1}{2}$ in., $r_2 = 6$ in., $a = 2$ ft, and $b = 1$ ft. C is allowed to descend 6 ft, thus turning the wheel, and then the brake is put on, with $P = 20$ lb. How much farther will C descend? (Neglect axle friction.) *Ans.* 10 *ft.*

17. A solid homogeneous cylinder 6 in. in diameter and weighing 100 lb is made to rotate about its axis at 200 rev/min and, while thus rotating, is placed in a trough formed by two intersecting planes each of which is inclined at 30° to the horizontal. The coefficient of friction between cylinder and each plane is 0.2. How many revolutions will the cylinder make before it stops, and how long will it take for it to stop? *Ans.* 0.367 *sec.*

18. Figure 232 represents a slender uniform rod, 8 ft long and weighing 15 lb, which is mounted at its quarter-point A on a frictionless horizontal axle. An elastic cord BC is attached to the end of the bar at B and to the fixed point C, which is on the same level as A. The normal length of this cord is 3 ft, and the cord stretches 6 in. for every pound of pull. The rod is released when in the vertical position shown, and is swung upward by the pull of the cord. Determine the linear velocity of the end B of the rod when the rod reaches the horizontal position.

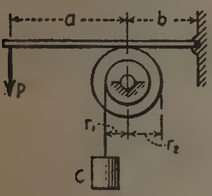

FIG. 231.

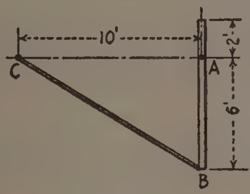

FIG. 232.

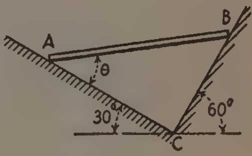

FIG. 233.

19. For the purposes of comparing their "running qualities," certain freight-car trucks were tested substantially as follows: Each one was made to roll down a steep incline to give it "initial velocity," and then it passed onto a moderate upgrade; the velocity was measured at two points on the upgrade; then the loss of kinetic energy was computed. These losses furnished a comparison. The upgrade was 0.38 per cent, and the points at which velocities were measured were 257.2 ft apart. One of these trucks (four-wheeled) weighed 18,150 lb; each pair of wheels and axle, 1800 lb. The diameter of wheels was 33 in.; the radius of gyration of a pair and axle was 0.81 ft. In one test the velocities at the two points were 14.95 and 11.05 ft/sec. Determine the average "truck resistance," a single imaginary force equivalent to actual resistances, not including gravity, on the truck. (Experiments by Professor L. E. Endsley for American Steel Foundries.) *Ans.* 50 *lb.*

20. Three bodies, A, B, and C, are allowed to roll for 100 ft down an inclined platform. A is a solid homogeneous cylinder; B is a solid homogeneous sphere; C is a hollow cylinder. Each body weighs 400 lb and has a diameter of 3 ft; the inner diameter of C is 2 ft. The ramp slopes at 30° to the horizontal, and all the bodies roll without slipping. Determine the velocity with which the center of each body will be moving at the end of the descent.

Art. 155. Conservation of Energy

1. The slender uniform bar AB, Fig. 233, is to be placed on the two smooth inclined planes in such a position as to be in equilibrium. Determine the angle θ corresponding to this position. (Express the distance $\bar{y}$ of the center of gravity of the bar above C as a function of θ, and then determine the value of θ that makes $\bar{y}$ minimum. Check your result graphically by ascertaining whether the three forces on the bar are concurrent.)

2. Figure 234 represents two weights suspended from a flexible, weightless cord, the distance between points of attachment being as indicated. Determine the angles α, β, and λ when the system is in equilibrium, using a graphical method similar to that described in Ex. 2 of Art. 155.

General

1. A 200-yd length of cable is coiled on a drum which is mounted on a horizontal axle at the head of a mine shaft. Fifty feet of the cable is allowed to run out, the drum is brought to rest momentarily and is then released and the remainder of the cable allowed to run out freely. The cable weighs 0.6 lb/ft and may be regarded as perfectly flexible; the drum is 3 ft in diameter, weighs 200 lb, and has a radius of gyration of 14 in.; the friction on the axle bearings is equivalent to a constant resisting torque of 15 ft-lb. Determine the rate at which the drum is rotating at the instant the cable has become completely unwound.

2. A flywheel of a 4-hp riveting machine fluctuates between 60 and 90 rev/min. Every 2 sec an operation occurs which requires seven-eighths of all the energy supplied for 2 sec. Find the moment of inertia of the wheel. (U. S. Civil Service examination.)

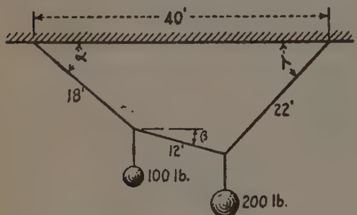

FIG. 234.

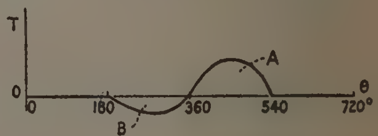

FIG. 235.

3. When a slender body, such as a pole, chimney, etc., is tipped over from an upright position, the motion is one of rotation about the point of contact of the body and the surface which supports the body until slip occurs at the point of contact. Assume that the slender body is of uniform section and that it is hinged to the supporting surface so that it cannot slip, and then determine the vertical and horizontal components of the supporting force for various positions of the tipping body. Draw curves showing how these components vary with the angular displacement of the pole from the vertical.

4. Referring to Prob. 3, assume that the slender uniform body rests on a horizontal floor and that the coefficient of friction between the body and the floor is μ. Determine the angle which the body makes with the horizontal at the instant that slipping occurs, assuming the body to have been tipped over from the vertical position. Determine the least value of μ that would entirely prevent slipping.

5. A single-cylinder oil engine is delivering 10 hp at 300 rev/min against a constant resisting torque. The form of the crank effort diagram (torque T plotted against angular displacement θ) is as shown in Fig. 235. The area A represents the positive work done on the piston during the power stroke, equal to 3300 ft-lb; the area B represents the negative work done on the piston during the compression stroke, equal to 1100 ft-lb. Find what the moment of inertia of the flywheel must be in order that the maximum fluctuation of speed from the mean may not exceed 5 per cent of the mean speed. (Make a diagram showing approximately how the flywheel kinetic energy changes during a four-stroke cycle. Note that the energy changes are -550, -1650, 2750, and -550 ft-lb in the successive strokes.)

6. A slender uniform rod is placed in an almost vertical position against a smooth vertical wall, its lower end on a smooth horizontal floor. It is then allowed to slip down. Ascertain

whether the upper end of the bar breaks contact with the wall during the fall and, if so, at what value of the angle θ between the bar and the horizontal.

7. In one form of high dive the diver simply stands at the edge of the platform and, holding his body straight and rigid, lets himself fall forward, without giving any spring at all. Assume the diver to weigh 180 lb and to be 6 ft tall; assume further that his center of gravity (arms upreached) is 3.2 ft above his feet and that the radius of gyration of his body is 1.6 ft about a right-left transverse axis through the center of gravity. What is the least height of dive for which the diver will be vertical when his head reaches the water level?

8. If a freight car is given a certain velocity and then left to coast along a perfectly smooth level track, will its speed change when it passes from a straight portion of the track to a curved portion, and vice versa? Explain your answer (*a*) by the principle of work and energy and (*b*) by the principle of force and acceleration.

9. A heavy block of weight W lb is dragged along a horizontal floor by means of a rope passing over a pulley, as shown in Fig. 236. If the coefficient of friction between block and floor is 0.4, how much work is done by the rope-pull while the block is dragged at uniform speed from A to B?

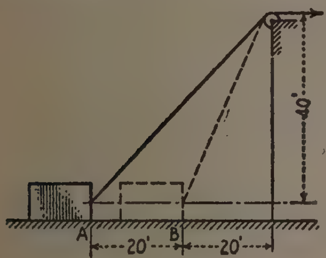

FIG. 236.

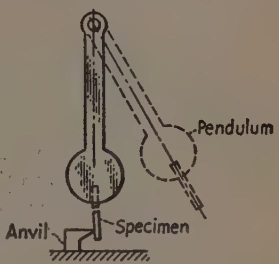

FIG. 237.

10. The exhaust gases of an airplane engine are discharged rearward under pressure and so exert on the plane a forward reaction that helps propel the plane. This force depends solely on the exhaust pressure and diameter of the exhaust manifold, and so the rate at which it does work on the plane increases with the velocity of flight in accordance with the relation Power = Fv (Art. 148). (One writer states that 40 per cent of the power that drives a pursuit plane may derive from this force.) Obviously, if the plane is made to move extremely fast, as in a power dive, the rate at which this force does work on the plane may actually exceed the total power the engine can develop, or even the total potential power of the fuel burned. Explain this apparent paradox.

11. In one kind of test to determine the toughness of metal under impact a small beam is supported in a horizontal position with each end against a vertical knife-edge, and a rigid pendulum is allowed to swing down and strike the beam at midlength, breaking it in two. From the initial and final angles the pendulum makes with the vertical, the loss in its kinetic energy due to impact can be computed; this is the work required to break the specimen and is taken as a measure of its toughness.

One experimenter modified the test thus: The specimen, instead of being stationary, was fixed in the pendulum, Fig. 237, and allowed to swing with it so as to strike a stationary anvil and thus be broken. Then, in order to ascertain whether the toughness of the metal was affected by the quickness or rate of breaking, the experimenter, instead of allowing the specimen to strike a stationary anvil, provided a timing device which fired a rifle so that at the instant the pendulum was vertical the bullet struck the specimen and broke it. The experimenter argued that in this case, just as in the original test, the loss of kinetic energy by the

pendulum was a measure of the work done in breaking the specimen. Do you think this claim is correct? If not, do you think the results of the modified test (using the bullet) indicated too little or too much energy expended in breaking the specimen?

CHAPTER XIV

Art. 159. Force-Momentum Relationship

1. Calculate the total pressure exerted by a 30 mi/hr wind against a sign board 10 ft square set at right angles to the direction of the wind, assuming that all the air intercepted by the sign board is deflected at right angles, and taking the weight of air as 0.0766 lb/ft³. (Actually, the force is considerably less than the value calculated in this way. To what do you think the disparity is due?) *Ans.* 462 *lb.*

2. A horizontal water jet 2 in. in diameter with a velocity of 60 ft/sec impinges against a fixed plane bent downward to the right at 20° with the horizontal and is deflected so as to flow along the plane. Determine the vertical and horizontal components of the plate reaction on the jet.

3. Suppose the plane in Prob. 2 to have a horizontal velocity of 10 ft/sec in the same direction as the jet; then solve. $m = 4000$

4. The table of a planer and the material fastened to it weigh 4000 lb. The table is driven by a rack for which the safe load is 80 lb. Find the shortest time in which the cutting speed of 25 ft/min can be reversed to a return speed of 50 ft/min. $\Delta v = 75$ fpm

5. Suppose that a machine gun fires 500 shots per minute, each bullet weighing 170 grains and being given a muzzle velocity of 2700 ft/sec. What average horizontal force is necessary to hold the gun in position? In this calculation, what minor factors are disregarded?

6. A certain $3\frac{1}{2}$-in. hose is conducting water at a velocity of 60 ft/sec. There is a circular bend of 180° in the hose; the radius of the bend is 8 ft. Assume water pressure at both ends of the bend to be 100 lb/in.² Determine the resultant water pressure on the bend. How much resultant outward pressure is there on each linear inch of hose along the bend?

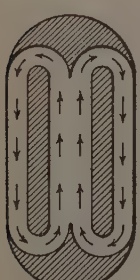

Fig. 238.

7. Water is flowing through a certain 6-in. pipe at a velocity of 4 ft/sec. Compute the resultant pressure of the water against a right-angle bend in the pipe. (Assume that the water pressure is 10 lb/in.² at both ends of the bend.)

8. Actually, the water pressure (referring to the preceding problem) is greater at the inlet end of the bend. Assume that the pressures are 10.4 and 10.0 lb/in.²; then solve.

9. Figure 238 represents schematically a longitudinal cross section of a variable-density wind tunnel. Compressed air is made to circulate as shown by the arrows, passing the model airplane which is mounted in the central tunnel. No air escapes or is introduced during operation. Assume the central tunnel to be 15 ft in diameter and circular in section, the air to be under 20 atmospheres of pressure, and the velocity of the air to be 150 mi/hr. What is the total tension (due to air motion only) in the tunnel structure on a transverse cross section? (Density of air at normal pressure is 0.002378 slug/ft³ and varies directly as the pressure.) *Ans.* 815,000 *lb.*

10. Water is flowing at a velocity of 20 ft/sec in a straight pipe 2 ft in diameter and 3000 ft long. Determine the linear momentum of this water column. How long a time would be required to close a valve at the discharge end without raising the average pressure more than 50 lb/in.²?

11. Assume that the car of Ex. 2 of Art. 107 is being given an acceleration of 5 ft/sec² in the direction of motion, and that it is moving 30 ft/sec when dumping starts. What is the propelling force 5 sec later?

12. A rocket weighing 30 lb carries 4.65 lb of fuel. The rocket travels 1070 ft/sec; the relative jet velocity is 6700 ft/sec; and the rate of fuel consumption is 23.2 lb/sec. Determine the efficiency and the propulsive force.

Art. 163. Force-Angular Momentum Relationship

1. A circular disk (grindstone) of radius r is made to rotate in a vertical plane about its axis with uniform angular velocity ω. While it so rotates sand is poured from above onto the edge of the disk at the uniform rate of w lb/sec. The edge of the disk is so rough that all the sand is thrown off tangentially with a velocity equal to the peripheral velocity of the disk. Determine the driving torque or couple necessary to keep the disk rotating under these circumstances, disregarding frictional resistance at the bearings.

2. An oak stick 8 in. square in section and 6 ft long, weighing 180 lb, is mounted so that it can swing on a horizontal axle. The bearing is frictionless and 1 ft from the upper end of the stick. A soft-nosed rifle bullet weighing 0.03 lb and having a muzzle velocity of 2400 ft/sec is fired against the stick, striking it horizontally at a point 1 ft above the lower end and penetrating to a depth of 6 in. Through what angle will the stick be made to swing by the impact? *Ans.* 4.3°.

3. A steel rotor spins on frictionless bearings and in a vacuum, so that there is no resistance to its rotation. If its temperature increases 200° F and the coefficient of thermal expansion for the steel is 0.0000065 per degree F, by what percentage will the angular velocity change?

4. The power of an operating hydraulic turbine equals the product of the angular velocity of the turbine and the rate at which angular momentum (about the axis of rotation) of the flowing water is changed in its passage through the turbine. Prove.

5. Refer to Ex. 2 of Art. 127, and determine the couple C_y by the force-angular momentum relationship.

6. Refer to Prob. 2 of Art. 122. Suppose the floor to be perfectly smooth. Determine the angular velocity of the spool and the linear velocity of its center when the 60-lb pull, applied as described, has acted for 5 sec, the spool being initially at rest. (Solve by force-momentum relationship, and check by finding the angular and linear accelerations.)

Art. 166. Impulse-Momentum Relationship

1. A block weighing 150 lb rests on a horizontal floor; the kinetic friction between block and floor is 0.2. A horizontal force of 60 lb always acting in the same direction is applied to the block for 10 sec. What is the velocity of the block at the end of that time?

2. A block slides down a 30° incline. If the coefficient of kinetic friction is 0.15, how long will it take for the velocity to change from 10 ft/sec to 40 ft/sec?

3. A block weighing 120 lb rests on a smooth horizontal floor. By means of a cord a force P, always equal to 60 lb, is applied to the block. At first P acts vertically, but the cord swings down with a uniform angular velocity of 0.5 rad/sec until it becomes horizontal. What is the velocity of the block at the instant P becomes horizontal? *Ans.* 32.2 *ft/sec.*

4. Assume that in Prob. 3 the coefficient of friction (static and kinetic) between floor and block is 0.2, and then solve.

5. Refer to Ex. 2 of Art. 168, where it was shown that the blast pressure from a bomb would not overturn the wall there described. Suppose that the wall simply rests on level ground, the coefficient of friction (static and kinetic) between wall and ground being 0.4. Determine the approximate distance the wall will be made to slide by the blast pressure.

Art. 168. Angular Impulse-Momentum Relationship

1. A homogeneous cylindrical disk, 4 ft in diameter and weighing 200 lb, is mounted on a frictionless horizontal axle collinear with the axis of the cylinder. A constant couple C whose moment is 50 ft-lb, in a plane normal to the axis of the cylinder, acts on the cylinder. How long will it take for C to increase the angular velocity of the cylinder from 100 rev/min to 200 rev/min? *Ans.* 2.6 *sec.*

2. Suppose that the couple C of Prob. 1 is not constant, but varies, its moment in foot-pounds being equal to $4t$ where t = time in seconds after the couple starts to act. If the cylinder is at rest when $t = 0$, how long will it take for it to attain an angular velocity of 100 rev/min?

Art. 169. Collision; Impact Forces

1. A $5\frac{1}{4}$-oz baseball having a horizontal velocity of 120 ft/sec is struck by a bat and given a velocity of 150 ft/sec in a direction at 150° to its initial velocity. If the time of contact between ball and bat is 0.025 sec, what are the direction and magnitude of the time-average force exerted by the bat on the ball?

2. A bullet moving 2000 ft/sec strikes a wooden block that is held rigidly in place, and penetrates to a depth of 8 in. The time required for this penetration to take place is 0.00055 sec. Calculate the space-average and the time-average value of the resisting force offered by the block.

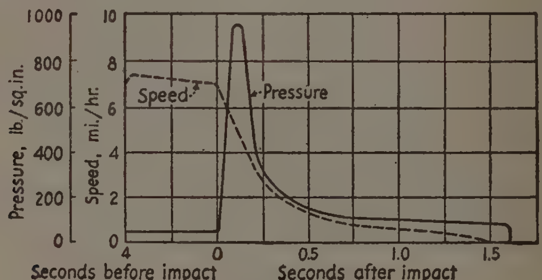

FIG. 239.

3. In connection with a lawsuit concerning an automobile accident, the following problem arose: One car (A) was standing on a level road with the brakes tightly set. A second car (B) ran into A from the rear, and as a result of the collision the two cars moved together along the road for a distance of 50 ft before stopping. Car A weighed 3600 lb; car B weighed 4200 lb. Tire marks on the pavement showed that the driver of B had applied his brakes and slid, with all four wheels locked, 20 ft before striking A. The collision knocked the driver of B unconscious, causing the brakes to be released, and also threw the gear into neutral. Experiment showed the coefficient of friction between the wheels of B and the pavement to be 0.3 and the rolling resistance of B to be 80 lb. For A, the coefficient of kinetic friction between the tires and pavement was found to be 0.32. The problem was to calculate, from these data, the velocity of B at the instant of collision. Solve, and explain the reasons why the result obtained can be regarded as only approximate.

General

1. A man in a light boat, on still water, runs to the bow of the boat and stops. After the man stops running, will the boat be moving forward, be moving rearward, or be at rest? Could the man, by alternately running or walking toward the bow and toward the stern, make the boat move a considerable distance? (The resistance of water to the movement of a boat increases with the velocity.)

2. Figure 239 is a part copy of a figure from a report on certain tests of a hydraulic (railway) buffer by Mr. Carl Schwartz, published in the *Journal of the American Society of Mechanical Engineers* for June, 1913. An abstract of the report is printed in *Engineering News* for Sept. 11, 1913. The buffer consists essentially of a cylinder 22 in. in diameter, and a piston; the working stroke is 11 ft. The buffer is firmly anchored at the stopping point, with the piston rod in the line of approach of the buffer of the car or locomotive to be stopped. The cylinder is grooved so as to allow water to pass by the piston during a stop.

The curve marked "speed" shows how the speed of the locomotive, in this instance, varied during a few seconds preceding impact and also during the impact. Thus the speed was 7.3 mi/hr about 3 sec before impact; then it decreased uniformly (while the locomotive coasted) up to the instant of impact, after which it decreased much more rapidly. The curve marked pressure shows how the hydraulic pressure behind the piston varied during the impact. Thus the initial pressure on each side of the piston was about 45 lb/in.^{2}; after the instant of impact the pressure shot up to a maximum of 925 lb/in.^{2} and then decreased to about 80. The entire travel of the piston in this case was 3 ft (not indicated in the figure). The locomotive weighed 100 tons.

Compute the time-average and the space-average force which stopped the locomotive, neglecting the effect of the train resistance. Estimate the train resistance from the retardation of the locomotive while coasting just before the impact, and then recompute the averages just mentioned. Measure the area under the pressure curve and interpret it. Does the shape of the curve suggest any improvement in the buffer?

3. A man is driving a car at 40 mi/hr around a curve of 300-ft radius when he suddenly runs onto an icy stretch where the coefficient of friction is practically zero. The road is level and has wide level shoulders, over which the ice extends. What will the car do?

4. Two hockey players, A and B, collide. At the moment of impact, A, who weighs 160 lb, is skating due north at a speed of 20 ft/sec; B, who weighs 170 lb, is skating north 30° east at a speed of 30 ft/sec. To prevent a fall, the skaters cling together. With what speed and in what direction are they moving immediately after collision?

5. According to the principle of conservation of angular momentum, a body cannot gain angular momentum about any axis unless external force on the body exerts angular impulse about that axis. Can you reconcile this with the fact that an aerialist can reverse his position while in the air, so as to be facing one way when he releases one trapeze and facing the opposite way when he grasps the second trapeze, or that a cat, dropped back down, will alight on its feet? Explain how it is possible for these maneuvers to be executed. (*Hint:* Note that the man and cat are not rigid bodies, and consider whether it is necessary for them to acquire any angular momentum in order to apparently turn as they do.) Explain also how it is possible for a child in a swing or a gymnast on the flying rings, starting from rest, to "build up" a swing without being pushed and without touching the ground.

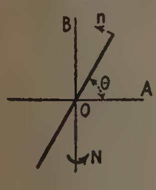

FIG. 240.

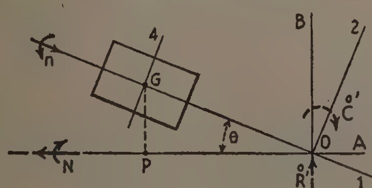

FIG. 241.

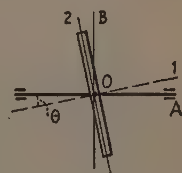

FIG. 242.

6. A ski jumper, while in the air, sometimes extends his arms laterally and rotates them vigorously in order to change the slope of his body. Which way should he rotate his arms in order to tilt his body forward? Explain why this scheme is effective.

CHAPTER XV

1. A slender rod is mounted at its midpoint O, Fig. 240, on a horizontal shaft so that it can be rotated in a vertical plane about the shaft. The shaft is mounted so that it can be rotated about a fixed vertical line OB. (The shaft does not show in the figure because it is now perpendicular to the paper.) O–ABC is a coordinate frame that rotates with the shaft about the vertical line.

Suppose that the rod is rotating at a constant speed n about the shaft, and that the shaft is rotating at a constant speed N about OB. Let I denote the moment of inertia of the rod about any line perpendicular to its axis at O, and take the moment of inertia about the axis of the rod as zero. Show that the inertia couples about OA, OB, and OC respectively are $-2INn\sin^2\theta$, $+INn\sin 2\theta$, $-\frac{1}{2}IN^2\sin 2\theta$.

2. The rectangle in Fig. 241 represents a homogeneous cylinder; G is its center of gravity and GO its axis; the diameter is 8 in. and the altitude 12 in.; it weighs 20 lb. The cylinder spins about its axis, and this axis rotates, or precesses, about the fixed line OP; $OG = 10$ in. and $\theta = 30°$. The velocities of spin and precession (n and N) are directed as indicated and are constant in magnitude; $n = 3000$ and $N = 1000$ rev/min. It is required to find the inertia system, for the cylinder, referred to O. *Ans. The resultant of the inertia system consists of a force* $\mathring{R}'$ *and couple* $\mathring{C}'$ *as indicated;* $\mathring{R}' = 2835$ *lb and* $\mathring{C}' = 2775$ *ft-lb.*

3. Figure 242 represents a flywheel imperfectly mounted on its shaft, the axis of the wheel being inclined to the axis of rotation. When the system is rotating, the wheel tends to bend the shaft. Show that the wheel is exerting a couple $\mathring{C}'$ on the shaft having a moment $\frac{1}{2}(I_1 - I_2)\sin 2\theta\cdot\omega^2$, where I_1 and I_2 denote principal moments of inertia of the wheel about axes 1 and 2, θ the small angle as indicated, and ω angular velocity of the wheel.

(The rotating flywheel has more than one fixed point; of these we choose O. Any point P of the wheel moves on a sphere whose center is O; P is confined to a certain path on the sphere. And so the motion of the wheel is a special or limited spherical motion. It could be proved that $h_1 = I_1\omega_1$ and $h_2 = I_2\omega_2$.)

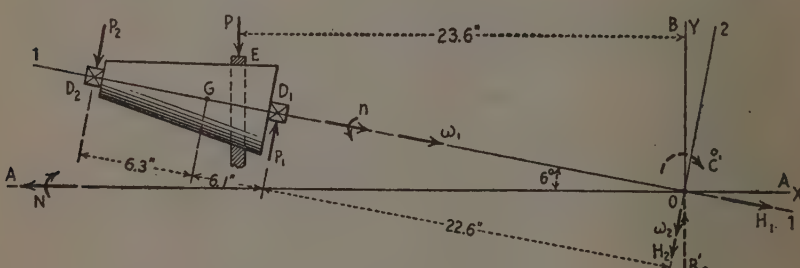

FIG. 243.

4. Refer to Fig. 329, Art. 181, and the accompanying description. The flywheel weighs 250 lb; its moment of inertia about its axis of rotation is 50,000 lb-in.^{2}, and about a diameter 20,000 lb-in.^{2}; $OG = 11$ ft. The speed of the wheel is 960 rev/min; the swing of the cab to or fro is made at a constant speed of 3.3 rev/min, except when starting and stopping which occur in 0.4 sec at constant acceleration. Calculate the inertia systems for the wheel in the starting, the stopping, and the intermediate period. *Ans. During the entire period there are: (i) An inertia couple* $\bar{C}_3' = 1085\ N$ *ft-lb, where N is speed of swinging in radians per second; the maximum value of this couple is* 375 *ft-lb. (ii) An inertia force (centrifugal)* $\bar{R}_n' = (250/32.2) \times 11 \times N^2$*; its maximum value is about* 10 *lb. During the starting and stopping periods there are also: (i) an inertia couple* $\bar{C}_2' = 3.7$ *ft-lb; (ii) an inertia force* $\bar{R}_t' = (250/32.2) \times 11 \times (dN/dt) \approx 75$ *lb.*

5. Suppose that the shaft of Prob. 1 is rotating, not about OB, Fig. 240, but about a fixed line parallel to and b distant from OB. Let the velocities of rotation be as in Prob. 1. Determine the inertia system referred to the center of gravity of the rod.

6. Refer to Prob. 2 and find the inertia system, for the cylinder, referred to G. *Ans. The resultant consists of a force* $\bar{R}'$ *acting from P to G and a couple* $\bar{C}'$ *in the plane of the figure;* $\bar{R}' =$

2835 *lb and* $\bar{C}' = 731$ *ft-lb, clockwise.* (*This inertia system and the one found in Prob.* 2 *are equivalent.*)

7. The flywheel of the engine of a certain automobile rotates clockwise, to the driver. What is the nature of the inertia couple (i) for or on the flywheel and (ii) on a wheel of the car when the car is turning a corner at constant speed and to the driver's right?

8. The trapezoid in Fig. 243 represents a frustum of a cone supported by radial bearings D_1 and D_2 fixed in a cage or housing which rotates about the fixed line OA. The frustum rolls on a fixed ring E which encircles the frustum or roller. Thus the roller spins about its axis $O1$, and this axis precesses about OA. The pressures P, P_1, and P_2 respectively at the ring and the bearings D_1 and D_2 due to the inertia forces are required. The following data are given: weight of roller = 13 lb; $I_1 = 0.00463$ and $I_2 = 3.172$ slug-ft^2; N (velocity of precession) = 100, and n (velocity of spin) = 300 rad/sec; dimensions as in the figure. G is the center of gravity of the roller. (Take P vertical, and P_1 and P_2 perpendicular to $O1$.) *Ans.* P = 1010, $P_1 = P_2 = 1280$ *lb.*

APPENDIX B

1. Show that the moment of inertia of a slender wire bent into a circular arc with respect to its axis of symmetry is

$$\tfrac{1}{2}mr^2\left(1 - \frac{\sin\alpha}{\alpha}\right)$$

where m is the mass of the wire, r the radius of the arc, and α the angle subtended, at the center of the circle, by the arc.

2. Show that the moment of inertia of a homogeneous right circular cone about its axis is $\tfrac{3}{10}mr^2$, where m is the mass of the cone and r the radius of the base.

3. Show that the moment of inertia of a straight slender rod about an axis through one end of and perpendicular to the rod is $\tfrac{1}{3}ml^2$, where m is the mass and l the length of the rod.

4. A solid piece of cast iron consists of two parts, a right-circular cylinder and a right-circular cone. The parts have a common base, 4 in. in diameter; their altitudes are 10 in. and 7 in. respectively. Determine the radius of gyration of the piece of cast iron with respect to its axis. *Ans.* 1.36 *in.*

5. A slender rod, 6 ft long and weighing 30 lb, has a right-angle bend at midlength. Find the moment of inertia of the rod about an axis through one end of the rod and perpendicular to the plane of the bent rod.

6. Calculate the moment of inertia of a cast-iron sphere 24 in. in diameter with respect to a line tangent to the sphere. *Ans.* 82 *sl-ft*^{2}.

7. A circular disk of cast iron 12 in. in diameter and 2 in. thick has a circular hole 6 in. in diameter through the disk. The axes of the disk and hole are parallel and 2 in. apart. Calculate the moment of inertia of the piece of cast iron with respect to the axis of the hole.

8. The outer and inner diameters respectively of the rim of a certain flywheel are 88 and 76 in.; the width is 12 in. The outer and inner diameters respectively of the hub are 8 and 4 in.; the width is 10 in. There are six spokes; the cross section of each is a 2-by-6-in. rectangle, the 2-in. side being parallel to the axis of the flywheel. Calculate the moments of inertia of the rim, of the hub, and of the group of spokes about the axis of the wheel.

9. The lengths of the edges of a homogeneous right parallelepiped are 2, 3, and 4 in. Calculate the radius of gyration of the solid with respect to a diagonal. *Ans.* 1.18 *in.*

10. Consider that the axis of a three-blade airplane propeller is a principal axis at the center of gravity G. What can you say about the inertia ellipsoid for the propeller at G? What can you say about the moments of inertia of the propeller with respect to various lines through G and perpendicular to its axis?

APPENDIX C

1. Refer to Fig. C1*d* (Art. C1). Suppose that the suspended body weighs 10 lb and that when the body is in its equilibrium position the stretch of the spring is 2 in. Calculate the natural frequencies of the system, p (rad/sec) and n (cyc/min); also the period T (sec). *Ans.* $p = 13.9$. $n = 132.6$.

2. Suppose that a vibration of the system of the preceding problem is started from a position of the suspended body 1.5 in. above the equilibrium position. Calculate the maximum velocity and the maximum acceleration in the ensuing vibration.

3. Referring to the preceding problem, suppose that the vibration is started from a position of no stretch. What is the maximum spring tension in the ensuing vibration?

4. A, B, C, and D, Fig. 244, are exactly alike; also the seven springs not drawn to scale. Compare the natural frequencies of vertical vibrations of the four systems. (The amplitudes are small so that no spring becomes slack.)

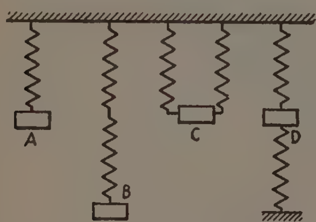

FIG. 244.

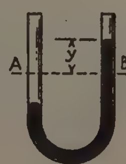

FIG. 245.

FIG. 246.

5. Figure 245 represents an oscillating column of liquid in a vertical U-tube; AB is the level of the ends of the column when at rest. Show that the oscillation is a shm and that the period is $2\pi\sqrt{l/2g}$, where l is the length of the column.

6. The inertia of the spring is not taken into account in the theory of Art. C3. It may be allowed for in Eq. 3 by adding to m one-third the mass of the spring. Make this correction to your solution of Prob. 1 for a spring weighing 0.8 lb (8 per cent of the weight of the suspended body). What is the percentage difference in the two answers for p?

7. The frequency formula $\sqrt{k/m}$ does not allow for the inertia of the springs, Fig. 246. Suitable allowances appear in the following formulas

$$\sqrt{\frac{k}{m + \frac{1}{4}m'}} \qquad \sqrt{\frac{k}{m + \frac{1}{2}m'}}$$

where m' denotes the mass of the spring. Calculate the percentage errors in $\sqrt{k/m}$ if $m' = 0.1m$ in each spring system.

8. (i) What is the effect on the frequency of the pendulum system in Fig. 247 of lowering the springs (increasing a)? (ii) Compare the natural frequencies of the system with and without springs. (iii) Compare the natural frequencies of the system as represented and inverted (upside down). (iv) Find the natural frequencies of the system as represented and inverted. (Regard the diameter of the bob as small compared to b, and neglect the weight of the pendulum rod.) *Ans. for inverted pendulum* $p^2 = \dfrac{2ka^2g}{Wb^2} - \dfrac{g}{b}$, *where k is the factor for one spring, and W the weight of the bob.*

9. Figure 248 represents a bent lever hinged at C and held in the position indicated by two springs, under tension and fastened to a support D. G is the center of gravity of the lever.

Show that the natural frequency p of the system for small oscillations is given by

$$p^2 = \frac{k_1a^2 + k_2b^2}{I}$$

where I is the moment of inertia of the lever about the hinge, and k_1 and k_2 are spring factors as indicated.

10. The disk of a torsion pendulum weighs 40 lb, and its diameter is 12 in. The torsional spring factor of the rod is 20 in.-lb/rad. Find the natural frequency of vibration of the pendulum. *Ans.* 31.3 *vib/min.*
(This answer was arrived at without allowing for the inertia of the pendulum rod, relatively very small. To allow for this inertia, when desired, regard I in Eq. 3, Art. C4, as the sum of the moment of inertia of the disk and one-third the moment of inertia of the rod, both about the axis of oscillation.)

11. Compare the natural frequencies of two torsional pendulums, Fig. C4*b*, that are just alike except that (*a*) the thicknesses of the disks are as 1 to 2; (*b*) the diameters of the disks are as 1 to 2.

12. The angle of twist of a shaft is given by

$$\theta = \frac{32Ml}{\pi E_s d^4}$$

where d denotes the diameter and l the length of the shaft, M the twisting moment applied at each end, and E_s the modulus of rigidity of the material of the shaft. Compare the spring factors of two shafts just alike except that (*a*) their lengths are as 1 to 2; (*b*) their diameters are as 1 to 2.

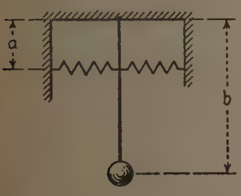

FIG. 247.

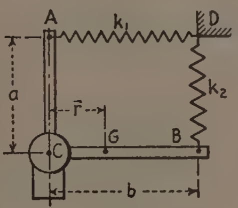

FIG. 248.

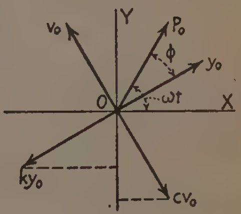

FIG. 249.

13. In a certain system, typified by Fig. C6, m and k respectively are 80 lb and 20 lb/in. From a record of a free vibration it was found that, in 6 cycles, the ratio of the first and last maximums was 2.50. Assume that the damping was viscous, and find the damping coefficient c. *Ans.* 1.18 *lb per ft/sec.*

14. Let Y_1, Y_2, Y_3, etc., denote successive maximums on a graph of a damped free vibration. Suppose that the ratios

$$\frac{Y_1}{Y_2}, \frac{Y_1}{Y_3}, \frac{Y_1}{Y_4}, \frac{Y_1}{Y_5}$$

respectively were ascertained from graphs of the vibrations of four different systems and that each ratio was 2! Find the damping ratio for each system, assuming that the damping was viscous.

15. For a given system typified in Fig. C9*a*, $k = 1000$ lb/in., $p = 8\pi$ rad/sec, $c/c_c = 0.2$, and the excitation is $P = 32 \sin 6\pi t$, where P is in pounds, t in seconds, and all angles in radians. Find the amplitude of the steady-state forced vibration in inches and the phase relation between the vibration and the excitation.

16. The vectors of Fig. 249 pertain to the forced vibration discussed in Art. C7. Their

magnitudes are y_0, P_0, v_0, etc. Suppose that all are rotating about O with constant angular velocity ω. Their projections on the y axis for any instant or position of the vectors represent various quantities, namely: the projection of vector y_0 represents the displacement $y = y_0 \sin(\omega t - \phi)$; the projection of vector v_0 represents the velocity $v = y_0\omega \cos(\omega t - \phi)$ (Art. 103); the projection of vector P_0 represents the excitation $P = P_0 \sin \omega t$; the projection of vector ky_0 represents the spring force $-ky$, without damping; and the projection of vector of cv_0 represents the damping force $-cv$.

As explained in Art. C7, the real spring force is $-(ky + cv)$. Show that its greatest value is

$$Q_0 = \sqrt{(ky_0)^2 + (cy_0\omega)^2} = \beta\sqrt{1 + \left(2\frac{c}{c_c}\cdot\frac{\omega}{p}\right)^2}\cdot P_0$$

17. When in operation, a certain machine (weight W) is subjected to a vertical excitation $P = P_0 \sin \omega t$. If the machine were resting on a perfectly rigid foundation, the pressure between machine and foundation would fluctuate between $W \pm P_0$, except for "give" or yielding in the machine itself. To lessen such fluctuation, the machine was cushioned on a nest of springs. Suppose that $W = 1000$ lb, $P_0 = 300$ lb, $k = 15{,}000$ lb/in., $\omega = 60\pi$ rad/sec (1800 cyc/min), and that the damping ratio c/c_c for the system is 0.2. Calculate the following: (i) static deflection due to W; (ii) static deflection due to P_0; (iii) natural frequency of the system, undamped; (iv) frequency ratio ω/p; (v) amplitude of the forced vibration; (vi) maximum and minimum pressures between machine and foundation. *Ans. Part* (*vi*) 1080 *and* 920 *lb.*

18. Because Q_0 (see Prob. 16) is the maximum value of the force, due to P, transmitted to the support of the spring, the ratio of that maximum to the maximum value of P is called the transmissibility of the system. Its value is

$$\frac{Q_0}{P_0} = \beta\sqrt{1 + \left(2\frac{c}{c_c}\frac{\omega}{p}\right)^2}$$

Show that (*a*) if $\omega/p > \sqrt{2}$, $Q_0/P_0 < 1$; (*b*) if $\omega/p < \sqrt{2}$, $Q_0/P_0 > 1$. ("Soft springs spell low transmissibility.")

19. A certain machine weighing 1000 lb will be subjected to a vertical excitation $P = P_0 \sin \omega t$ having a frequency of 1800 cyc/min. A spring cushion is to be provided which will insure a transmitted force not greater than $\frac{1}{10}P_0$. Assume a damping coefficient $c = 0.2c_c$, and calculate the spring factor k for the cushion. *Ans.* 4130 *lb/in.*

20. Imagine a disk rigidly mounted at the middle of a shaft supported in bearings at its ends. Suppose that the center of gravity is not in the line or axis of the bearings, so that when the shaft is rotated it is subjected to a centrifugal force $P_0 = m\bar{r}\omega^2$, where m is the mass of the disk, $\bar{r}$ the distance from the axis of the bearings to the center of gravity of the disk, and ω the angular velocity of the rotation. The system is subjected to vertical and horizontal excitations

$$P_0 \sin \omega t \qquad \text{and} \qquad P_0 \cos \omega t$$

Suppose the shaft to be guided so that it can bend in a vertical plane only. In the steady state, the displacement of the center of the disk is given by

$$y = \beta\frac{P_0}{k}\sin(\omega t - \phi)$$

where k is the spring factor for the system. Suppose the shaft to be guided so that it can vibrate in a horizontal plane only; the displacement of the center of the disk is given by

$$x = \beta\frac{P_0}{k}\cos(\omega t - \phi)$$

Show that the path of the center of the disk is a circle if there are no guides, and that the radius of the circle is $\beta(\omega/p)^2\bar{r}$. When the system has settled down into the steady state, there is no damping, and the expression above for deflection of shaft becomes

$$\frac{(\omega/p)^2}{1-(\omega/p)^2}\bar{r}$$

This becomes very large as the frequency ω of rotation approaches the natural frequency p of transverse vibration of the system; and so $\omega = p$ is a *critical* speed of the system.

21. The weight of the suspended body in Fig. C13 is 8.05 lb; the spring factor is 10 lb/in. Suppose that the box is given a vertical simple harmonic vibration whose amplitude and frequency respectively are 0.05 in. and 1 cyc/sec. Determine the amplitudes (x_0 and y_0) of the steady-state forced vibration of the body relative to the box and relative to the earth. (Assume that the damping is very small so that it may be neglected in your calculations.) *Ans.* $x_0 = 0.0046$ *and* $y_0 = 0.0546$ *in.*

22. The natural frequency p of a vibrometer is 100 cyc/min; its damping ratio is 0.2. At what frequency of forced vibration would the instrument indication of amplitude be 3 per cent in error? *Ans. About* 560 *cyc/min.*

SINES, angles 0° to 45°. Example, sin 33.3° = 0.5490

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											0.0000	90°	
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	17
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	17
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	0523	87	17
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	0698	86	17
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	0.0872	85	17
5	0.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	1045	84	17
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	1219	83	17
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	1392	82	17
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	1564	81	17
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	0.1736	80°	17
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	79	17
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	78	17
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2250	77	17
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	76	17
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	0.2588	75	17
15	0.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	74	17
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	73	17
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	72	17
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	71	17
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	0.3420	70°	16
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	69	16
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	68	16
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	67	16
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	66	16
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	0.4226	65	16
25	0.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	64	16
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	63	16
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	62	16
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	61	15
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	0.5000	60°	15
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	59	15
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	58	15
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	57	15
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	56	15
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	0.5736	55	14
35	0.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	54	14
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	53	14
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	52	14
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	51	14
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	0.6428	50°	13
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	49	13
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	48	13
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	47	13
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	46	13
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	0.7071	45°	12
45°	0.7071												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COSINES, angles 45° to 90°. Example, cos 66.6° = 0.3971

SINES, angles 45° to 90°. Example, sin 66.6°=0.9178

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9		Angle	Avge. diff.
											0.7071	45°	
45°	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	7193	44	12
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	7314	43	12
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	7431	42	12
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	7547	41	12
49	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	0.7660	40°	11
50°	0.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	7771	39	11
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	7880	38	11
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	7986	37	11
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	8090	36	10
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	0.8192	35	10
55	0.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	8290	34	10
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	8387	33	10
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	8480	32	9
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	8572	31	9
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	0.8660	30°	9
60°	0.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	8746	29	9
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	8829	28	8
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	8910	27	8
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	8988	26	8
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	0.9063	25	7
65	0.9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	9135	24	7
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	9205	23	7
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	9272	22	7
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	9336	21	6
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	0.9397	20°	6
70°	0.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	9455	19	6
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	9511	18	6
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	9563	17	5
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	9613	16	5
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	0.9659	15	5
75	0.9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	9703	14	4
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	9744	13	4
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	9781	12	4
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	9816	11	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0.9848	10°	3
80°	0.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	9877	9	3
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	9903	8	3
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	9925	7	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	9945	6	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	.9962	5	2
85	0.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	9976	4	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	9986	3	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	9994	2	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0.9998	1	0
89	0.9998	9999	9999	9999	9999	0000	0000	0000	0000	0000	1.0000	0°	0
90°	1.0000												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COSINES, angles 0° to 45°. Example, cos 33.3°=0.8358

TANGENTS, angles 0° to 45°. Example, tan 33.3° = 0.6569

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											.0000	90°	
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	17
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	17
2	0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	0524	87	17
3	0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	0699	86	18
4	0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	0.0875	85	18
5	0.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	1051	84	18
6	1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	1228	83	18
7	1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	1405	82	18
8	1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	1584	81	18
9	1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	0.1763	80°	18
10°	0.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	79	18
11	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	78	18
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	77	18
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	76	18
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	0.2679	75	19
15	0.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	2867	74	19
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	73	19
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	72	19
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	71	19
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	0.3640	70°	20
20°	0.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	69	20
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	68	20
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	67	21
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	66	21
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	0.4663	65	21
25	0.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	64	21
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	63	22
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	62	22
28	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	61	23
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	0.5774	60°	23
30°	0.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6009	59	24
31	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	58	24
32	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	57	25
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	56	25
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	0.7002	55	26
35	0.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	54	26
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	53	27
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	52	28
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	51	28
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	0.8391	50°	29
40°	0.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	49	30
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	48	31
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	47	32
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	0.9657	46	33
44	0.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.0000	45°	34
45°	1.0000												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COTANGENTS, angles 45° to 90°. Example, cot 66.6° = 0.4327

TANGENTS, angles 45° to 90°. Example, tan 66.6°=2.311

Angle	°.0	°.1	°.2	°.3	°.4	°.5	°.6	°.7	°.8	°.9			Avge. diff.
											1.0000	45°	
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	0355	44	35
46	0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	0724	43	37
47	0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	1106	42	38
48	1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	1504	41	40
49	1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	1.1918	40°	41
50°	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	2349	39	43
51	2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	2799	38	45
52	2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	3270	37	47
53	3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	3764	36	49
54	3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	1.4281	35	52
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	4826	34	55
56	4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	5399	33	57
57	5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	6003	32	60
58	6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	6643	31	64
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	1.7321	30°	67
60°	1.732	1.739	1.746	1.753	1.760	1.767	1.775	1.782	1.789	1.797	1.804	29	7
61	1.804	1.811	1.819	1.827	1.834	1.842	1.849	1.857	1.865	1.873	1.881	28	8
62	1.881	1.889	1.897	1.905	1.913	1.921	1.929	1.937	1.946	1.954	1.963	27	8
63	1.963	1.971	1.980	1.988	1.997	2.006	2.014	2.023	2.032	2.041	2.050	26	9
64	2.050	2.059	2.069	2.078	2.087	2.097	2.106	2.116	2.125	2.135	2.145	25	9
65	2.145	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	2.246	24	10
66	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	2.356	23	11
67	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	2.475	22	12
68	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	2.605	21	13
69	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	2.747	20°	14
70°	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	2.904	19	16
71	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	3.078	18	17
72	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.251	3.271	17	19
73	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	3.487	16	22
74	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.706	3.732	15	24
75	3.732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	4.011	14	28
76	4.011	4.041	4.071	4.102	4.134	4.165	4.198	4.230	4.264	4.297	4.331	13	32
77	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	4.705	12	37
78	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	5.145	11	44
79	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	5.671	10°	53
80°	5.671	5.730	5.789	5.850	5.912	5.976	6.041	6.107	6.174	6.243	6.314	9	
81	6.314	6.386	6.460	6.535	6.612	6.691	6.772	6.855	6.940	7.026	7.115	8	
82	7.115	7.207	7.300	7.396	7.495	7.596	7.700	7.806	7.916	8.028	8.144	7	
83	8.144	8.264	8.386	8.513	8.643	8.777	8.915	9.058	9.205	9.357	9.514	6	
84	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20	11.43	5	
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95	14.30	4	
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46	19.08	3	
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27	28.64	2	
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08	57.29	1	
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0	∞	0°	
90°	∞												
		°.9	°.8	°.7	°.6	°.5	°.4	°.3	°.2	°.1	°.0	Angle	Avge. diff.

COTANGENTS, angles 0° to 45°. Example, cot 33.3°=1.5224

INDEX